Abstract. In this current article, we intend to investigate $k$-almost Yamabe and gradient $k$-almost Yamabe solitons inside the setting of three-dimensional Kenmotsu manifolds.

1. Introduction

In [11] several years ago, Hamilton publicized the concept of Yamabe soliton. According to the author, a Riemannian metric $g$ of a complete Riemannian manifold $(M^n, g)$ is called a Yamabe soliton if it obeys

\begin{equation}
\frac{1}{2} \mathcal{L} W g = (r - \lambda) g,
\end{equation}

where $W, \lambda, r$ and $\mathcal{L}$ indicates a smooth vector field, a real number, the well-known scalar curvature and Lie-derivative respectively. Here, $W$ is termed as the soliton field of the Yamabe soliton. A Yamabe soliton is called shrinking or expanding according as $\lambda > 0$ or $\lambda < 0$, respectively whereas steady if $\lambda = 0$. Yamabe solitons have been investigated by several geometers in various contexts (see, [2], [3], [10], [17], [20]). The so called Yamabe soliton becomes the almost Yamabe soliton if $\lambda$ is a $C^\infty$ function. In [1], Barbosa and Ribeiro introduced the above notion which was completely classified by Seko and Maeta in [16] on hypersurfaces in Euclidean spaces.

The Yamabe soliton reduces to a gradient Yamabe soliton if the soliton field $W$ is gradient of a $C^\infty$ function $\gamma : M^n \rightarrow \mathbb{R}$. In this occasion, from (1) we have

\begin{equation}
\nabla^2 \gamma = (r - \lambda) g,
\end{equation}

where $\nabla^2 \gamma$ indicates the Hessian of $\gamma$. The idea of gradient Yamabe soliton was generalized by Huang and Li [12] and named as quasi-Yamabe gradient soliton.
According to Huang and Li, \( g \) (Riemannian metric) obeys the equation

\[
\nabla^2 \gamma = \frac{1}{m} d\gamma \otimes d\gamma + (r - \lambda) g,
\]

where \( \lambda \in \mathbb{R} \) and \( m \) is a positive constant. If \( m = \infty \), the foregoing equation reduces to Yamabe gradient soliton.

A few years ago in [14], taking \( \lambda \) as a \( C^\infty \) function, Pirhadi and Razavi investigated an almost quasi-Yamabe gradient soliton. They got a few fascinating formulas and produce a necessary and sufficient condition under which an arbitrary compact almost Yamabe soliton is necessarily gradient. Recently, Chen [5] has studied almost quasi-Yamabe solitons within the context of almost Cosymplectic manifolds.

According to Chen [4], a Riemannian metric is said to be a \( k \)-almost Yamabe soliton if there exists a smooth vector field \( W \), a \( C^\infty \) function \( \lambda \) and a nonzero function \( k \) such that

\[
\frac{k}{2} \mathcal{L}_W g = (r - \lambda) g
\]

holds. We denote the \( k \)-almost Yamabe soliton by \((g, W, k, \lambda)\). If \( W = D\gamma \), the previous equation reduces to gradient \( k \)-almost Yamabe soliton \((g, \gamma, k, \lambda)\). The \( k \)-almost Yamabe soliton is called closed if the 1-form \( W^\flat \) is closed. The \( k \)-almost Yamabe soliton becomes trivial if \( W \equiv 0 \), otherwise nontrivial. Furthermore, when \( \lambda = \text{constant} \), the previous equation gives the \( k \)-Yamabe soliton.

The above works motivate us to study \( k \)-almost Yamabe soliton in 3-dimensional Kenmotsu manifolds. Precisely, we prove the following results:

**Theorem 1.1.** There does not exist \( k \)-almost Yamabe soliton with soliton field pointwise collinear with the characteristic vector field in a Kenmotsu manifold \( M^3 \).

For \( W \) being orthogonal to the characteristic vector field, we have

**Theorem 1.2.** If the metric of a three-dimensional Kenmotsu manifold \( M^3 \) is a \( k \)-Yamabe soliton with \( W \) being orthogonal to \( \xi \), then the manifold is of constant sectional curvature \(-1\) and the \( k \)-Yamabe soliton is expanding with \( \lambda = -6 \).

**Theorem 1.3.** If a Kenmotsu manifold \( M^3 \) admits a closed \( k \)-almost Yamabe soliton, then \( M^3 \) is isometric to a Euclidean space \( \mathbb{R}^3 \).

**Theorem 1.4.** There does not exist nontrivial \( k \)-almost gradient Yamabe soliton on a Kenmotsu manifold \( M^3 \).

2. Preliminaries

Let \( M^{2n+1} \) be a connected almost contact metric manifold endowed with an almost contact metric structure \((\phi, \xi, \eta, g)\), that is, \( \phi \) is an \((1,1)\)-tensor field,
$\xi$ is a vector field, $\eta$ is a 1-form and $g$ is a compatible Riemannian metric such that
\[
\phi^2(E) = -E + \eta(E)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta\phi = 0,
g(\phi E, \phi F) = g(E, F) - \eta(E)\eta(F),
g(E, \xi) = \eta(E)
\]
for all $E, F \in \Gamma(TM)$.

If the following condition is fulfilled in an almost contact metric manifold
\[
(\nabla_E \phi) F = g(\phi E, F)\xi - \eta(F)\phi E,
\]
then $M$ is called a Kenmotsu manifold [13], where $\nabla$ denotes the Levi-Civita connection of $g$.

From the antecedent equation it is clear that
\[
(5) \quad \nabla_E \xi = E - \eta(E)\xi
\]
and
\[
(\nabla_E \eta) F = g(E, F) - \eta(E)\eta(F).
\]

In addition, the curvature tensor $R$ and the Ricci tensor $S$ satisfy
\[
R(E, F)\xi = \eta(E)F - \eta(F)E,
R(\xi, E)F = \eta(F)E - g(E, F)\xi,
R(\xi, E)\xi = E - \eta(E)\xi,
S(E, \xi) = -2\eta(E).
\]

From [8], we know that for a Kenmotsu manifold $M^3$
\[
(6) \quad R(E, F)Z = \frac{r + 4}{2}[g(F, Z)E - g(E, Z)F] - \frac{r + 6}{2}[g(F, Z)\eta(E)\xi - g(E, Z)\eta(F)\xi + \eta(F)\eta(Z)E - \eta(E)\eta(Z)F],
\]
\[
(7) \quad S(E, F) = \frac{1}{2} [(r + 2)g(E, F) - (r + 6)\eta(E)\eta(F)]
\]
where $S$, $R$ and $r$ are the Ricci tensor, the curvature tensor and the scalar curvature of the manifold respectively. Kenmotsu manifolds have been studied by several authors such as Pitis [15], De and De ([6], [7]) De, Yildiz and Yaliniz [9] and many others.

**Definition 2.1.** A vector field $W$ on an $n$ dimensional Riemannian manifold $(M, g)$ is said to be conformal if
\[
(8) \quad \mathcal{L}_W g = 2\rho g,
\]
$\rho$ being the conformal coefficient. If the conformal coefficient is zero then the conformal vector field is a Killing vector field.

In the following we write $\rho = \frac{r - \lambda}{k}$. Therefore we have the subsequent lemma:
Lemma 2.2 ([19]). On an \(2n+1\)-dimensional Riemannian manifold endowed with a \(k\)-almost Yamabe soliton, the following relations are satisfied:

\[
(\mathcal{L}_WS)(E,F) = -(2n-1)g(\nabla_E D\rho, F) - (\Delta \rho)g(E,F),
\]
\[
\mathcal{L}_W r = -2\rho r - 4n\Delta \rho
\]
for \(E,F \in \mathfrak{X}(M)\), \(D\) being the gradient operator and \(\Delta = \text{div}D\) being the Laplacian operator of \(g\).

3. Proofs of Theorems

3.1. Proof of Theorem 1.1

Here we suppose that the potential vector field \(W\) is pointwise collinear with the characteristic vector field \(\xi\) (i.e., \(Z = c\xi\), where \(c\) is a function on \(M\)).

Then from (4) we derive

\[
k\{g(\nabla_E c\xi, F) + g(\nabla_F c\xi, E)\} = 2(r - \lambda)g(E,F).
\]

Utilizing (5) in (9), we get

\[
2kc[\eta(E,F) - \eta(E)\eta(F)] + (Ec)\eta(F) + (Fc)\eta(E) - 2(r - \lambda)g(E,F) = 0.
\]

Replacing \(F\) by \(\xi\) in (10) gives

\[
( Ec ) + (\xi c)\eta(E) - 2(r - \lambda)\eta(E) = 0.
\]

Putting \(E = \xi\) in (11) yields

\[
\xi c = (r - \lambda).
\]

Substituting the value of \(\xi c\) in (11) we infer

\[
dc = (r - \lambda)\eta.
\]

Applying \(d\) on (13) and using Poincare lemma \(d^2 \equiv 0\), we have

\[
(r - \lambda)d\eta + (dr)\eta - (d\lambda)\eta = 0.
\]

Taking wedge product of (14) with \(\eta\), we obtain

\[
(r - \lambda)\eta \wedge d\eta = 0.
\]

Since \(\eta \wedge d\eta \neq 0\), we infer

\[
r - \lambda = 0.
\]

Utilizing (16) in (13) gives \(dc = 0\) i.e., \(c = \text{constant}\). Then (4) yields \(\mathcal{L}_\xi g = 0\) i.e., \(\xi\) is Killing vector field. But in a Kenmotsu manifold we know that \(\xi\) can never be a Killing vector field by (5). This finishes the proof.
3.2. Proof of Theorem 1.2

From (4) we have

$$\frac{k}{2}(g(\nabla_E W, F) + g(W, \nabla_F W)) = (r - \lambda)g(E, F)$$ for all $E, F$.

Letting $E = F = \xi$ gives $r = \lambda$ since $W$ is orthogonal to $\xi$ and Equation (5) implies $\nabla_\xi \xi = 0$. Since $\lambda$ is constant, by [20, Lemma 3.2] we know $r = \lambda = -6$. Further, it follows from (6) that $R(E, F)Z = -[g(F, Z)E - g(E, Z)F]$ for all vector fields $E, F, Z$. This means that $M^3$ is of constant sectional curvature $-1$. This completes the proof.

3.3. Proof of Theorem 1.3

Lie differentiating (7) along $W$ and utilizing Lemma 2.1, we have

$$-2g(\nabla_E D\rho, F) - 2(\Delta \rho)g(E, F)$$
$$= \mathcal{L}_W \{g(E, F) - \eta(E)\eta(F)\} + (r + 2)(\mathcal{L}_W g)(E, F)$$
$$- (r + 6)(\mathcal{L}_W \eta)(E)\eta(F) - (r + 6)\eta(E)(\mathcal{L}_W \eta)(F)$$
$$= -2(\rho \rho + 4\Delta \rho)[g(E, F) - \eta(E)\eta(F)] + 2(r + 2)g(E, F)$$
$$- (r + 6)[2\rho \eta + g(E, \mathcal{L}_W \xi)\eta(F) - (r + 6)][2\rho \eta(F) + g(F, \mathcal{L}_W \xi)\eta(E)$$
$$- (r + 6)[g(E, \mathcal{L}_W \xi)\eta(F) + g(F, \mathcal{L}_W \xi)\eta(E)]$$
$$\xi$$

from which we get

$$\nabla_E D\rho = (\Delta \rho - 2\rho)E - (\rho \rho - 2\Delta \rho + 12\rho)\eta(E)\xi$$
$$+ \left(\frac{\rho}{2} + 3\right)[g(E, \mathcal{L}_W \xi)\xi + \eta(E)\mathcal{L}_W \xi].$$

(17)

Now setting $E = \xi$ yields

$$\nabla_\xi D\rho = (-\frac{\rho \rho}{2} - \Delta \rho + 7\rho)\xi + \left(\frac{\rho}{2} + 3\right)\mathcal{L}_W \xi.$$ (18)

Let us assume that $(g, W, k, \lambda)$ is a closed $k$-almost Yamabe soliton. Then from (4) we can easily get $\nabla_E W = \rho E$. Thus utilizing (5) we infer

$$\mathcal{L}_W \xi = W - \eta(W)\xi - \rho \xi.$$ (19)

Moreover, for any vector fields $E, F$, we easily find that

$$R(E, F)W = E(\rho)F - F(\rho)E.$$ (20)

Contracting the previous equation over $E$, we have $QW = -2D\rho$. Now differentiating the above expression along $\xi$ gives $\nabla_\xi D\rho = \rho \xi$, since $Q\xi = -2\xi$ in a 3-dimensional Kenmotsu manifold. Hence from (18) we obtain

$$\left(\frac{\rho}{2} + 3\right)[F - \eta(F)\xi] = (-\Delta \rho + 3\rho)\xi.$$ (21)

This gives $\Delta \rho = 3\rho$. 
Substituting the above value and (19) into (17) yields
\[
\nabla_E D \rho = \rho E.
\]
Therefore, from [18, Theorem 2], we conclude that \( M^3 \) is isometric to a Euclidean space \( \mathbb{R}^n \).

3.4. Proof of Theorem 1.4

Let us consider a \( k \)-almost Yamabe gradient soliton \((g, \gamma, k, \lambda)\) on a Kenmotsu manifold \( M^3 \). Then equation (4) can be written as
\[
k\nabla_E D \gamma = (r - \lambda) F.
\]
Executing the covariant derivative of (23) along \( E \), we obtain
\[
k\nabla_E \nabla_F D \gamma = (E(r - \lambda))F + (r - \lambda) \nabla_E F
\]
and
\[
(24) \quad \frac{1}{k}(Ek)(r - \lambda)F.
\]
Exchanging \( E \) and \( F \) in (24), we get
\[
k\nabla_F \nabla_E D \gamma = (F(r - \lambda))E + (r - \lambda) \nabla_F E
\]
and
\[
(25) \quad \frac{1}{k}(Fk)(r - \lambda)E
\]
Utilizing (23)-(26) and together with \( R(E, F)W = \nabla_E \nabla_F W - \nabla_F \nabla_E W -
\nabla_{[E,F]}W \), we infer
\[
k^2 R(E, F) D \gamma = (r - \lambda)[E, F].
\]
Executing the inner product of (27) with \( \xi \) yields
\[
k^2 g(R(E, F) D \gamma, \xi) = -k^2\{(F\gamma)\eta(E) - (E\gamma)\eta(F)\}\}
\]
Combining equation (28) and (29), we get
\[
k^2 g(R(E, F) D \gamma, \xi) = -k^2\{(F\gamma)\eta(E) - (E\gamma)\eta(F)\}\}
\]
Now replacing \( F \) by \( \xi \), give that
\[
k^2 [E\gamma - (\xi\gamma)\eta(E)] = 0.
\]
Since \( k \neq 0 \), we immediately have

\[(32) \quad D\gamma = (\xi\gamma)\xi.\]

From the preceding equation we can write \( d\gamma = (\xi\gamma)\eta \), where \( d \) stands for the exterior differentiation. Taking exterior derivative of the previous equation yields \( d^2\gamma = d(\xi\gamma)\wedge\eta + (\xi\gamma)d\eta \). Utilizing Poincare lemma \( d^2 \equiv 0 \) in the foregoing equation and then executing wedge product with \( \eta \) we obtain \( (\xi\gamma)\eta \wedge d\eta = 0 \). Hence \( \xi\gamma = 0 \), since \( \eta \wedge d\eta \neq 0 \) in a Kenmotsu manifold. Thus we conclude that \( d\gamma = 0 \) and therefore \( \gamma \) is constant. This finishes the proof.

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