SYMMETRY AND UNIQUENESS OF EMBEDDED MINIMAL HYPERSURFACES IN $\mathbb{R}^{n+1}$

Sung-Ho Park

Abstract. In this paper, we prove some rigidity results about embedded minimal hypersurface $M \subset \mathbb{R}^{n+1}$ with compact $\partial M$ that has one end which is regular at infinity. We first show that if $M \subset \mathbb{R}^{n+1}$ meets a hyperplane in a constant angle $\geq \pi/2$, then $M$ is part of an $n$-dimensional catenoid. We show that if $M$ meets a sphere in a constant angle and $\partial M$ lies in a hemisphere determined by the hyperplane through the center of the sphere and perpendicular to the limit normal vector $n_M$ of the end, then $M$ is part of either a hyperplane or an $n$-dimensional catenoid.

We also show that if $M$ is tangent to a $C^2$ convex hypersurface $S$, which is symmetric about a hyperplane $P$ and $n_M$ is parallel to $P$, then $M$ is also symmetric about $P$. In special, if $S$ is rotationally symmetric about the $x_{n+1}$-axis and $n_M = e_{n+1}$, then $M$ is also rotationally symmetric about the $x_{n+1}$-axis.

1. Introduction

In [7], Schoen defined the notion of an end $E$ of a minimal hypersurface $M \subset \mathbb{R}^{n+1}$ being regular at infinity, and showed that a complete minimal immersion $M \subset \mathbb{R}^{n+1}$ with two ends, which are regular at infinity, is either an $n$-dimensional catenoid or a pair of hyperplanes [7]. In $\mathbb{R}^3$, Osserman showed that an end of a complete minimal surface is regular at infinity if and only if the end has finite total curvature and is embedded [5].

In [1], Choe used the Weierstrass representation formula for minimal surfaces in $\mathbb{R}^3$ and the fact that the Gauss map of a minimal surface in $\mathbb{R}^3$ is meromorphic to show that a minimal surface meeting a plane in a constant angle can be reflected across the plane. In special, Choe showed that if a complete minimal surface has finite total curvature and one end of the surface meets a plane in a constant angle, then the minimal surface is a catenoid. In [6], the authors showed that a minimal hypersurface in $\mathbb{R}^{n+1}$ meeting a sphere in a constant
angle and staying in a half space, determined by a hyperplane passing through
the center of the sphere, is part of an n-dimensional catenoid or a hyperplane.

We generalize the above results using a variation of the Alexandrov’s reflection argument based on the spherical reflection developed in [6]. Actually, we prove some rigidity and symmetry results about embedded minimal hypersurface $M \subset \mathbb{R}^{n+1}$ with one end, which is regular at infinity, meeting a hyperplane or a sphere in a constant angle. Throughout the paper, we assume that $\partial M$ is compact. First we show that if $M \subset \mathbb{R}^{n+1}$ meets a hyperplane $\Pi$ in a constant angle $\gamma \geq \pi/2$, then $M$ is part of an $n$-dimensional catenoid. (See §2 for the choice of $\gamma$.) Next we show that if $M$ meets a sphere in a constant angle and $\partial M$ lies in a hemisphere determined by the hyperplane perpendicular to $n_M$ (the limit normal vector of the end), then $M$ is part of either a hyperplane or an $n$-dimensional catenoid.

We also show that if $M \subset \mathbb{R}^{n+1}$ is tangent to a $C^2$ convex hypersurface $S$, which is symmetric about a hyperplane $P$ and $n_M$ is parallel to $P$, then $M$ is also symmetric about $P$. If $S$ is rotationally symmetric about the $x_{n+1}$-axis, then $M$ is symmetric about each hyperplane containing $x_{n+1}$-axis and parallel to $n_M$. Moreover, if $n_M = e_{n+1}$, then $M$ is rotationally symmetric.

2. Complete minimal hypersurfaces meeting a hyperplane or a sphere in constant angle

An end $E$ of a minimal hypersurface in $\mathbb{R}^{n+1}$ is regular at infinity if i) after a suitable rotation, $E$ is the graph of a function $u$ having bounded slope on the exterior of some bounded domain in $\Pi = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$, and ii) for the coordinates $x = (x_1, \ldots, x_n)$ on $\Pi$, $u$ satisfies the following asymptotic behavior for $|x|$ large: if $n = 2$

$$u(x) = a \log |x| + b + \frac{c_1 x_1 + c_2 x_2}{|x|^2} + O\left(|x|^{-2}\right),$$

if $n \geq 3$

$$u(x) = b + a|x|^{2-n} + \sum_{j=1}^{n} c_j x_j |x|^{-n} + O\left(|x|^{-n}\right)$$

for constants $a, b, c_j$ [7]. If $n = 2$ and $a \neq 0$, then $E$ is asymptotic to an end of a catenoid, and is called catenoidal. If $n = 2$ and $a = 0$ or $n \geq 3$, then $E$ is asymptotic to a (hyper)plane. If $n = 2$, then $E$ is called planar. The limit unit normal vector $n_M$ of the end is the limit of the unit normal vector of the end as $|x| \to \infty$.

Let

$$\Pi_t = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = t\},$$

$$\Pi_{[a,b]} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : a \leq x_{n+1} \leq b\}.$$
Let $\Pi_0^+ = \Pi_{[0, \infty)}$. Hence $\Pi = \Pi_0$, $\Pi^+ = \Pi_0^+$ and $\Pi^- = \Pi_0^-$. For a nonzero vector $u$, let

$$\Pi_u = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : (x_1, \ldots, x_{n+1}) \cdot u = 0\}.$$

We use similar notations for $\Pi_u$. Let $e_{n+1} = (0, \ldots, 0, 1)$.

In the hyperplane case, we assume that $M$ meets $\Pi_1$. In the sphere case, we assume that $M$ meets the unit sphere $S^n(O, 1)$. We note that $M$ divides the half space $\Pi_1^+$ (in the hyperplane case) or $\mathbb{R}^{n+1} \setminus B^{n+1}(O, 1)$ (in the sphere case) into two parts. Let $U_M$ be the component of $\Pi_1^+ \setminus M$ or $\mathbb{R}^{n+1} \setminus B^{n+1}(O, 1)$ which stays above $M$ for $x \in \Pi$ with large $|x|$. The unit normal vector $\nu$ on $M$ is chosen to point into $\overline{U_M}$, and $n_M$ is the limit of $\nu$. Note that $n_M \cdot e_{n+1} = 0$. The contact angle $\gamma$ between $M$ and $\Pi_1$ or $S^n(O, 1)$ is measured between the outward conormals $\eta$ along $\partial M$ and $\eta_{U_M}$ of $\Pi_1 \cap \overline{U_M}$ or $S^n(O, 1) \cap \overline{U_M}$ along the boundary.

We note that, outside some compact set, $M$ is a graph over $\Pi_{n_M}$. If $M$ is not flat and has an asymptotic hyperplane $\Pi_{n_M}^1$ and meets a sphere or a convex hypersurface in a constant contact angle, then $\partial M$ lies on one side of $\Pi_{n_M}^1$. Otherwise, one may use the maximum principle for the 2nd order elliptic pde [3] to see that $\eta \cdot n_M > 0$ and $\eta \cdot n_M < 0$ at the points $p_1$ and $p_2$ where $x \cdot n_M$ attains maximum and minimum on $\partial M$ respectively. Hence, $\gamma = \pi/2$ must have different signs at $p_1$ and $p_2$. In the following, we assume that either $a > 0$ in (1) or (2), or $\partial M$ lies below $\Pi_{n_M}^1$.

For completeness, we prove the following.

**Lemma 2.1.** Let $M$ be an embedded minimal hypersurface in $\mathbb{R}^{n+1}$ ($n \geq 2$) having one regular at infinity end and meeting $\Pi_1$ in a constant angle with compact $\partial M$. Then the limit unit normal $n_M$ of $M$ is perpendicular to $\Pi_1$.

**Proof.** The flux of $M$ along $\partial M$ is

$$\text{Flux}(\partial M) = \int_{\partial M} \eta.$$ 

If $n = 2$, then $\text{Flux}(\partial M) = c n_M$. Moreover, $c \neq 0$ if the end is catenoidal, and $c = 0$ if the end is planar [2]. Similarly, when $n \geq 3$, one may use (2) to see that $\text{Flux}(\partial M) = c n_M$ with $c \neq 0$ if $a \neq 0$ and $c = 0$ if $a = 0$. Clearly, $n_M$ is perpendicular to the asymptotic hyperplane of the end.

Along $\partial M$, we have $\eta = -(\sin \gamma) e_{n+1} + (\cos \gamma) \eta_{U_M}$. Since $\gamma$ is constant, we use the divergence theorem to get

$$\int_{\partial M} \eta = \int_{\partial M} (-(\sin \gamma) e_{n+1} + (\cos \gamma) \eta_{U_M}) = -(\sin \gamma) \text{Vol}(\partial M)e_{n+1}.$$ 

If $n = 2$ and the end is planar or $n > 2$ and $a = 0$ in (2), then we have $\gamma = 0$. Hence $M$ is a hyperplane and consequently $M = \Pi_1$, which contradicts that $\partial M$ is compact. Otherwise, we have $n_M$ and $e_{n+1}$ are parallel. Hence, $n_M \perp \Pi_1$. □
The spherical reflection

$$SR_1 : \mathbb{R}^{n+1} \setminus \{O\} \rightarrow \mathbb{R}^{n+1}$$

about the unit sphere $S^n(O, 1)$ is defined by

$$SR_1(x) = \frac{x}{|x|^2}.$$  

It is easy to see that for an end $\mathcal{E}$ of (1) or (2), $SR_1(\mathcal{E}) \cup \{O\}$ is $C^1$ at $O$. From [4], we recall the following.

**Lemma 2.2.** For a smooth hypersurface $M \subset \mathbb{R}^{n+1}$ and $x \in M$, the mean curvature $H$ of $SR_1(M)$ at $SR_1(x)$ is given by

$$\tilde{H}(SR_1(x)) = H(x)|x|^2 + 2x \cdot \nu,$$

where $H(x)$ is the mean curvature of $M$ at $x$ with respect to the unit normal $\nu$ of $M$.

For a unit vector $v \in \Pi_{n+1}$, let $V_0$ be the hyperplane passing through $O$ and containing the $(n-1)$-dimensional plane perpendicular to $v$ in $\Pi_{n+1}$, and making angle $\theta$ with $v$. ($\theta$ is chosen in such a way that $V_0$ is above $v$ for small $\theta$.) Let $R_\theta$ be the reflection about $V_0$. For a subset $D \subset \mathbb{R}^{n+1}$ and $0 \leq \theta < \pi/2$, let $D^\circ_\theta$ be the subset of $D$ on or below $V_0$, and let $D^\circ_\theta$ be the subset of $D$ on or above $V_0$.

**Theorem 2.3.** Let $M$ be an embedded minimal hypersurface in $\mathbb{R}^{n+1}$ having one end, which is regular at infinity. Suppose that $M \subset \Pi^+_1$ meets the hyperplane $\Pi_1$ in a constant angle $\gamma > \pi/2$ along $\partial M$, which is compact. Then $M$ is part of an $n$-dimensional catenoid.

**Proof.** From Lemma 2.1, we may assume that $M \subset \Pi^+_1$. We first show that $M$ is a graph over $\Pi_1$. Fix $t_1 > 0$ so that $M_{[t_1, \infty)} \cap \Pi_{[t_1, 2, \infty)}$ is a graph over $\Pi$ in (1) or (2), and the projection $p : M \rightarrow \Pi$ is one-to-one on $\{(x, x_{n+1}) \in M : x_{n+1} \geq t_1/2\}$.

Suppose that $M_{[1, t_1]}$ is not a graph over $\Pi$. Decreasing $t$ from $t_1$ to 1, there exists $t_0$ such that the reflection $M^{R_{[t_0, t_1]}}_{[t_0, t_1]}$ about $\Pi_{t_0}$ meets $M_{[1, t_0]}$ for the first time i) tangentially at an interior point of $M_{[1, t_0]}$, ii) tangentially at a boundary point of $M_{[1, t_0]}$ or iii) transversally at a point in $\partial M$. If i) or ii) holds, then $M^{R_{[t_0, t_1]}}_{[t_0, t_1]}$ and $M_{[1, t_0]}$ coincide by the comparison principles of the 2nd order elliptic pdes. Hence $\Pi_{t_0}$ is a symmetry plane of $M$. Since $M^{R_{[t_0, t_1]}}_{[t_0, t_1]}$ is a graph over $\Pi$, we should have $\gamma < \pi/2$, which is a contradiction. If iii) holds, then $x_{n+1}$ of $M^{R_{[t_0, t_1]}}_{[t_0, t_1]}$ has an interior local minimum, which is impossible.

Using a parallel translation, we may assume that $N \equiv (0, \ldots, 0, 1) \notin \partial M$. We note that $\tilde{M} = SR_1(M) \cup \{O\}$ is $C^1$ at $O$, $SR_1(M) \subset B^{n+1}(N', 1/2)$. Using
a parallel translation and a proper homothety, we may assume that \( SR_t(\partial M) \) lies in the upper hemisphere \( \bar{S}^+ \) of \( S^n(N', 1/2) \). Let
\[
M = SR_t(\partial U_M) \cup \{O\}.
\]

For each unit vector \( v \in \Pi \), we show that \( V_{\pi/2} \) is symmetry hyperplane of \( M \). Let \( W_t \subset \Pi_1 \) be the \((n-1)\)-plane perpendicular to \( v \) where \( t = v \cdot y \) for \( y \in W_t \). For big \( t > 0 \), \( W_t \cap \partial M = \emptyset \). Decreasing \( t \), there exists \( t_0 \) such that the reflection of \( (\partial M)^{t_0} = \partial M \cap \left( \bigcup_{t \geq t_0} W_t \right) \) about \( W_{t_0} \) is tangent to \( (\partial M)^{t_0} = \partial M \cap \left( \bigcup_{t \leq t_0} W_t \right) \) for the first time at some point, say \( x_0 \), in \( (\partial M)^{t_0} \).

Using a parallel translation and a homothety centered at \( N \), we may assume that \( t_0 = 0 \) and \( \partial M \subset B^n(N, 1) \subset \Pi_1 \). We have \( V_0 \cap M = \{O\}, W_0 \subset V_0 \) and
\[
(SR_1)^{-1} \circ R_\theta \circ SR_1 = R_\theta.
\]

Since \( M \) is a graph over \( \Pi_1 \), we have \( R_\theta((\partial M)_{\theta_0}^+) \cap M_{\theta_0}^+ = \emptyset \) for \( \theta < \pi/2 \), and \( R_\theta((\partial M)_{\theta_0}^-) \cap M_{\theta_0}^- = \{O\} \) for \( \theta > \pi/2 \). Increasing \( \theta \) from 0 to \( \pi/2 \), there exists \( \theta_0 \) such that \( R_{\theta_0}(M_{\theta_0}^-) \) and \( M_{\theta_0}^+ \) meets for the first time i) tangentially at \( x \in \text{int}(M_{\theta_0}^+) \) or ii) at \( x \in \partial \left( M_{\theta_0}^+ \right) \) tangentially. Let \( x_0 \in M_{\theta_0}^- \) for which \( R_{\theta_0}(x_0) = x \).

We show that \( \theta_0 = \pi/2 \). Otherwise, \( x \in M_{\theta_0}^+ \) and \( x \neq O \). Suppose that i) holds. Since \( R_{\theta_0}(\partial M_{\theta_0}^+) \cap M_{\theta_0}^- = \emptyset \) for \( \theta < \pi/2 \), we have \( x_0 \in \text{int}(M_{\theta_0}^-) \), and \( R_{\theta_0}(M_{\theta_0}^-) \) and \( M_{\theta_0}^+ \) are tangent at \( x \). Applying \((SR_1)^{-1}\), we see that \( R_{\theta_0}(M_{\theta_0}^-) \) and \( M_{\theta_0}^+ \) are tangent at \( SR^{-1}_1(x) \) and \( R_{\theta_0}(M_{\theta_0}^-) \) lies on one side of \( M_{\theta_0}^+ \) near \( SR^{-1}_1(x) \). Since \( R_{\theta_0}(M_{\theta_0}^-) \) and \( M_{\theta_0}^+ \) are both minimal, \( R_{\theta_0}(M_{\theta_0}^-) \) and \( M_{\theta_0}^+ \) coincide by the comparison principles for the 2nd order elliptic pde's. Hence \( V_{\theta_0} \) is a symmetry hyperplane of \( M \), which contradicts \( n_M = e_{n+1} \). If ii) holds, then \( x \in \text{int}(M) \cap V_0 \). It is easy to see that \( V_{\theta_0} \) is also a symmetry hyperplane of \( M \) as above, which is a contradiction.

Hence \( \theta_0 = \pi/2 \). By the choice of \( x_0 \) and the fact that \( \gamma \) is constant, \( R_{\pi/2}(M_{\pi/2}^-) \) and \( M_{\pi/2}^+ \) are tangent at \( SR_1(x_0) \in M_{\pi/2}^+ \setminus \{O\} \). Note that \( SR_1(x_0) \) might be a corner point on \( M_{\pi/2}^+ \cap V_{\pi/2} \). One may apply the comparison principles for the 2nd order elliptic pde's at a boundary point or at a corner point \([4]\) to see that \( R_{\pi/2}(M_{\pi/2}^-) \) and \( M_{\pi/2}^+ \) coincide as above. Hence \( V_{\pi/2} \) is a symmetry hyperplane of \( M \). Since \( v \in \Pi \) is arbitrary, \( M \) is rotationally symmetric.

In [6], the authors showed that an embedded minimal hypersurface \( M \subset \mathbb{R}^{n+1} \) with one regular at infinity end that meets \( S^n(O, 1) \) in a constant angle is a hyperplane or an \( n \)-dimensional catenoid if \( M \) stays in a half space determined by a hyperplane passing through \( O \).

The following lemma is a generalization of the result in [6]. In the following lemma, the end of \( M \) is a graph of a function \( u \) on the exterior of some compact
Lemma 2.4. Let \( H \) be an embedded minimal hypersurface in \( \mathbb{R}^{n+1} \) with one regular at infinity end, which is a graph of a function \( u \) as in (1) on the exterior of some compact set in \( H_{nM} \) and \( nM \cdot e_{n+1} > 0. \) Suppose that \( H \) meets \( S^n(O,1) \) in a constant contact angle and lies outside of \( S^n(O,1). \) If \( \partial H \) lies in the upper hemisphere and either \( n = 2 \) and \( a > 0 \) in (1) or \( \partial H \) lies below the asymptotic hyperplane \( \Pi_{nM}^1 \) of the end, then \( H \) is either a hyperplane or an \( n \)-dimensional catenoid.

Proof. We may assume that \( \partial H \) is not flat. There is \( \delta > 0 \) such that \( \partial H \subset \Pi_\delta^+ \).
Clearly, \( H^+ = H \cap \Pi^+ \) is a graph over \( \Pi. \) Note that \( \tilde{M} = SR_1(M) \cup \{O\} \) is \( C^1 \) with \( T_O \tilde{M} = H_{nM}. \) Clearly, \( \tilde{M}^+ \) lies above the reflection \( R(\tilde{M}^-) \) of \( \tilde{M}^- \) about \( \Pi, \) and is transversal to \( R(\tilde{M}^-) \) along the boundary.

We fix a unit vector \( v \in \Pi. \) Let \( U_\theta \) be the hyperplane containing the \( (n-1) \)-plane perpendicular to \( v \) in \( \Pi \) and making angle \( \theta \) with \( v. \) There exists \( \theta^0 \) such that \( U_{\theta^0} \perp H_{nM}. \) Increasing \( \theta \) from 0 to \( \theta^0, \) there is \( \theta_v \) for which \( \tilde{R}_{\theta_v}(\tilde{M}_\theta^+) \) and \( \tilde{M}_\theta^+ \) meet tangentially for the first time either at an interior point \( \tilde{x} \) of \( \tilde{M}_\theta^+ \) or at a point on \( \partial \tilde{M}_\theta^+. \) If \( \theta_v = \theta^0 \) and \( \tilde{x} = O, \) then we repeat the above process with \( -v \) instead of \( v. \) Then we get a new \( \theta_v \) and \( \tilde{x} \) such that either \( \theta_v = \theta^0 \) or \( \theta_v = \theta^0 \) and \( \tilde{x} = O. \) Since \( \tilde{R}_{\theta_v}(\tilde{M}_\theta^+) \) and \( \tilde{M}_\theta^+ \) are tangent at \( \tilde{x} \neq O, \) \( \tilde{R}_{\theta_v}(\tilde{M}_\theta^+) \) and \( \tilde{R}_{\theta_v}(\tilde{M}_\theta^+) \) are tangent at \( \tilde{x} \). We see that \( \tilde{R}_{\theta_v}(\tilde{M}_\theta^+) \) and \( \tilde{R}_{\theta_v}(\tilde{M}_\theta^+) \) coincide and \( U_{\theta_v} \) is a symmetry hyperplane of \( H. \) Moreover, we should have \( \theta_v = \theta^0. \) Since \( v \) is arbitrary, \( H \) is rotationally symmetric about the line parallel to \( n_M. \)

In the following theorem, \( \partial H \) lies in the lower hemisphere, while \( \partial H \) lies in the upper hemisphere for \( |x| \) large. The reflection \( R(\tilde{M}^-) \) of \( \tilde{M}^- \) about \( \Pi \) may not be a graph over \( \Pi. \)

Theorem 2.5. Let \( M \subset \mathbb{R}^{n+1} \) be an embedded minimal hypersurface having one end, which is regular at infinity with \( n_M = e_{n+1}. \) Suppose that \( \partial H \) lies outside of \( S^n(O,1) \) in a constant contact angle \( \gamma \) along \( \partial H, \) which is compact and lies in the lower hemisphere \( S^n(O,1) \cap \Pi^- \). Then \( M \) is part of an \( n \)-dimensional catenoid or part of a hyperplane.

Proof. Since \( M \) is minimal and \( \partial H \) lies in \( \Pi_{n+1} \) for \( |x| \) large, \( M \cap \Pi_{-1} = \emptyset \) unless \( M = \Pi_{-1}. \) We assume that \( M \) is not flat. There exists \( 0 < \delta < 1 \) such that \( M \subset \Pi_{n+1}^+ \) and \( \partial M \cap \Pi_{-1}^+ \neq \emptyset. \) It follows that \( \gamma < \pi/2. \) Fix \( t_1 > 0 \) such that \( M_{t_1/2} \) is a graph over \( \Pi, \) and the projection \( p : M \to \Pi \) is one-to-one on \( \{(x,x_{n+1}) \in M : x_{n+1} \geq t_1/2\}. \) We may assume that \( M_{t_1} \) is connected.

Step I) We show that \( M \) is a graph over \( \Pi. \) Suppose that \( M_{t_1} \) is not a graph over \( \Pi. \) Decreasing \( t \) from \( t_1 \) to \( -1, \) there exists \( t_0 \) such that the reflection
\[ M_{[t_0, t_1]}^{R_{[t_0]}} \text{ of } M_{[t_0, t_1]} \text{ about } \Pi_{t_0} \text{ meets } M_{[-1, t_0]} \text{ for the first time \( i \) tangentially at an interior point of } M_{[-1, t_0]} \text{; \( ii \) tangentially at a boundary point of } M_{[-1, t_0]} \text{ or \( iii \) transversally at a point } x_f \in \partial M. \]

If \( i \) or \( ii \) holds, then \( M_{[t_0, t_1]}^{R_{[t_0]}} \) and \( M_{[-1, t_0]} \) coincide by the comparison principles for the 2nd order elliptic pdes. Since \( \partial M \subset \Pi^- \) and \( M \) lies outside of \( S^n(O, 1) \), both \( M_{[-1, t_0]} \) and \( M_{[t_0, t_1]}^{R_{[t_0]}} \) cannot be a graph over \( \Pi \). This is a contradiction.

Now suppose that \( iii \) holds. Then \( M_{[t_0, t_1]}^{R_{[t_0]}} \cap B^{n+1}(O, 1) \neq \emptyset \). Clearly, \( M_{[t_0, \infty]} \) is a graph over \( \Pi \) and \( M_{[t_0, t_1]} \) is connected. We first show that the projection \( p : M_{[t_0, t_1]} \rightarrow B^n(O, 1) \subset \Pi \) is onto. Otherwise, there is \( q \in B^n(O, 1) \setminus p(M_{[t_0, t_1]}) \). Let \( M_{t_0}^q \) be the component of \( M \cap \Pi_{t_0} \) containing \( (q, t_0) \) inside and no other component of \( M \cap \Pi_{t_0} \). Let \( M_{-1, t_0}^q \) be the component of \( M_{[-1, t_0]} \) having \( M_{t_0}^q \) as boundary. Since \( M_{[t_0, t_1]} \) is a graph over \( \Pi \) and \( q \in B^n(O, 1) \setminus p(M_{[t_0, t_1]}) \) and
\[ M_{[t_0, t_1]}^{R_{[t_0]}} \cap B^{n+1}(O, 1) \neq \emptyset, \]
we have \( p(M_{t_0}^q) \cap B^n(O, 1) \neq \emptyset \). Moreover, \( M_{[-1, t_0]}^q \) lies between \( M_{[t_0, t_1]}^{R_{[t_0]}} \) and \( S^n(O, 1) \). Since \( \partial M \) lies in the lower hemisphere and \( M_{t_0}^q \) surrounds \( (q, t_0) \) in \( \Pi_{t_0} \) and \( p^{-1}(q, 0) \cap M = \emptyset \), it follows that \( M_{[-1, t_0]}^q \) cannot exist. Hence \( p : M_{[t_0, t_1]} \rightarrow B^n(O, 1) \) is onto.

Using the Sard’s theorem, we assume that \( M_0 = M \cap \Pi \) is regular. (One may use \( \Pi' \) instead of \( \Pi \) for small \( \epsilon > 0 \).) If \( M_0 \) is connected and encloses \( B^n(O, 1) \subset \Pi \), then, for some point \( a \in B^n(O, 1) \), \( p^{-1}(a) \cap (M \cap \Pi^+ \cap \Pi^-) \) should contain at least 2 points. In this case, \( iii \) cannot happen. Hence \( M_0 \) either consists of at least 2 components or is connected and encloses a region disjoint from \( B^n(O, 1) \subset \Pi \). It follows that \( T = \{x \in M : \Pi \cap M \text{ contains at least } 2 \text{ points}\} \) is not empty.

Let dist\((O, x)\), for \( x \in T \), attains maximum at \( \hat{P} \) and let \( w = \overline{OP}/|\overline{OP}| \). For small \( \epsilon > 0 \), let \( w_\epsilon = \hat{w} - \epsilon \). We apply the Alexandrov’s reflection argument to \( M \) using the hyperplanes \( \Pi_{w_\epsilon}^+ \). Let \( R_s \) be the reflection about \( \Pi_{w_\epsilon}^+ \). For large \( s \), \( \Pi_{w_\epsilon}^+ \) is disjoint from \( M_{[-1, t_1]} \), and \( R_s(M_{w_\epsilon}^+) \), where \( M_{w_\epsilon}^+ = \{x \in M : x \cdot w_\epsilon > s\} \), stays above \( M_{w_\epsilon}^- = \{x \in M : x \cdot w_\epsilon \leq s\} \). Decreasing \( s \), there exists \( s_0 > 1 \) such that \( R_s(M_{w_\epsilon}^+) \) meets \( M_{w_\epsilon}^- \) for the first time either \( i \) tangentially at a point of \( M_{w_\epsilon, s_0}^- \) or \( ii \) transversally at \( x_{w_\epsilon} \in \partial M \). It is easy to see that \( i \) is impossible for sufficiently small \( \epsilon > 0 \).

We show that \( ii \) is impossible for sufficiently small \( \epsilon > 0 \). Since \( P \in T \), \( \overline{OP} \cap M \) contains a point \( P' \in \Pi_{w_\epsilon}^+ \) for some small \( \delta > 0 \). It follows that, for sufficiently small \( \epsilon > 0 \), the line \( P + tw_\epsilon, t \in \mathbb{R} \), intersects \( M \) at a point \( \hat{P} \in \Pi_{w_\epsilon}^+ \) close to \( P' \), for some \( \delta > 0 \). Let \( B = \frac{\hat{P}}{2} \). Then
\[ \text{dist}\( (B, \Pi_{w_\epsilon}^+) < s_0 \) and \( \text{dist}(P, \Pi_{w_\epsilon}^+) \leq \text{dist}(P, B) \).
\]
On the other hand, for sufficiently small \( \epsilon > 0 \)
\[ \text{dist}(P, B) \leq \text{dist}(\hat{P}, \Pi_{w_\epsilon}^+) \leq \text{dist}(x_{w_\epsilon}, \Pi_{w_\epsilon}^+) \].
Hence
\[ s_0 - R_{s_0}(P) \cdot w_s/|w_s| \leq s_0 - x_0 \cdot w_s/|w_s|. \]

Let \( w^\perp \) be a vector with \( w^\perp \perp w_s, w^\perp \perp (\Pi \cap \Pi_{s_0}^\perp) \) and \( w^\perp \cdot e_{n+1} > 0 \).

Suppose that \( \Pi_{w^\perp}^\perp \) passes through \( P \). From the choice of \( P \), we see that the function \( s_0 - x \cdot w_s/|w_s| \) on \( R_{s_0}(M_{w^\perp,s_0}) \cap \Pi_{w^\perp}^\perp \) attains maximum at \( P \). From (5), it follows that \( s_0 - x \cdot w_s/|w_s| \) on \( R_{s_0}(M_{w^\perp,s_0}) \) attains an interior local maximum, which is a contradiction. Hence iii) cannot happen, which completes the proof of Step I).

Step II) We show that \( M \) is rotationally symmetric. The proof is similar to the proof of Lemma 2.4. \( \tilde{M} = SR_1(M) \cup \{O\} \) is \( C^1 \) with \( T_0 \tilde{M} = \Pi \), and meets \( S^n(O,1) \) in constant angle \( \gamma \). Since \( M \) is a graph over \( \Pi \), \( R_0(M^\perp_0) \) lies above \( M^\perp_0 \), and is transversal to \( M^\perp_0 \) along \( R_0(M^\perp_0) \cap M^\perp_0 \). Otherwise, there is either i) a point \( \tau \in (R_0(M^\perp_0) \cap M^\perp_0 \setminus \Pi \) or ii) \( \tau \in M^\perp_0 \cap \Pi \) where \( R_0(M^\perp_0) \) and \( M^\perp_0 \) are tangent. In both cases, \( M \) cannot be a graph over \( \Pi \).

Fix a unit vector \( v \in \Pi \). Increasing \( \theta \) from 0 to \( \pi/2 \), there is \( \theta_v \) for which \( R_{\theta_v}(M^\perp_0) \) and \( M^\perp_0 \) meet tangentially for the first time either at an interior point \( \tilde{x} \) of \( M^\perp_0 \) or at a point on \( \partial M^\perp_0 \). If \( \theta_v = \pi/2 \) and \( \tilde{x} = O \), then we repeat the above process with \( -v \) instead of \( v \). Then we get a new \( \theta_{-v} \) and \( \tilde{x} \) such that either \( \theta_{-v} \neq -\pi/2 \) or \( \theta_{-v} = \pi/2 \) and \( \tilde{x} \neq O \). As in the proof of Lemma 2.4, we see that \( V_0 \) is a symmetry hyperplane of \( M \), and \( \theta_0 = \pi/2 \). Since \( v \) is arbitrary, \( M \) is rotationally symmetric about the \( x_{n+1} \)-axis.

If \( \partial M \) intersects both \( \Pi^+ \) and \( \Pi^- \), then the projection \( p \) might not be onto \( B^n(O,1) \subset \Pi \). It would be interesting to prove Theorem 2.5 without conditions on \( \partial M \).

3. Minimal hypersurfaces tangent to a convex rotational hypersurface

We assume that either \( a > 0 \) in (1) or the asymptotic hyperplane \( \Pi_{a,M}^\perp \) stays above \( \partial M \) as in §2, and \( n_M \cdot e_{n+1} \geq 0 \).

**Theorem 3.1.** Let \( S \subset R^{n+1} \) be a convex \( C^2 \) hypersurface symmetric about a hyperplane \( P \). Let \( M \subset R^{n+1} \) be an embedded minimal hypersurface with one end which is regular at infinity. Assume \( \partial M \) is compact and \( M \) is tangent to \( S \). If \( n_M \) is parallel to \( P \), then \( M \) is symmetric about \( P \).

Since we assume that \( \partial M \) is compact, \( S \) can be assumed to be compact. We fix the unit normal vector \( \nu_S = \tilde{H}_S/|\tilde{H}_S| \) on \( S \), where \( \tilde{H}_S \) is the mean curvature vector of \( S \). Let \( B \) be the convex body bounded by \( S \). Then \( M \) divides \( R^{n+1} \setminus B \) into two parts. As in §2, let \( U_M \) be the component of \( R^{n+1} \setminus B \) that stays above \( M \).

**Proof of Theorem 3.1.** There is \( t_0 \) such that \( M \subset \Pi_{n,M,t_0}^+ \) and \( M \cap \Pi_{n,M,t_0} \subset \partial M \). It follows that that \( \nu = \nu_S \) along \( \partial M \).
By translating in the direction of \( n_M \), we may assume that \( S \cup M \) lies in \( \Pi_{n_M}^+ \) and outside of \( B^{n+1}(O,1) \). Let \( M = SR_1(M \cup (S \setminus \partial U_M)) \cup \{O\} \), which is \( C^1 \) and complete. Since \( M \) and \( S \) are embedded respectively, \( M \) is also embedded except for the points where \( SR_1(M) \) and \( SR_1(S) \) are tangent. We note that \( M \) is Alexandrov embedded, that is, bounds an embedded region in \( \mathbb{R}^{n+1} \).

For each unit vector \( v \in \Pi_{n_M} \) perpendicular to \( P \), we show that \( V_{\pi/2} \) is a symmetry hyperplane of \( M \). Note that \( V_0 = \Pi_{n_M} \) and \( V_0 \cap M = \{O\} \). Increasing \( \theta \) from 0 to \( \pi/2 \), there exists \( \theta_v \leq \pi/2 \) for which \( R_{\theta_v}(M_{\theta_v}^-) \) meets \( M_{\theta_v}^+ \) tangentially for the first time at \( x_f \in M_{\theta_v}^+ \).

First we show that \( \theta_v = \pi/2 \). Suppose that \( \theta_v < \pi/2 \). Since \( SR_1(S) \) is symmetric about \( V_{\pi/2} \), the point \( x_v \in M_{\theta_v}^- \) for which \( R_{\theta_v}(x_v) = x_f \) is an interior point of \( SR_1(M) \). The mean curvatures of \( R_{\theta_v}(M_{\theta_v}^-) \) and \( M_{\theta_v}^+ \) satisfy
\[
H_{M_{\theta_v}^+}(x_f) \leq H_{R_{\theta_v}(M_{\theta_v}^-)}(x_f).
\]
We note that \( \bar{H}(x_f) = H_{M_{\theta_v}^+}(x_f) \) and \( \bar{H}(x_v) = H_{R_{\theta_v}(M_{\theta_v}^-)}(x_f) \).

Let \( \bar{x} = (SR_1)^{-1}(x_f) \) and \( \bar{x}_v = (SR_1)^{-1}(x_v) \). By (3),
\[
\bar{H}(x_f) = H(\bar{x})(|\bar{x}|^2 + 2\bar{x} \cdot \nu(\bar{x})�\]
and
\[
\bar{H}(x_v) = 2\bar{x}_v \cdot \nu'(\bar{x}_v),
\]
where \( \nu' = \nu \) if \( x_v \in SR_1(M) \) and \( \nu' = \nu_S \) if \( x_v \in SR_1(S) \). Since \( SR_1 \) is conformal and \( R_{\theta_v}(M_{\theta_v}^-) \) and \( M_{\theta_v}^+ \) are tangent at \( x_f \), we have \( \bar{x} \cdot \nu_M(\bar{x}) = \bar{x}_v \cdot \nu'(\bar{x}_v) \). If \( x_f \not\in \bar{S} \setminus \bar{M} \), then \( H(\bar{x}) > 0 \) by the choice of \( \nu_S \) and
\[
\bar{H}(x_v) < \bar{H}(x_f).
\]
This is a contradiction. Hence \( x_f \in SR_1(M) \). Note that \( R_{\theta_v}(M_{\theta_v}^-) \) is tangent to \( M_{\theta_v}^+ \) at \( x_v \) and lies on one side of \( M_{\theta_v}^+ \). Since \( R_{\theta_v}(M_{\theta_v}^-) \) and \( M_{\theta_v}^+ \) are both minimal, they coincide. Hence \( V_{\theta_v} \) is a symmetry hyperplane of \( M \) and \( n_M \in V_{\theta_v} \neq V_{\pi/2} \), which is a contradiction.

Now we show that \( V_{\pi/2} \) is a symmetry hyperplane of \( M \). If \( V_{\pi/2} \) is not a symmetry hyperplane of \( M \), then we repeat the above argument with \( -v \). We must have \( \theta_{-v} < \pi/2 \), which is a contradiction. Therefore \( V_{\pi/2} \) is a symmetry hyperplane of \( M \).

Suppose that \( M \) satisfies the conditions of the above Theorem.

**Corollary 3.2.** If \( S \) is rotationally symmetric about the \( x_{n+1} \)-axis, then \( M \) is symmetric about each hyperplane containing \( x_{n+1} \)-axis and parallel to \( n_M \). If \( n_M = e_{n+1} \), then \( M \) is either a hyperplane or an \( n \)-dimensional catenoid.

**Proof.** If \( S \) is rotationally symmetric about the \( x_{n+1} \)-axis, then \( S \) is symmetric about each hyperplane containing the \( x_{n+1} \)-axis. Therefore \( M \) is symmetric about each hyperplane containing the \( x_{n+1} \)-axis and parallel to \( n_M \).
If \( n_M = e_{n+1} \), then \( M \) is symmetric about each hyperplane containing \( x_{n+1} \)-axis. Hence \( M \) is rotationally symmetric about the \( x_{n+1} \)-axis. □

References


Sung-Ho Park
Graduate School of Education
Hankuk University of Foreign Studies
Seoul 02450, Korea
Email address: sunghopark@hufs.ac.kr