ESTIMATES FOR THE HIGHER ORDER RIESZ TRANSFORMS RELATED TO SCHRÖDINGER TYPE OPERATORS

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Abstract. We consider the Schrödinger type operator $L_k = (-\Delta)^k + V_k$ on $\mathbb{R}^n (n \geq 2k + 1)$, where $k = 1, 2$ and the nonnegative potential $V$ belongs to the reverse Hölder class $RH_s$ with $n/2 < s < n$. In this paper, we establish the $(L^p, L^q)$-boundedness of the higher order Riesz transform $T_{\alpha, \beta} = V^{2\alpha} \nabla^2 L_k^{1-\beta} (0 \leq \alpha \leq 1/2 < \beta \leq 1, \beta - \alpha \geq 1/2)$ and its adjoint operator $T^*_{\alpha, \beta}$ respectively. We show that $T_{\alpha, \beta}$ is bounded from Hardy type space $H^1_{L^2}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$ and $T^*_{\alpha, \beta}$ is bounded from $L^{p_1}(\mathbb{R}^n)$ into $BMO$ type space $BMO_{L^1}(\mathbb{R}^n)$ when $\beta - \alpha > 1/2$, where $p_1 = \frac{n}{1/(\beta - \alpha) - 2}$, $p_2 = \frac{n}{n - 4/(\beta - \alpha) + 2}$. Moreover, we prove that $T_{\alpha, \beta}$ is bounded from $BMO_{L^1}(\mathbb{R}^n)$ to itself when $\beta - \alpha = 1/2$.

1. Introduction and results

For $s > 1$, a nonnegative locally $L^s$-integrable function $V$ is said to belong to the reverse Hölder class $RH_s$ if there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left( \frac{1}{|B|} \int_B V(y)^s dy \right)^{1/s} \leq C \int_B V(y) dy$$

holds for every ball $B \subset \mathbb{R}^n$. It is obvious that $V$ is a doubling measure if $V \in RH_s$ with $s > 1$.

Given a potential $V \in RH_s$ with $s > n/2$, we define the auxiliary function (see [7])

$$\rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x, r)} V(y) dy \leq 1 \right\}, \quad x \in \mathbb{R}^n,$$

where $B(x, r)$ denotes the ball centered at $x$ with radius $r$. It is well known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

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We consider the Schrödinger type operator

\[ \mathcal{L}_k = (-\Delta)^k + V^k, \quad n \geq 2k + 1, \quad k = 1, 2. \]

When \( V \) is a nonnegative polynomial, Zhong [10] showed the \( L^p \) boundedness of the operators \( V^{2j/3} V^j \mathcal{L}_2^{-1} \), where \( j = 0, 1, 2, 3, 4 \). For the potential \( V \) which belongs to \( RH_s, n/2 < s < n \), and there exists a constant \( C \) such that \( V(x) \leq C \rho(x)^{-2} \), Sugano in [8] established estimates of the fundamental solutions for \( \mathcal{L}_2 \) and showed the \( L^p \) boundedness of the operators \( V^{2j/3} V^j \mathcal{L}_2^{-1} \), where \( j = 0, 1, 2, 3 \). When \( V \in RH_s \) with \( s > n/2 \), Wang in [9] obtained the \( (L^p, L^q) \)-boundedness of the operator \( V^{2\alpha} \mathcal{L}_2^\beta \) for \( 0 < \alpha \leq \beta \leq 1 \).

Let \( V \) belong to \( RH_s \) with \( n/2 < s < n \). We concentrate on the higher order Riesz transform

\[ T_{\alpha, \beta} = V^{2\alpha} \nabla^2 \mathcal{L}_2^{-\beta}, \quad 0 \leq \alpha \leq 1/2 \leq \beta \leq 1, \quad \beta - \alpha \geq 1/2, \]

and its adjoint operator \( T_{\alpha, \beta}^* \). Obviously, if \( (\alpha, \beta) = (0, \frac{1}{2}) \), \( T_{\alpha, \beta} \) just is the transform \( \mathcal{R} = \nabla^2 \mathcal{L}_2^{-1} \). Liu and Dong in [5] investigated the \( L^p \) and weak \( (1, 1) \) estimates of \( \mathcal{R} \); Liu et al. in [6] obtained the \( L^p \) boundedness of the commutators of \( \mathcal{R} \).

We first establish the following \((L^p, L^q)\)-boundedness.

**Theorem 1.1.** Suppose \( V \in RH_s \) with \( n/2 < s < n \). Let \( 0 \leq \alpha \leq 1/2 < \beta \leq 1, \beta - \alpha \geq 1/2, \) and \( \frac{1}{p} = \frac{2\alpha + 2}{s} - \frac{2}{n} \).

(i) If \( \frac{p'}{q'} < \frac{n}{4(\beta - \alpha) - 2} \) and \( \frac{1}{q} = \frac{1}{p} - \frac{4(\beta - \alpha) - 2}{n} \), then

\[ \|T_{\alpha, \beta}^*(f)\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}; \]

(ii) If \( 1 < p < \frac{1}{\frac{n}{4(\beta - \alpha) - 2} + \frac{1}{p} - \frac{4(\beta - \alpha) - 2}{n}} \) and \( \frac{1}{q} = \frac{1}{p} - \frac{4(\beta - \alpha) - 2}{n} \), then

\[ \|T_{\alpha, \beta}(f)\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}. \]

Let us recall the concept of Hardy spaces related to Schrödinger type operators.

The Schrödinger type operators \( \mathcal{L}_k = (-\Delta)^k + V^k \) \((k = 1, 2)\) generate \( C_0 \) semigroups \( \{e^{-t\mathcal{L}_k}\}_{t \geq 0} \). The maximal function with respect to the semigroup \( \{e^{-t\mathcal{L}_k}\}_{t \geq 0} \) are given by

\[ M^{\mathcal{L}_k} f(x) = \sup_{t > 0} |e^{-t\mathcal{L}_k} f(x)|. \]

By [1, 3], a function \( f \in L^1(\mathbb{R}^n) \) is said to be in \( H^1_{\mathcal{L}_k}(\mathbb{R}^n)(k = 1, 2) \) if the semigroup maximal function \( M^{\mathcal{L}_k} f \) belongs to \( L^1(\mathbb{R}^n) \). The norm of such a function is defined by

\[ \|f\|_{H^1_{\mathcal{L}_k}(\mathbb{R}^n)} = \|M^{\mathcal{L}_k} f\|_{L^1(\mathbb{R}^n)}. \]

It was showed that \( H^1_{\mathcal{L}_k}(\mathbb{R}^n) = H^1_{\mathcal{L}_1}(\mathbb{R}^n) \) with equivalent norms (see Theorem 1.1 in [1]).
Theorem 1.2. Suppose \( V \in RH_s \) with \( n/2 < s < n \). Let \( 0 \leq \alpha \leq 1/2 < \beta \leq 1, \beta - \alpha \geq 1/2 \). Then
\[
\|T_{\alpha,\beta}(f)\|_{L^{p_2}(\mathbb{R}^n)} \leq C\|f\|_{H^1_\alpha(\mathbb{R}^n)},
\]
where \( p_2 = \frac{n}{\alpha - \beta + 1} \).

The dual space of \( H^1_\alpha(\mathbb{R}^n) \) is the \( BMO \) type space \( BMO_{\mathcal{C}_1}(\mathbb{R}^n) \) (see [2]). Let \( f \) be a locally integrable function on \( \mathbb{R}^n \) and \( B = B(x,r) \). Set \( f_B = \frac{1}{|B|} \int_B f(y)dy \) and \( f(B,V) = f_B \) if \( r < \rho(x) \); \( f(B,V) = 0 \) if \( r \geq \rho(x) \). We say \( f \in BMO_{\mathcal{C}_1}(\mathbb{R}^n) \) if
\[
\|f\|_{BMO_{\mathcal{C}_1}(\mathbb{R}^n)} = \sup_B \frac{1}{|B|} \int_B |f(y) - f(B,V)|dy < \infty.
\]
It follows from [2] that \( \|f\|_{BMO_{\mathcal{C}_1}(\mathbb{R}^n)} \) is actually a norm which makes \( BMO_{\mathcal{C}_1}(\mathbb{R}^n) \) a Banach space. Since \( H^1(\mathbb{R}^n) \subset H^1_\alpha(\mathbb{R}^n) \), we conclude by duality that \( BMO_{\mathcal{C}_1}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n) \).

By Theorem 1.2 and duality, we get:

Corollary 1.3. Given \( V \in RH_s \) with \( n/2 < s < n \). Let \( 0 \leq \alpha \leq 1/2 < \beta \leq 1 \). If \( \beta - \alpha > 1/2 \), we have
\[
\|T^*_\alpha,\beta(f)\|_{BMO_{\mathcal{C}_1}(\mathbb{R}^n)} \leq C\|f\|_{L^n(\mathbb{R}^n)},
\]
where \( p_1 = \frac{n}{\beta - \alpha - 1} \).

In case \( \beta - \alpha = 1/2 \), we have:

Theorem 1.4. Given \( V \in RH_s \) with \( n/2 < s < n \), and \( 0 \leq \alpha \leq 1/2 < \beta \leq 1 \). If \( \beta - \alpha = 1/2 \), we have
\[
\|T^*_\alpha,\beta(f)\|_{BMO_{\mathcal{C}_1}(\mathbb{R}^n)} \leq C\|f\|_{BMO_{\mathcal{C}_1}(\mathbb{R}^n)}.
\]

2. Preliminaries

Throughout this section, we always assume \( V \in RH_s \) with \( n/2 < s < n \). Let us first recall some important properties concerning the auxiliary function.

Lemma 2.1 ([4]). There exist constants \( C \) and \( l_0 > 0 \) such that
\[
\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq C \left(1 + \frac{r}{\rho(x)}\right)^{l_0}.
\]

Lemma 2.2 ([7]). For \( 0 < r < R < \infty \), we have
\[
\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq C \left(\frac{R}{r}\right)^{n/s-2} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy.
\]
Lemma 2.3 ([7]). There exist constants $C$ and $k_0 \geq 1$ such that
\[
C^{-1} \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{-k_0} \leq \frac{\rho(y)}{\rho(x)} \leq C \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{\frac{1}{1 + k_0}}
\]
for all $x, y \in \mathbb{R}^n$.

A ball $B(x, \rho(x))$ is called critical. Assume that $Q = B(x_0, \rho(x_0))$. For $x \in Q$, Lemma 2.3 tells us that $\rho(x) \sim \rho(y)$, if $|x - y| < C\rho(x)$.

Let $W_\beta = \nabla^2 L_2^{-\beta}$, and let $K_\beta, K'_\beta$ be the kernels of $W_\beta$ and $W'_\beta$, respectively. Then $K'_\beta(x, z) = K_\beta(z, x)$ and we have the following estimates.

Lemma 2.4. Suppose $1/2 < \beta \leq 1$.

(i) For every positive integer $N$ there exists a constant $C_N$ such that
\[
|K'_\beta(x, z)| \leq \frac{C_N}{(1 + \frac{|x - z|}{\rho(x)})^N} \frac{1}{|x - z|^{n-4\beta}} \times \left( \int_{B(z, |x - z|/4)} \frac{V(\xi)^2}{|\xi - z|^{n-2}} d\xi + \frac{1}{|x - z|^2} \right).
\]
Moreover, the inequality above also holds with $\rho(x)$ replaced by $\rho(z)$.

(ii) For every positive integer $N$ and some $\delta > 0$ there exists a constant $C_N$ such that
\[
|K'_\beta(x, z) - K'_\beta(y, z)| \leq \frac{C_N}{(1 + \frac{|x - z|}{\rho(x)})^N} \frac{|x - y|^{\delta}}{|x - z|^{n-4\beta + \delta}} \left( \int_{B(z, |x - z|/4)} \frac{V(\xi)^2}{|\xi - z|^{n-2}} d\xi + \frac{1}{|x - z|^2} \right),
\]
whenever $|x - y| < \frac{1}{10} |x - z|$. Moreover, the inequality above also holds with $\rho(x)$ replaced by $\rho(z)$.

Proof. Let $\Gamma_{L_2}(x, y, \lambda)$ be the fundamental solution of $L_2 + \lambda$, where $\lambda \geq 0$. When $\lambda = 0$, it follows from Theorem 2 in [8] that for any positive integer $N$ there exists a positive constant $C_N$ such that
\[
0 \leq \Gamma_{L_2}(x, y, 0) \leq \frac{C_N}{(1 + \frac{|x - y|}{\rho(x)})^N} \frac{1}{|x - y|^{n-4}}.
\]

When $\lambda > 0$, from [5] we have
\[
0 \leq \Gamma_{L_2}(x, y, \lambda) \leq \frac{C_N}{(1 + \lambda^2 |x - y|^2)^N} \frac{1}{\left(1 + \frac{|x - y|}{\rho(x)}\right)^N} \frac{1}{|x - y|^{n-4}}.
\]

By the functional calculus, we may write, for any $1/2 < \beta < 1$,
\[
L_2^{-\beta} = \frac{\sin \pi \beta}{\pi} \int_0^\infty \lambda^{-\beta} (L_2 + \lambda)^{-1} d\lambda.
\]
Let $f \in C_0^\infty$. It follows from $(\mathcal{L}_2 + \lambda)^{-1} f(x) = \int_{\mathbb{R}^n} \Gamma_{\mathcal{L}_2}(x, z, \lambda) f(z) \, dz$ that

$$W_\beta(f)(x) = \nabla^2 \mathcal{L}_2^{-\beta}(f)(x) = \int_{\mathbb{R}^n} K_\beta(x, z) f(z) \, dz.$$  

Then

$$W_\beta^*(f)(x) = \int_{\mathbb{R}^n} K_\beta^*(x, z) f(z) \, dz,$$

where

$$K_\beta^*(x, z) = \frac{\sin \pi \beta}{\pi} \int_0^\infty \lambda^{-\beta} \nabla^2 \Gamma_{\mathcal{L}_2}(z, x, \lambda) \, d\lambda$$

for $1/2 < \beta < 1$, and

$$K_\beta^*(x, z) = \nabla^2 \Gamma_{\mathcal{L}_2}(z, x, 0)$$

for $\beta = 1$.

Fix $x_0, z_0 \in \mathbb{R}^n$. Let $u(z) = \Gamma_{\mathcal{L}_2}(z, x_0, \lambda)$ and $R = \frac{|x_0 - z_0|}{4}$. By the proof of Lemma 9 in [5] we know

$$\nabla^2 u(z_0) \leq \frac{C}{(1 + \lambda^2 R^2)^N (1 + \frac{R}{\rho(x_0)})} \left\{ \frac{1}{R^{n-4}} \int_{B(z_0, R)} \frac{V(\xi)^2 \, d\xi}{|\xi - z_0|^{n-2}} + \frac{1}{R^{n-2}} \right\}.$$

Then, for $\beta = 1$ we have

$$|K_\beta^*(x_0, z_0)| = |\nabla^2 \Gamma_{\mathcal{L}_2}(z_0, x_0, 0)| \leq \frac{C}{(1 + \frac{R}{\rho(x_0)})} \left\{ \frac{1}{R^{n-4}} \int_{B(z_0, R)} \frac{V(\xi)^2 \, d\xi}{|\xi - z_0|^{n-2}} + \frac{1}{R^{n-2}} \right\}.$$

Note that

$$\int_0^\infty \frac{\lambda^{-\beta}}{(1 + \lambda^2 R^2)^N} \, d\lambda \leq CR^{4\beta-4}.$$

So, for $1/2 < \beta < 1$, we get

$$|K_\beta^*(x_0, z_0)| \leq \frac{C_N}{(1 + \frac{R}{\rho(x_0)})} \left\{ \int_{B(z_0, |u - z_0| < R)} \frac{V(\xi)^2 \, d\xi}{|u - z_0|^{n-2}} + \frac{1}{R^2} \right\}.$$

(ii) Fix $x, z \in \mathbb{R}^n$. Let $R = \frac{|x - z|}{8}$ and $\delta = 4 - 2n/s$. By the functional calculus we have

$$|K_\beta^*(x, z) - K_\beta^*(y, z)| \leq C \int_0^\infty \lambda^{-\beta} |\nabla^2 \Gamma_{\mathcal{L}_2}(z, x, \lambda) - \nabla^2 \Gamma_{\mathcal{L}_2}(z, y, \lambda)| \, d\lambda.$$

From the proof of Lemma 3.2 in [6] we get

$$|\nabla^2 \Gamma_{\mathcal{L}_2}(z, x, \lambda) - \nabla^2 \Gamma_{\mathcal{L}_2}(z, y, \lambda)| \leq \frac{C R^{n-4} |x - y|^{\delta}}{(1 + \lambda^2 R^2)^N (1 + \frac{R}{\rho(x)})} \left\{ \frac{1}{R^{n-4}} \int_{B(z, 5R)} \frac{V(\xi)^2 \, d\xi}{|\xi - z|^{n-2}} + \frac{1}{R^{n-2}} \right\}.$$
Note that
\[ \int_0^\infty \frac{\lambda^{-\beta}}{(1 + \lambda^{\frac{1}{2}} R^2)^N} d\lambda \leq CR^{4\beta - 4}. \]

Then
\[ |K_\beta^*(x, z) - K_\beta^*(y, z)| \leq \frac{C|x - y|^\delta}{(1 + \frac{R}{\rho(x)}) R^{n - 4\beta + \delta}} \left\{ \int_{B(z, 5R)} \frac{V(\xi) d\xi}{|u - z|^\alpha - 2 + 1} \right\}. \]

This concludes the proof of Lemma 2.4. □

Let \( \gamma \geq 1, f \in L^1_{\text{loc}}(\mathbb{R}^n) \). For \( 0 \leq \sigma < n/\gamma \), the fractional Hardy-Littlewood maximal function \( M_{\sigma, \gamma} \) is defined by
\[ M_{\sigma, \gamma}(f)(x) = \sup_{|B|} \left( \frac{1}{|B|^{1 - \frac{1}{\gamma}}} \int_B |f(y)|^\gamma dy \right)^{\frac{1}{\gamma}}. \]

**Lemma 2.5.** Suppose \( 1 \leq \gamma < p < \infty \), \( B = B(x, r) \). If \( f \in BMO_{\mathcal{L}_1}(\mathbb{R}^n) \), then
\[ \|M_{\sigma, \gamma}f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}. \]

At the end of this section we give a characterization of function space \( BMO_{\mathcal{L}_1}(\mathbb{R}^n) \).

**Lemma 2.6** ([2]). Let \( 1 \leq p < \infty, B = B(x, r) \). If \( f \in BMO_{\mathcal{L}_1}(\mathbb{R}^n) \), then
\[ \sup_B \left( \frac{1}{|B|} \int_B |f(y) - f(B, V)|^p dy \right)^{1/p} \leq C\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}. \]

A function \( f \in BMO_{\mathcal{L}_1}(\mathbb{R}^n) \) if and only if there exists a suitable constant \( c_B \) depending on \( B \) and satisfying \( c_B = 0 \) whenever \( r \geq \rho(x) \) such that
\[ \sup_B \left( \frac{1}{|B|} \int_B |f(y) - c_B|^p dy \right)^{1/p} < \infty \]
and
\[ \|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} \leq C \sup_B \left( \frac{1}{|B|} \int_B |f(y) - c_B|^p dy \right)^{1/p}. \]

3. Proof of main results

First we prove the following lemma.

**Lemma 3.1.** Suppose \( V \in RH_s \) with \( n/2 < s < n \). Let \( 0 \leq \alpha \leq 1/2 < \beta \leq 1, \beta - \alpha \geq \frac{1}{2} \). Then there exists a constant \( C \) such that
\[ |T_{\alpha, \beta}^*f(x)| \leq CM_{\gamma, \rho_0}(f)(x) \]
for all \( f \in C_0^\infty(\mathbb{R}^n) \), where \( \frac{1}{p_0} = 1 - \frac{1}{p_0}, \frac{1}{p_\alpha} = \frac{2\alpha + 2}{s} - \frac{2}{s}, \) and \( \gamma = 4(\beta - \alpha) - 2 \).
Proof. Let \( r = \rho(x), C_j = \{ z : 2^{j-1}r < |z-x| \leq 2^jr \} \). We choose \( t \) such that
\[
1/t = 2/s - 2/n.
\]
Then \( 1/t + 1/p'_n + (2\alpha)/s = 1 \). By Hölder inequality,
\[
|T_{\alpha,n}(f)(x)| \leq \sum_{j=-\infty}^{+\infty} \int_{C_j} |K_{\alpha}^n(x,z)|V(z)^{2\alpha}|f(z)|dz
\]
\[
\leq C \sum_{j=-\infty}^{+\infty} (2^jr)^n \left( \frac{1}{(2^jr)^n} \int_{C_j} |K_{\alpha}^n(x,z)|dz \right)^{1/t}
\]
\[
\times \left( \frac{1}{(2^jr)^n} \int_{B(x,2^jr)} V(z)^sdz \right)^{2\alpha/s} \left( \frac{1}{(2^jr)^n} \int_{B(x,2^jr)} |f(z)|^{p'_n}dz \right)^{1/p'_n}.
\]
Due to \( V \in RH_s \), we have
\[
\left( \frac{1}{(2^jr)^n} \int_{B(x,2^jr)} V(z)^sdz \right)^{2\alpha/s} \leq C \left( \frac{1}{(2^jr)^n} \int_{B(x,2^jr)} V(z)dz \right)^{2\alpha}
\]
\[
\leq C (2^jr)^{-4\alpha} \left( \frac{(2^jr)^2}{(2^jr)^n} \int_{B(x,2^jr)} V(z)dz \right)^{2\alpha}.
\]
Let \( I_2(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}} \). By (i) of Lemma 2.4, Minkowski inequality and Hardy-Littlewood-Sobolev inequality, we obtain
\[
(2^jr)^n \left( \frac{1}{(2^jr)^n} \int_{C_j} |K_{\alpha}^n(x,z)|dz \right)^{1/t}
\]
\[
\leq C (1 + 2^jr)^{N} \left( (2^jr)^2 \int_{C_j} (I_2(V^{2\alpha}B(x,2^j+1r))(z))^{1/t} + (2^jr)^{2-2} \right)
\]
\[
\leq C (1 + 2^jr)^{N} \left( 2^{2n/j} \left( \frac{1}{(2^jr)^n} \int_{B(x,2^j+1r)} V(z)^sdz \right)^{2/s} + (2^jr)^{2-2} \right)
\]
\[
\leq C (1 + 2^jr)^{N} \left( \frac{(2^jr)^2}{(2^jr)^n} \int_{B(x,2^jr)} V(z)dz \right)^2 + 1 \right).
\]
For case \( j \geq 1 \), by Lemma 2.1 we have
\[
\left( \frac{(2^jr)^2}{(2^jr)^n} \int_{B(x,2^jr)} V(z)dz \right) \leq C 2^j\alpha.
\]
For the case \( j \leq 0 \), by Lemma 2.2 we get
\[
\left( \frac{(2^jr)^2}{(2^jr)^n} \int_{B(x,2^jr)} V(z)dz \right) \leq C 2^{j(2-2/n)}\frac{1}{r^{n-2}} \int_{B(x,r)} V(z)dz \leq C 2^{j(2-2/n)}.
\]
Then, taking $N \geq 2l_0(\alpha + 1)$ we get

$$|T_{\alpha,\beta}^* (f)(x)| \leq C \left( \sum_{j=1}^{\infty} \frac{1}{(2^j)^{N/2l_0(\alpha + 1)}} + \sum_{j=-\infty}^{0} (2^j)^{2\alpha/(2 - \frac{\alpha}{n})} \right) \times \frac{1}{(2^j)^{2-4(\beta - \alpha)}} \left( \frac{1}{(2^j)^{n}} \int_{B(x,2^j r)} |f(z)|^p' dz \right)^{1/p'_\alpha}$$

$$\leq C \left( \frac{1}{(2^j)^{n-2(\beta - \alpha) - 2p'_\alpha}} \int_{B(x,2^j r)} |f(z)|^p' dz \right)^{1/p'_\alpha} \leq C M_{\gamma,p'_\alpha}(f)(x).$$

**Proof of Theorem 1.1.** By Lemma 2.5 and Lemma 3.1 we know $\|T_{\alpha,\beta}^* (f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}$ holds if $p'_\alpha < p < \frac{n}{4(\beta - \alpha) - 2}$ and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta - \alpha) - 2}{n}$. By duality, we get $\|T_{\alpha,\beta}^* (f)\|_{L^{q'}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}$ for $\frac{n}{n - 4(\beta - \alpha) - 2} < q < p$ and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta - \alpha) - 2}{n}$. These conditions are equivalent to

$$1 < p < \frac{1}{p'_\alpha} + \frac{4(\beta - \alpha) - 2}{n} \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{4(\beta - \alpha) - 2}{n}.$$ 

Theorem 1.1 is proved.

**Proof of Theorem 1.2.** Since $H^1_{\Delta_x}(\mathbb{R}^n) = H^1_{\Delta_x}(\mathbb{R}^n)$ and their norms are equivalent, we only need to show that the operator $T_{\alpha,\beta}$ maps the Hardy space $H^1_{\Delta_x}(\mathbb{R}^n)$ continuously into $L^{p_2}(\mathbb{R}^n)$, where $p_2 = \frac{n}{n - 4(\beta - \alpha) - 2}$. Firstly, we review the concept of $(1,q)_\rho$-atom.

Let $1 < q \leq \infty$. A measurable function $a$ is called a $(1,q)_\rho$-atom associated with the ball $B(x,\rho)$ if $r < \rho(x)$ and the following conditions hold:

1. supp $a \subset B(x,\rho)$ for some $x \in \mathbb{R}^n$ and $r > 0$,
2. $\|a\|_{L^q(\mathbb{R}^n)} \leq |B(x,\rho)|^{1/q - 1}$,
3. if $r < \rho(x)/4$, then $\int_{B(x,\rho)} a(x) dx = 0$.

It follows from [3] that the Hardy space $H^1_{\Delta_x}(\mathbb{R}^n)$ admits the following atomic decomposition:

**Lemma 3.2.** $f \in H^1_{\Delta_x}(\mathbb{R}^n)$ if and only if $f$ can be written as $f = \sum_j \lambda_j a_j$, where $a_j$ are $(1,q)_\rho$-atoms and $\sum_j |\lambda_j| < \infty$. Moreover

$$\|f\|_{H^1_{\Delta_x}} \sim \inf \left\{ \sum_j |\lambda_j| \right\},$$
where the infimum is taken over all atomic decompositions of \( f \) into \((1,q)_{\rho}-\)atoms.

Since \( 0 \leq \alpha \leq 1/2 < \beta \leq 1, \beta - \alpha \geq 1/2 \), we can choose \( q_1 \) and \( q_2 \) such that

\[
1 < q_1 < \frac{1}{p_n} + \frac{4(\beta - \alpha) - 2}{n}
\]

and

\[
\frac{1}{q_2} - \frac{1}{q_1} = \frac{4(\beta - \alpha) - 2}{n}.
\]

By Lemma 3.2 we only need to prove \( \|T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} \leq C \) holds for any \((1,q_1)_{\rho}\)-atom, where the constant \( C > 0 \) is independent of \( a \).

Assume that \( \text{supp} a \subset B(x_0,r), r < \rho(x_0) \). Then

\[
\|T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} \leq \|\chi_{16B} T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} + \|\chi_{(16B)^c} T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} = I_1 + I_2.
\]

By Hölder inequality, Theorem 1.1 and \( \frac{1}{p_2} = 1 - \frac{4(\beta - \alpha) - 2}{n} \), we have

\[
I_1 = \|\chi_{16B} T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)}
\leq C |16B|^{\frac{1}{p_2} - \frac{1}{q_2}} \left( \int_{\mathbb{R}^n} |T_{\alpha,\beta} a(x)|^{q_2} dx \right)^{1/q_2}
\leq C |16B|^{\frac{1}{p_2} - \frac{1}{q_2}} \left( \int_{B} |a(x)|^{q_1} dx \right)^{1/q_1}
\leq C |16B|^{\frac{1}{p_2} - \frac{1}{q_2}} |B|^{\frac{1}{q_2} - 1} \leq C.
\]

We divided into two cases for the estimate of \( I_2 : r \geq \rho(x_0)/4 \) and \( r < \rho(x_0)/4 \).

Case I: \( r \geq \rho(x_0)/4 \). In this case, we have \( r \sim \rho(x_0) \). For \( z \in B(x_0,r) \), we have \( \rho(z) \sim \rho(x_0) \). By Lemma 2.4,

\[
\int_B |K_\beta(x,z) a(z)| \, dz
\]

\[
\leq C \int_B \frac{|a(z)|}{\rho(z)^{N}} \left( 1 + \frac{|x-z|}{\rho(z)} \right)^{N} \int_B \frac{|a(z)|}{|x-z|^{n-4\beta+2}} \int_{B(x,|z-x|/4)} V(\xi)^2 dx \, dz
\]

For \( z \in B \), \( x \in C_k = \{ x : 2^k r < |x - x_0| \leq 2^{k+1} r, k \geq 5 \} \), we have

\[
\int_B |K_\beta(x,z) a(z)| \, dz
\]

\[
\leq C \int_B \frac{1}{(1 + 2^k)^n (2^k r)^{n-4\beta+2}} \int_B |a(z)| \, dz
\]
\[ + \frac{1}{(1 + 2^k)^{N(2^k)^{n-4}\gamma}} \mathcal{I}_2(V^2 \chi_{B(x_0, 2^{k+1}r)}(x)) \int_B |a(z)|dz. \]

Then
\[
I_2 = \left( \int_{(16B)^c} |T_{\alpha, \beta}a(x)|^{p_2} dx \right)^{1/p_2}
\leq \left( \sum_{k \geq 5} \int_{C_k} V(x)^{2\alpha p_2} \left( \int_B |K_\beta(x, z)a(z)|dz \right)^{p_2} dx \right)^{1/p_2}
\leq C \left( \sum_{k \geq 5} \frac{(2^k)^{4(\beta - n - 2)p_2 + n}}{1 + 2^k} \frac{1}{|2^kB|} \int_{2^kB} V(x)^{2\alpha p_2} dx \right)^{1/p_2}
\times \int_B |a(z)|dz
= I_{21} + I_{22}.
\]

Notice
\[
p_2 = \frac{n}{n - 4(\beta - \alpha) + 2} < \frac{n}{4\alpha} < \frac{s}{2\alpha}.
\]

By Hölder inequality, \( V \in RH_s \) and Lemma 2.1 we have
\[
(1) \quad \frac{1}{|2^kB|} \int_{2^kB} V(x)^{2\alpha p_2} dx \leq C(2^k)^{-4\alpha p_2} \left( 1 + \frac{2^k}{\rho(x_0)} \right)^{2l_0\alpha p_2}.
\]

Since \( a \) is a \((1, q_1)_{\tau'}\)-atom, we have
\[
(2) \quad \int_B |a(y)|dy \leq 1.
\]

Due to \( 4(\beta - \alpha) - n - 2 + \frac{n}{p_2} = 0 \), by (1) and (2) we obtain
\[
I_{21} \leq C \left( \sum_{k \geq 5} \frac{(2^k)^{4(\beta - n - 2)p_2 + n}}{1 + 2^k} \right)^{1/p_2} \leq C \left( \sum_{k \geq 5} \frac{1}{(2^k)^{Np_2 - 2l_0\alpha p_2}} \right)^{1/p_2}.
\]

Taking \( N \) large enough such that \( N > 2l_0\alpha \), we get \( I_{21} \leq C \).

It is easy to check that \( \frac{1}{p_2} > \frac{1}{p_2} = \frac{2\alpha + 1}{1 + \frac{1}{r} = \frac{2}{s} - \frac{2}{n} } \). By Hölder inequality, Hardy-Littlewood-Sobolev inequality and \( V \in RH_s \), we obtain
\[
\frac{1}{|2^kB|} \int_{2^kB} V(x)^{2\alpha p_2} (\mathcal{I}_2(V^2 \chi_{2^kB})(x))^{p_2} dx
\leq C \left( \frac{1}{|2^kB|} \int_{2^kB} V(x)^{s} dx \right)^{2p_2\alpha/s} \left( \frac{1}{|2^kB|} \int_{2^kB} (\mathcal{I}_2(V^2 \chi_{2^kB})(x))^t dx \right)^{p_2/t}
\]
Then
\[
\frac{1}{|2^k B|} \int_{2^k B} V(x)^{2\alpha p_2} (\mathcal{I}_2 (V^2 \chi_{2^k B})(x))^{p_2} dx \\
\leq C(2^k r)^{-2p_2(2\alpha + 1)} \left( 1 + \frac{2^k r}{\rho(x_0)} \right)^{2p_2(\alpha + 1)}.
\]
Note that \((4\beta - n)p_2 + n - 2p_2(2\alpha + 1) = p_2(4(\beta - \alpha) - n - 2) + n = 0\). Then, taking \(N > 2l_0(\alpha + 1)\), we get
\[
I_{22} \leq C \left( \sum_{k \geq 5} \left( 1 + 2^k \right)^{N \beta - 2p_2(\alpha + 1)} \right)^{1/p_2} \leq C.
\]

Case II: \(r < \rho(x_0)/4\). Let \(z \in B, x \in C_k, k \geq 5\). By (ii) of Lemma 2.4, for some \(\delta > 0\), we have
\[
\left| K_\beta(x, z) - K_\beta(x, x_0) \right| \\
\leq \frac{C_N}{\left( 1 + \frac{|x-z|}{\rho(x_0)} \right)^N \left| x - z \right|^{n-4\beta + \delta} + 2} \\
+ \frac{C_N}{\left( 1 + \frac{|x-z|}{\rho(x_0)} \right)^N \left| x - z \right|^{n-4\beta + \delta}} \int_{B(z,|x-z|/4)} \frac{V(\xi)^2}{|\xi - x|^{n-2}} d\xi \\
\leq C \left( \frac{1}{\left( 1 + \frac{2^k r}{\rho(x_0)} \right)^N (2^k r)^{n+\delta-4\beta+2}} \\
+ \frac{1}{\left( 1 + \frac{2^k r}{\rho(x_0)} \right)^N (2^k r)^{n+\delta-4\beta+3}} \mathcal{I}_2(V^2 \chi_{2^k+1 B})(x) \right).
\]

By the vanishing condition of \(a\) and (2),
\[
I_2 = \left( \int_{(16B)^c} \left| T_{0,\beta} a(x) \right|^{p_2} dx \right)^{1/p_2} \\
\leq C \left( \sum_{k \geq 5} \int_{C_k} V(x)^{2\alpha p_2} \left( \int_{B} \left| (K_\beta(x, z) - K_\beta(x, x_0))a(z) \right| dz \right)^{p_2} dx \right)^{1/p_2} \\
\leq CI'_{21} + I'_{22},
\]
where

\[ I'_{21} = \left( \sum_{k \geq 5} \frac{1}{(1 + \frac{2^k r}{\rho(x_0)})^N p_2 (2^{k r})^{(n+\delta-4\beta+2)p_2}} \int_{2^k B} V(x)^{2\alpha p_2} dx \right)^{1/p_2} \]

and

\[ I'_{22} = \left( \sum_{k \geq 5} \frac{1}{(1 + \frac{2^k r}{\rho(x_0)})^N p_2 (2^{k r})^{(n+\delta-4\beta)p_2}} \int_{2^k B} V(x)^{2\alpha p_2} (\mathcal{I}_2(V^2 x_{2^k B})(x))^{p_2} dx \right)^{1/p_2}. \]

Since \(4(\beta - \alpha) - n - 2 + \frac{n}{p_2} = 0\), by (1) we obtain

\[ I'_{21} \leq C \left( \sum_{k \geq 5} \frac{1}{(1 + \frac{2^k r}{\rho(x_0)})^N p_2 - 2\alpha \rho p_2} \right)^{1/p_2} \]

\[ \leq C \left( \sum_{k \geq 5} \frac{1}{2^k \rho p_2} \right)^{1/p_2} \leq C. \]

By (3) and \(n - (n - 4\beta)p_2 + 2p_2(2\alpha + 1) = 0\), we get

\[ I'_{22} \leq C \left( \sum_{k \geq 5} \frac{1}{(1 + \frac{2^k r}{\rho(x_0)})^N p_2 - 2\alpha (\alpha + 1)p_2} \right)^{1/p_2} \]

\[ \leq C \left( \sum_{k \geq 5} \frac{1}{2^k \rho p_2} \right)^{1/p_2} \leq C. \]

This completes the proof of Theorem 1.2. \(\Box\)

**Proof of Theorem 1.4.** If \(\beta - \alpha = 1/2\), it follows from Theorem 1.1 that

\[ \|T_{\alpha,\beta} f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \]

for \(p' < p < \infty\).

Let \(f \in BMO_{L^1}(\mathbb{R}^n)\), \(B = B(x_0, r)\), and let \(B^* = B(x_0, 2r)\). Write

\[ f = f \chi_{B^*} + f \chi_{B(x_0, r)} = f_1 + f_2. \]

Consider the case \(r \geq \rho(x_0)\). Owing to the fact that \(T_{\alpha,\beta}^*\) is bounded on \(L^p(\mathbb{R}^n)\), we have

\[ \frac{1}{|B|} \int_B |T_{\alpha,\beta}^*(f_1)(x)| dx \leq C \left( \frac{1}{|B|} \int_B |T_{\alpha,\beta}^*(f_1)(x)|^p dx \right)^{1/p} \]

\[ \leq C \left( \frac{1}{|B^*|} \int_{B^*} |f(x)|^p dx \right)^{1/p}. \]
By Lemma 2.6 we get

\[ \frac{1}{|D|} \int_{D} |T_{\alpha,\beta}^*(f_1)(x)|\,dx \leq C \|f\|_{BMO_{L_1}(\mathbb{R}^n)}. \]

For any \( x \in B(x_0, r) \), by Lemma 2.3, we have \( \rho(x) \sim \rho(x_0) \). Then

\[ |T_{\alpha,\beta}^*(f_2)(x)| \]

\[ \leq \int_{B(x_0, r)} |K_{\alpha}^*(x, z)|V(z)^{2\alpha}|f(z)|\,dz \]

\[ \leq C \int_{B(x_0, r)} \frac{V(z)^{2\alpha}}{(1 + \frac{|x-z|}{\rho(x_0)})^n} |f(z)|\,dz \]

\[ + \int_{B(x_0, r)} \frac{V(z)^{2\alpha}}{(1 + \frac{|x-z|}{\rho(x_0)})^n} \int_{B(z, |x-z|/4)} \frac{V(\xi)^2}{|\xi-z|^{n-2\beta+2}}\,d\xi\,dz \]

\[ \leq C \sum_{k=2}^{\infty} \frac{1}{(2^k r)^{4\alpha}} \int_{2^k r} V(z)^{2\alpha}|f(z)|I_{2}(V^2\chi_{2^{k+1}B})(z)\,dz \]

\[ + C \sum_{k=2}^{\infty} \frac{1}{(2^k r)^{4\alpha}} \int_{2^k r} V(z)^{2\alpha}|f(z)|I_{2}(V^2\chi_{2^{k+1}B})(z)\,dz \]

\[ = J_1 + J_2. \]

Note that \( \frac{1}{p'} + \frac{2\alpha}{t} + \frac{1}{4} = 1 \), and \( \frac{1}{t} = \frac{2}{n} - \frac{4\alpha}{n} \); by Hölder inequality and Hardy-Littlewood-Sobolev inequality we get

\[ \frac{1}{(2^k r)^n} \int_{2^k r} V(z)^{2\alpha}|f(z)|I_{2}(V^2\chi_{2^{k+1}B})(z)\,dz \]

\[ \leq C \left( \frac{1}{(2^k r)^n} \int_{2^k r} V(z)^{s}dz \right)^{2\alpha/s} \left( \frac{1}{(2^k r)^n} \int_{2^k r} |f(z)|^{p'}\,dz \right)^{1/p'} \]

\[ \times \left( \frac{1}{(2^k r)^n} \int_{2^k r} I_{2}(V^2\chi_{2^{k+1}B})(z)|f|'\,dz \right)^{1/t} \]

\[ \leq C \left( \frac{1}{(2^k r)^n} \int_{2^k r} V(z)\,dz \right)^{2\alpha} \|f\|_{BMO_{L_1}(\mathbb{R}^n)} \]

\[ \times \left( \frac{1}{(2^k r)^n} \int_{2^k r} V(z)^{s}dz \right)^{2/s} (2^k r)^{rac{2\alpha}{s} - \frac{n}{s}} \]

\[ \leq C (2^k r)^{-4(n+1)} \left( 1 + \frac{2^k r}{\rho(x_0)} \right)^{2\alpha/(n+1)} (2^k r)^{\frac{2\alpha}{n} - \frac{n}{2}} \|f\|_{BMO_{L_1}(\mathbb{R}^n)}. \]

Because \( 4\beta - 4(n+1) + \frac{2\alpha}{n} - \frac{n}{2} = 0 \), then, taking \( N > l_0(n+1) \) we get

\[ J_2 \leq C \sum_{k=2}^{\infty} \frac{1}{(2^k r)^{N-2l_0(n+1)}} \|f\|_{BMO_{L_1}(\mathbb{R}^n)}. \]
Thus, for any $x \in (x_0, r)$, we get
\[
\left| T_{\alpha, \beta}^{*}(f_2)(x) \right| \leq C\|f\|_{BMO_{C_1}(\mathbb{R}^n)}.
\]

Consequently,
\[
\frac{1}{|B|} \int_B |T_{\alpha, \beta}^{*}(f_2)(x)|dx \leq C\|f\|_{BMO_{C_1}(\mathbb{R}^n)}.
\]

Let us consider the case $r < \rho(x_0)$. We set $B_1 = B(x_0, 2\rho(x_0))$ and write
\[
f = f \chi_{B_1} + f \chi_{(B_1)^c} = f_1 + f_2.
\]

Similar to the estimates for $|T_{\alpha, \beta}^{*}(f_2)(x)|$, we have
\[
|T_{\alpha, \beta}^{*}(f_2^1)(x)| \leq C\|f\|_{BMO_{C_1}(\mathbb{R}^n)}.
\]

Then
\[
\frac{1}{|B|} \int_B |T_{\alpha, \beta}^{*}(f_2^1)(x) - (T_{\alpha, \beta}^{*}(f_2^1))_B|dx \leq C\|f\|_{BMO_{C_1}(\mathbb{R}^n)}.
\]

For any $x \in B(x_0, r)$, let $B_{x,k} = B(x, 2^{k-1}\rho(x_0))$. It is obvious that
\[
|f(B_{x,k}, V)| = 0
\]
for $k = 0, 1, 2$. Notice
\[
|f(B_{x,3}, V) - f(B_{x,2}, V)| = |f(B_{x,3}, V)|
\[
\leq C \frac{1}{|B(x, \rho(x_0))|} \int_{B(x, \rho(x_0))} |f(z)|dz \\
\leq C \|f\|_{BMO_{\epsilon_1}(\mathbb{R}^n)}.
\]
So, for \( k = 3, 4, \ldots \), we have
\[
|f(B_{x,k}, V) - f(B_{x,k-1}, V)| \leq C \|f\|_{BMO(\mathbb{R}^n)} \leq C \|f\|_{BMO_{\epsilon_1}(\mathbb{R}^n)}.
\]
Then, for \( k = 3, 4, \ldots \), we get
\[
|f(B_{x,k}, V)| \leq C k \|f\|_{BMO_{\epsilon_1}(\mathbb{R}^n)}.
\]
Hence, for any \( p > 1 \) and \( k = 0, 1, 2, \ldots \), we have
\[
\left( \frac{1}{|B_{x,k}|} \int_{B_{x,k}} |f(z)|^p dz \right)^{1/p} \\
\leq C \left( \frac{1}{|B_{x,k}|} \int_{B_{x,k}} |f(z) - f(B_{x,k}, V)|^p dz \right)^{1/p} + |f(B_{x,k}, V)| \\
\leq C(k + 1) \|f\|_{BMO_{\epsilon_1}(\mathbb{R}^n)}.
\]
For any \( x \in B \),
\[
|T_{0, \beta}^n(f^{1/2}_2)(x)| \\
\leq C \int_{B^n} |K_{\beta}(x, z)| V(z)^{2n} |f(z)|dz \\
\leq C \sum_{k=0}^{\infty} \int_{B_{x,k}\setminus B_{x,k+1}} \frac{V(z)^{2n}}{ \left(1 + \frac{|z-x|}{\rho(x_0)}\right)^N |x-z|^{n-4\beta+2}} |f(z)|dz \\
+ \sum_{k=0}^{\infty} \int_{B_{x,k}\setminus B_{x,k+1}} \frac{V(z)^{2n}}{ \left(1 + \frac{|z-x|}{\rho(x_0)}\right)^N |x-z|^{n-4\beta}} \int_{B(|z|, |x-z|/4)} \frac{V(\xi)^2}{|\xi-z|^{n-2}} d\xi dz \\
\leq C \sum_{k=0}^{\infty} \frac{(2^{-k} \rho(x_0))^{4\beta-2}}{|B_{x,k}|} \int_{B_{x,k}} V(z)^{2n} |f(z)|dz \\
+ \sum_{k=0}^{\infty} \frac{(2^{-k} \rho(x_0))^{4\beta}}{|B_{x,k}|} \int_{B_{x,k}} V(z)^{2n} |f(z)|T_2(V^2 \chi_{B_{x,k}})(z)dz \\
= K_1 + K_2.
\]
Note that \( 2\alpha < s \), by Hölder inequality and \( \beta - \alpha = 1/2 \), we get
\[
K_1 \leq C \sum_{k=0}^{\infty} (2^{-k} \rho(x_0))^{4\alpha} \frac{1}{|B_{x,k}|} \int_{B_{x,k}} V(y)^{2\alpha} |f(y)|dy \\
\leq C \sum_{k=0}^{\infty} (2^{-k} \rho(x_0))^{4\alpha} \left( \frac{1}{|B_{x,k}|} \int_{B_{x,k}} V(y)^s dy \right)^{2\alpha/s}.
\begin{align*}
\times \left( \frac{1}{|B_{x,k}|} \int_{B_{x,k}} |f(y)|^{\frac{n}{\alpha'}} \, dy \right)^{1/(\frac{n}{\alpha'})^2}
\leq C \|f\|_{BMO_{\mathbb{C}_1}(\mathbb{R}^n)} \sum_{k=0}^{\infty} (k+1) \left( \frac{1}{(2^{2-k}\rho(x_0))^{n-2}} \int_{B_{x,k}} V(y) \, dy \right)^{2\alpha}.
\end{align*}

It follows from Lemma 2.2 that
\[ \frac{1}{(2^{2-k}\rho(x_0))^{n-2}} \int_{B_{x,k}} V(y) \, dy \leq C 2^{-k\delta} \]
for \( k = 2,3, \ldots \), where \( \delta = 2 - \frac{n}{s} > 0 \). Then \( K_1 \leq C \|f\|_{BMO_{\mathbb{C}_1}(\mathbb{R}^n)}. \)

Pay attention to \( \frac{1}{2^k} + \frac{2s}{s} + \frac{1}{r} = 1, \frac{1}{r} = \frac{2}{s} - \frac{n}{r}, \) By Hölder inequality and Hardy-Littlewood-Sobolev inequality we get
\begin{align*}
&\frac{1}{|B_{x,k}|} \int_{B_{x,k}} V(z)^{2\alpha} |f(z)| I_2 (V^2 \chi_{B_{x,k}})(z) \, dz \\
&\leq C \left( \frac{1}{|B_{x,k}|} \int_{B_{x,k}} V(z)^s \, dz \right)^{2\alpha/s} \left( \frac{1}{|B_{x,k}|} \int_{B_{x,k}} |f(z)|^{\alpha'} \, dz \right)^{1/\alpha'} \\
&\times \left( \frac{1}{|B_{x,k}|} \int_{B_{x,k}} |I_2 (V^2 \chi_{B_{x,k}})(z)| \, dz \right)^{1/t} \\
&\leq C (k+1) \|f\|_{BMO_{\mathbb{C}_1}(\mathbb{R}^n)} \left( \frac{1}{|B_{x,k}|} \int_{B_{x,k}} V(z) \, dz \right)^{2\alpha} \\
&\times \left( \frac{1}{|B_{x,k}|} \int_{B_{x,k}} V(z)^s \, dz \right)^{2/s} \frac{|B_{x,k}|^{s-1}}{s} \\
&\leq C (k+1) \|f\|_{BMO_{\mathbb{C}_1}(\mathbb{R}^n)} (2^{2-k}\rho(x_0))^{-4n-2} \\
&\times \left( \frac{1}{(2^{2-k}\rho(x_0))^{n-2}} \int_{B_{x,k}} V(z) \, dz \right)^{2\alpha+2}.
\end{align*}

Then
\[ K_2 \leq C \|f\|_{BMO_{\mathbb{C}_1}(\mathbb{R}^n)} \sum_{k=0}^{\infty} (k+1)2^{-k\delta(2\alpha+2)} \leq C \|f\|_{BMO_{\mathbb{C}_1}(\mathbb{R}^n)}. \]

Combining the estimates for \( K_1 \) and \( K_2 \), we have proved the inequality
\[ |T_{\alpha,\beta}^*(f_1^z)(x)| \leq C \|f\|_{BMO_{\mathbb{C}_1}(\mathbb{R}^n)} \]
holds for any \( x \in B(x_0, r), r < \rho(x_0) \). Then
\[ \frac{1}{|B|} \int_B |T_{\alpha,\beta}^*(f_1^z)(x) - (T_{\alpha,\beta}^*(f_1^z))_B| \, dx \leq C \|f\|_{BMO_{\mathbb{C}_1}(\mathbb{R}^n)}. \]

This finishes the proof of Theorem 1.4. \qed
References


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