A PROOF OF THE CONJECTURE OF MAZUR-RUBIN-STEIN

HAE-SANG SUN

Abstract. We present a concise proof of the conjecture of Mazur-Rubin-Stein on the distribution of modular symbols.

1. Introduction

Let $f$ be a cusp form of a level $N$ and weight 2. For $r \in \mathbb{Q} \cap (0, 1)$, one defines period integrals

$$m_{\pm}(r) = \int_r^{i\infty} 2\pi i f(z)dz \pm \int_{-r}^{i\infty} 2\pi i f(z)dz.$$

Developing a conjecture on the Diophantine stability, Mazur and Rubin [6] establish heuristics on the distribution of period integrals $m_{\pm}(r)$ for a new form and propose several conjectures:

1. The random variable $m_{\pm}$ on the rationals with the fixed denominator $M$, is asymptotically Gaussian.
2. For a divisor $g$ of $N$, there exist constants $C_f$ and $D_{f,g}$ called the variance slope and the variance shift, respectively, such that the difference between variance of $m_{\pm}$ and $C_f \log M$ converges to $D_{f,g}$ when $g$ is the G.C.D. of $M$ and $N$.
3. The integer-valued random variable $m_{\pm}/\Omega_f^\pm$ for suitable periods $\Omega_f^\pm$ is equi-distributed modulo $p$.

Petridis-Risager [7] prove average versions of (1) and (2) using a theory of Eisenstein series whose coefficients are the moments of period integrals. Even more, they give explicit expressions for the variance slopes $C_f$ and the shifts $D_{f,g}$ in terms of special values of the symmetric square $L$-function of $f$. Lee-Sun [5] also presents a proof of the average version of the conjectures including the statement (3) by studying the dynamics of continued fractions. Using a
theory of shifted convolution, Blomer et al. [1] obtain the second moment of 
$m_\pm$, i.e., the statement (2).

The above conjecture of Mazur-Rubin implies that the period integrals are
distributed with a certain regularity. Therefore, Mazur, Rubin, and Stein [6]
propose another conjecture:

Conjecture A (Mazur-Rubin-Stein). For $0 < x \leq 1$, one has
\[
\lim_{M \to \infty} \frac{1}{M} \sum_{1 \leq r \leq Mx} m_\pm \left( \frac{r}{M} \right) = \sum_{n=1}^{\infty} \frac{a_n(f)e_\pm(2nx)}{n^2},
\]

where $e_\pm$ is given by
\[
e_\pm(x) = \int_0^x \exp(2\pi it) \pm \exp(-2\pi it) dt.
\]

In this paper, we prove:

**Theorem 1.1.** Let $N$ be square-free. For any $\epsilon > 0$ and $0 \leq x \leq 1$, we have
\[
(1.1) \quad \frac{1}{M} \sum_{1 \leq r \leq Mx} m_\pm \left( \frac{r}{M} \right) = \sum_{n=1}^{\infty} \frac{a_n(f)e_\pm(2nx)}{n^2} + O_f,\epsilon \left( \frac{N^{1/2+\epsilon}}{M^{1/4-\epsilon}} \right),
\]

where the implicit constant in the error term depends only on $\epsilon$ and $f$; and independent of $f$ if $f$ is a newform.

The condition on $N$ originates from the functional equation of the $L$-functions. During preparation of the manuscript, N. Diamantis informed us that a proof of Theorem 1.1 even for a general level is obtained by Diamantis-Hoffstein-Kiral-Lee [2] using the functional equation for general levels. Let us remark that even though the level is limited, a virtue of our paper is the brevity of the proof. The role of smooth approximation of bump functions in Diamantis-Hoffstein-Kiral-Lee [2], is played by Lemma 3.1 in present paper.

**Acknowledgements.** The author is grateful to Ashay Burungale for bring his attention to the conjecture of Mazur-Rubin-Stein. He is also grateful to Barry Mazur for clarifying his understanding on the conjectures of Mazur-Rubin. The author would like to thank Nikoals Diamantis for kindly sending us a preprint about the Mazur-Rubin-Stein conjecture, a part of which inspires us to improve the error in the main result. He is grateful to anonymous referee for providing valuable suggestions to improve the manuscript.

2. Approximate functional equations of additive twists

This section is a summary of Kim-Sun [4, Section 2].

Let $f \in S_{2k}(N,\delta)$ for a Nebentypus $\delta$. Let $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ be the Fricke involution. Note that $f|W_N \in S_{2k}(N,\delta)$. For $x \in \mathbb{Q}$, let us set
\[
t(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.
\]
Let \( e(x) = \exp(2\pi i x) \). For \( f(z) = \sum_{n \geq 1} a_n e(nz) \) and \( s \in \mathbb{C} \) with \( \Re(s) > k + 1/2 \), the partial \( L \)-function is given by

\[
L(s, f, x) := \sum_{n \geq 1} \frac{a_n(f)e(nx)}{n^s}.
\]

It can be easily verified that

\[
L(1, f, x) = -2\pi i \int_x^{i\infty} f(z) dz.
\]

Let \( q > 0 \) be an integer that is not necessarily prime to \( N \). Let \( Q \) be the least common multiple of \( N \) and \( q^2 \). Let us set \( d = \gcd(q^2, N) \), \( N_0 = \frac{N}{d} \) and assume that \( \gcd(N_0, d) = 1 \). Hence \( Q = q^2 N_0 d \). Let \( \delta \) be decomposed as

\[
\delta = \delta_1 \delta_2
\]

corresponding to \((\mathbb{Z}/N\mathbb{Z})^\times \cong (\mathbb{Z}/N_0\mathbb{Z})^\times \times (\mathbb{Z}/d\mathbb{Z})^\times\). For \( x, y \in \mathbb{Z} \) with \( xd - \frac{N}{d} y = 1 \), let us set

\[
W_d = \begin{pmatrix} dx & y \\ N & d \end{pmatrix}.
\]

The matrix \( W_d \) is a normalizer of \( \Gamma_0(N) \). Let us set

\[
W_{N,d} = W_N W_d.
\]

Then \( W_{N,d} \) commutes with the Hecke operators \( T(n) \) when \( \gcd(n, N) = 1 \) and \( f|W_{N,d} \in S_{2k}(N, \overline{\delta_1 \delta_2}) \). Furthermore, we have \( f|W_{N,d}^2 = \delta(dx - N_0 y^2)f = \overline{\delta_2}(-N_0)f \). Note that if \( f \) is a newform, then \( f|W_{N,d} = \zeta f \) for a \( \zeta \in \mathbb{C} \) with \( \zeta^2 = \overline{\delta_2}(-N_0) \).

Let \( \Phi \) be an infinitely differentiable function on \((0, \infty)\) with compact support and \( \int_0^\infty \Phi(y) \frac{dy}{y} = 1 \), and set \( \kappa(t) = \int_0^\infty \Phi(y) y^t \frac{dy}{y} \). Let us set

\[
F_{1,s}(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \kappa(t) \Gamma(s + t) x^{-t} \frac{dt}{t} \quad \text{and}
\]

\[
F_{2,s}(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \kappa(-t) \Gamma(s + t) x^{-t} \frac{dt}{t}.
\]

With those settings, we have the asymptotics of \( F_{i,s} \) and the approximate functional equation of which proofs are given in Kim-Sun [4, (2.5), (2.6), (2.7)]:

**Proposition 2.1.** (1) For each \( i \), we have

\[
F_{i,s}(x) = O(\Gamma(\Re(s) + j)x^{-j}) \quad \text{for all} \ j \geq 1 \quad \text{as} \ x \to \infty.
\]

(2.2)

\[
F_{i,s}(x) = \Gamma(s) + O\left( \Gamma\left(\Re(s) - \frac{1}{2}\right)x^\frac{1}{2} \right) \quad \text{as} \ x \to 0.
\]

(2.3)
(2) For an integer \( a > 0 \) with \( \gcd(a, q) = 1 \), choose an integer \( u > 0 \) such that
\[
u \equiv -(aN_0)^{-1} \pmod{\frac{a^2}{q}}.
\] Then \( L(s, f, \frac{a}{q}) \) satisfies the approximate functional equation
\[
\Gamma(s)L\left(s, f, \frac{a}{q}\right) = \sum_{n=1}^{\infty} \frac{a_n(f)e\left(\frac{nM}{q}\right)}{n^s} F_{1,s}\left(\frac{n}{y}\right) + i^{2k} \delta_1(q) \delta_2(N_0) \left(\frac{Q}{4\pi^2}\right)^{k-s} \sum_{n=1}^{\infty} \frac{a_n\left[|W_{N,d}|e\left(\frac{nM}{q}\right)\right] \delta_2(u)}{n^{2k-s}} F_{2,2k-s}\left(\frac{4\pi^2 ny}{Q}\right),
\]
where \( y > 0 \) is a real number.

3. Proof of Theorem 1.1

Let \( f \) be a cusp form of weight 2 and a square-free level \( N \) with a Nebentypus \( \delta \). Note that we have
\[
e_+(x) = \frac{\sin(2\pi x)}{\pi} \quad \text{and} \quad e_-(x) = \frac{i(1 - \cos(2\pi x))}{\pi}.
\]
For integers \( n > 0 \) and \( M \geq 2 \) and \( 0 < x \leq 1 \), let us set
\[
U(x, M; n) := \frac{1}{M} \sum_{r=1}^{M} \left\{ e\left(\frac{rn}{M}\right) \pm e\left(-\frac{rn}{M}\right) \right\}.
\]
We need an estimate on the sum.

**Lemma 3.1.** Let \( M \geq 2 \) and \( n \geq 1 \) be integers. Let \([n]\) be the least positive residue of \( n \) modulo \( M \) and assume that \( n \) is not divisible by \( M \). Then for a real \( x \) with \( 0 \leq x \leq 1 \) we have
\[
U(x, M; n) = e_+([n]x) + O\left(\frac{1}{M}\right).
\]

**Proof.** We may assume that \( n < M \). Since \( U(x, M; n) \) is a geometric series, it is equal to
\[
\frac{e\left(\frac{n}{M}\right)e\left(\frac{|Mx|n}{M}\right) - 1}{M(e\left(\frac{n}{M}\right) - 1)}.
\]
Here note that \([n] = n\) as \( n < M \). Since \(|Mx|/M = x - \theta/M\) with \( 0 \leq \theta < 1 \), the last expression is equal to
\[
\frac{(e(xn) - 1) \mp (e(-xn) - 1) + O\left(\frac{n^2}{M}\right)}{2\pi in + O\left(\frac{n^2}{M}\right)} + O\left(\frac{1}{M}\right).
\]
Hence we finish the proof. \( \square \)
We need an estimate on an exponential sum that can be obtained from one on the Kloosterman sum (see Heath-Brown [3]):

**Proposition 3.2.** Let $I$ be a sub-interval of $[0, 1)$. Let $a$, $b$, and $M \geq 2$ be integers. Then one has

$$\sum_{r \in \mathbb{Z}} e\left(\frac{ar + br'}{M}\right) \ll \gcd(a, b, M)^{1/2} M^{1/2 + \epsilon},$$

where $\Sigma'$ is the sum over the integers $1 \leq r \leq M$ with $\gcd(r, M) = 1$.

For an integer $M$ and Dirichlet character $\delta_2$ of modulus $N/\gcd(M^2, N)$, let us set

$$V(x, M; n) = \sum_{r \leq Mx \atop \gcd(r, M) = 1} \delta_2(r') \left\{ e\left(\frac{r'n}{M}\right) \pm e\left(\frac{-r'n}{M}\right) \right\},$$

where $r'$ is the inverse of $r$ modulo $M$. We also need:

**Lemma 3.3.** For two integers $M \geq 2$, $n \geq 1$ and a real $0 \leq x \leq 1$, we have

$$V(x, M; n) \ll N^{1/2} \gcd(n, M)^{1/2} M^{1/2 + \epsilon}.$$

**Proof.** The case of $M \mid n$ is obvious. Let us assume $M \nmid n$ and set $N_0 = N/\gcd(M^2, N)$, $M = N_0 M_0$. Since $\delta_2$ is a periodic function of a period $N_0$, it can be written as

$$\delta_2(s) = \sum_{j=1}^{N_0} c_j e\left(\frac{s M_0}{M}\right)$$

form some $c_j$ with $\sum_{j=1}^{N_0} |c_j|^2 \leq 1$. Then, we obtain

$$V(x, M; n) = \sum_{j=1}^{N_0} c_j \sum_{r \leq Mx \atop \gcd(r, M) = 1} \left\{ e\left(\frac{r'(n + j M_0)}{M}\right) \pm e\left(\frac{-r'(n - j M_0)}{M}\right) \right\}.$$  

From Cauchy-Schwartz inequality and Proposition 3.2, we finish the proof. □

Weil and Deligne have shown that for a cusp form $f$ and $\epsilon > 0$, there exists a constant $b_f > 0$ dependent only on $f$ such that

$$a_n(f) \ll \epsilon b_f n^{1/2 + \epsilon}.$$  

(3.1)

Note that if $f$ is a newform, then $b_f$ is independent of $f$ and can be chosen as $b_f = 1$.

We are ready to give:

**Proof of Theorem 1.1.** First note that

$$\sum_{r \leq Mx} m_{\pm} \left(\frac{r}{M}\right) = \sum_{g \mid M} \sum_{\substack{r \leq x \atop \gcd(g, r) = 1}} m_{\pm} \left(\frac{s}{g}\right).$$  

(3.2)
From (2.1), (2.4), and (3.2), we have

\[
M \sum_{r=1}^{M} \frac{m_{\pm}}{r M} = \sum_{n=1}^{\infty} \frac{a_n(f)U(x, M; n)}{n} F_{1.1} \left( \frac{n}{y} \right) + \frac{i2k}{M} \sum_{g | M} \delta_1(g) \delta_2(N/e) \sum_{n=1}^{\infty} \frac{a_n(f)W_N(e)V(x, g; n)}{n} F_{2.1} \left( \frac{4\pi^2 ny}{N g^2/e} \right),
\]

where \( e = \gcd(g^2, N) \) and \( \delta_1, \delta_2 \) in the sum \( \sum_{g | M} \) are the Dirichlet characters of moduli \( N/e \) and \( e \), respectively. Here note that \( e \) and \( N/e \) are relatively prime since \( N \) is square-free. Let us rewrite (3.3) as

\[
S_1 + \frac{1}{M} \sum_{g | M} \delta_1(g) \delta_2(N/e) S_2(g).
\]

First of all, consider the first sum \( S_1 \). We split it into two parts,

\[
S_1 = S_{1, \leq y} + S_{1, > y},
\]

a sum over \( n \leq y \) and one over \( n > y \), respectively. Observe that by Lemma 3.1 and the estimate \( F_{1.1} \left( \frac{n}{y} \right) = O(1) \), the sum over \( n \leq y \) is equal to

\[
\sum_{n \leq y} \frac{a_n(f)\epsilon_+(\lfloor \frac{n}{y} \rfloor)}{n} F_{1.1} \left( \frac{n}{y} \right) + O \left( \frac{1}{M} \sum_{n \leq y} \frac{|a_n(f)|}{n} \right) + O \left( \sum_{n \leq y} \frac{|a_n(f)|}{n} \right).
\]

By (2.3) and the bound (3.1), this is equal to

\[
\sum_{n \leq y} \frac{a_n(f)\epsilon_+(\lfloor \frac{n}{y} \rfloor)}{n} F_{1.1} \left( \frac{n}{y} \right) + O \left( \frac{bf}{y^{1/2+\epsilon}} \right).
\]

Observe that for a real number \( b > 0 \) and \( y > M \), we obtain

\[
\sum_{n \leq y} \frac{b f (y^{1/2+\epsilon})}{n^{1/2}} + O \left( \sum_{n \leq y} \frac{n^{1/2+\epsilon}}{n} \right).
\]

Therefore, with \( b = \epsilon \), the error terms of (3.4) are equal to \( O \left( \frac{bf(y^{1/2+\epsilon})}{M} \right) \).

Separating the first summand of (3.4) into three parts, namely (1) \( n < M \), (2) \( M < n < y \) with \( M \nmid n \), and (3) \( M < n < y \) with \( M \mid n \), the first summand of (3.4) equals

\[
\sum_{n < M} \frac{a_n(f)\epsilon_+(nx)}{n^2} + O \left( \sum_{M < n \leq y} \frac{b f}{n^{1/2-\epsilon}} \right) + O \left( \frac{bfy^{1/2+\epsilon}}{M} \right).
\]
\[
= \sum_{n < M} a_n(f)e_{\pm}(nx) \frac{1}{n^2} + O \left( \frac{b_f(\log M) y^{1/2 + \epsilon}}{M} \right).
\]
Hence, finally, (3.4) is equal to
\[
S_{1, \leq y} = \sum_{n < M} a_n(f)e_{\pm}(nx) \frac{1}{n^2} + O \left( \frac{b_f(\log M) y^{1/2 + \epsilon}}{M} \right).
\]
Note that the sum over \( n > y \) in the first sum of (3.3) also can be calculated in a similar way but using (2.2) instead of (2.3). It is equal to
\[
S_{1, > y} = \sum_{n > y} a_n(f)U(M, x; n) F_{1, 1} \left( \frac{n}{y} \right) = O \left( \frac{b_f y^{1/2 + \epsilon} \log M}{M} \right).
\]
Now let us consider the sum \( S_2(g) \). We also divide it into two parts:
\[
S_2(g) = S_{2, \leq y}(g) + S_{2, > y}(g),
\]
a sum over \( n \leq Ag^2/y \) and one over \( n > Ag^2/y \), respectively for \( A = N/(4\pi^2\epsilon) \).
By Lemma 3.3 and (2.3), we obtain \( V(x, g; n) \ll N^{1/2} \gcd(g, n)^{1/2} y^{1/2 + \epsilon} \) and the sum over \( n \leq Ag^2/y \) is
\[
S_{2, \leq y}(g) \ll b_f N^{1/2} g^{1/2 + \epsilon} \sum_{n \leq Ag^2/y} \gcd(n, g)^{1/2} n^{-1/2 - \epsilon}.
\]
The last sum is equal to
\[
\sum_{d \mid g} d^{-1/2} \sum_{n \leq Ag^2/y, \gcd(n, g/d) = 1} \frac{1}{m^{1/2 - \epsilon}} \ll \left( \frac{Ag^2}{y} \right)^{1/2 + \epsilon} \sum_{d \mid g} \frac{1}{d^{1/2}}.
\]
Since \( \sum_{d \mid g} d^{-1/2} \ll g^\epsilon \), the sum \( S_{2, \leq y}(g) \) is equal to
\[
S_{2, \leq y}(g) = O \left( \frac{N^{1/2} b_f A^{1/2 + \epsilon} g^{3/2 + 3\epsilon}}{y^{1/2 + \epsilon}} \right).
\]
In a similar way as the last calculations together with (2.2), the sum over \( n > Ag^2/y \) is equal to
\[
S_{2, > y}(g) = O \left( \frac{N^{1/2} b_f A^{1/2 + \epsilon} g^{3/2 + 3\epsilon}}{y^{1/2 + \epsilon}} \right).
\]
Therefore, from the inequality \( \sum_{g \mid M} g^a \ll M^{a + \epsilon} \) for \( a > 0 \), we obtain
\[
\frac{1}{M} \sum_{g \mid M} |S_2(g)| \ll \frac{N^{1/2} b_f A^{1/2 + \epsilon} g^{3/2 + 3\epsilon}}{y^{1/2 + \epsilon}} \ll \frac{b_f N^{1/2} M^{1/2 + 4\epsilon}}{y^{1/2 + \epsilon}}.
\]
In total, setting
\[
y = N^{1+\epsilon} M^{3/2 + 3\epsilon},
\]
we complete the proof with a new \( \epsilon > 0 \).

Let \( E_M \) be the average on the rationals in \((0, 1)\) with the denominator \( M \).
Corollary 3.4. Let $N$ be square-free. For $\epsilon > 0$, we have

$$E_M[m_{\pm}] \ll_f N^{1/2+\epsilon} M^{-1/4+\epsilon},$$

where the implicit constant in the error term is independent of $f$ if $f$ is a newform.

Proof. Let us set

$$G^\pm(f; x) = \sum_{n=1}^{\infty} a_n(f) e^\pm (nx) \frac{c_\pm}{n^2}.$$

From Möbius inversion formula, we obtain

$$\phi(M) E_M[m_{\pm}] = \sum_{r \equiv 1 \pmod{M}} m_{\pm} \left( \frac{r}{M} \right) = \sum_{d|M} \mu \left( \frac{M}{d} \right) \sum_{s=1}^{d} m_{\pm} \left( \frac{s}{d} \right).$$

By Theorem 1.1, this equals

$$\sum_{d|M} \mu \left( \frac{M}{d} \right) \left[ dG^\pm(f; 1) + O \left( \frac{b_f d N^{1/2+\epsilon}}{d^{1/4-\epsilon}} \right) \right].$$

Since $\phi(M) \gg M / \log M$, $q^{3/4+\epsilon} + 1 \ll e q^{3/4+2\epsilon}$, and

$$\sum_{d|M} \left| \mu \left( \frac{M}{d} \right) \right| d^{3/4+\epsilon} = \prod_{q \nmid M} \left( 1 + q^{3/4+\epsilon} \right) \ll e^{M^{3/4+2\epsilon}},$$

we obtain the proof of the Corollary with a new $\epsilon > 0$. $\square$

References


