GREEN FUNCTIONS FOR FLOW VELOCITY OF
STATIONARY STOKES SYSTEMS

JONGKEUN CHOI

ABSTRACT. We establish existence and uniqueness of Green functions for
flow velocity of stationary Stokes systems, under a continuity assumption
of weak solutions to the system, in a bounded domain such that the di-
vergence equation is solvable there. We also obtain pointwise bounds of
the Green functions.

1. Introduction

This paper is a continuation of [6], in which the authors studied Green func-
tions for the flow velocity of stationary Stokes systems with variable coefficients
in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 3$. In [6], they established the exis-
tence of the Green function satisfying pointwise bound away from the boundary
of the domain under the regularity assumption that weak solutions of the system
are H"older continuous in the interior of the domain. They also obtained global
pointwise bound for the Green function under the additional assumption that
weak solutions of Dirichlet problems are locally bounded up to the boundary.
Such type of regularity assumptions were introduced in [12, 13] to deal with
fundamental solutions and Green functions for elliptic systems with irregular
coefficients. We refer the reader to [4, 5, 7, 2] and the references therein for
some work in this direction. We also refer to [10, 3] for Green functions in two
dimensional domains without any regularity assumptions.

In this paper, we extend the results in [6] to domains such that the divergence
equation is solvable there. This solvability assumption is sufficiently general to
allow the domain $\Omega$ to be, for example, a John domain. Hence, the class of
domains we consider includes Lipschitz domains, Reifengerg flat domains, and
Semmes-Kenig-Toro (SKT) domains. Note that, in establishing the existence of
the Green function, we use a weaker condition that weak solutions of the system
are continuous in the interior of the domain. For further details, see Theorem
2.6.

Received January 16, 2021; Accepted January 25, 2021.
2010 Mathematics Subject Classification. 35J08, 35J57, 35R05.
Key words and phrases. Green function, stationary Stokes system, measurable coefficients.
This work was supported by a 2-Year Research Grant of Pusan National University.

©2021 The Youngnam Mathematical Society
(pISSN 1226-6973, eISSN 2287-2833)
As an application of our results combined with $W^{1,q}$-estimates for the Stokes system established in [8], we have that if the coefficients of the Stokes system are merely measurable in one direction, which may differ depending on the local coordinates, and have small mean oscillations in the other directions (variably partially BMO) and the domain is a John domain, then the Green function exists and satisfies the pointwise bound away from the boundary of the domain. Moreover, the Green function satisfies the pointwise bound globally if the domain is Reifenberg flat. Note that Stokes systems with variably partially BMO coefficients can be used to describe the motion of inhomogeneous fluids with density dependent viscosity and two fluids with interfacial boundaries; see [8, 9] and the references therein.

The remainder of the paper is organized as follows. In Section 2, we state our main results along with some definitions and assumptions. Section 3 is devoted to the construction of approximated Green functions. In Section 4, we provide the proofs of the main theorems.

2. Main results

Throughout the paper, we denoted by $\Omega$ a bounded domain in the Euclidean space $\mathbb{R}^d$, where $d \geq 3$. For any $x \in \Omega$ and $R > 0$, we write $\Omega_R(x) = \Omega \cap B_R(x)$, where $B_R(x)$ is a usual Euclidean ball of radius $R$ centered at $x$. For $q \in [1, \infty]$, we define

$$\tilde{L}^q(\Omega) = \{u \in L^q(\Omega) : (u)_\Omega = 0\},$$

where $L^q(\Omega)$ is the set of all measurable functions on $\Omega$ that are $q$th integrable and $(u)_\Omega$ is the average of $u$ over $\Omega$, i.e.,

$$(u)_\Omega = \frac{1}{|\Omega|} \int_{\Omega} u \, dx.$$

We also denote by $W^{1,q}(\Omega)$ the usual Sobolev space and $W^{1,q}_0(\Omega)$ the completion of $C^\infty_0(\Omega)$ in $W^{1,q}(\Omega)$, where $C^\infty_0(\Omega)$ is the set of all infinitely differentiable functions with compact supports in $\Omega$.

Let $\mathcal{L}$ be a second-order elliptic operator in divergence form

$$\mathcal{L}u = D_\alpha (A^{\alpha\beta} D_\beta u)$$

acting on column vector-valued functions $u = (u^1, \ldots, u^d)^T$ defined on the domain $\Omega$, where the coefficients $A^{\alpha\beta} = A^{\alpha\beta}(x)$ are $d \times d$ matrix-valued functions on $\mathbb{R}^d$ satisfying the strong ellipticity condition, that is, there is a constant $\lambda \in (0, 1]$ such that for any $x \in \mathbb{R}^d$ and $\xi_\alpha \in \mathbb{R}^d$, $\alpha \in \{1, \ldots, d\}$, we have

$$|A^{\alpha\beta}| \leq \lambda^{-1}, \quad \sum_{\alpha, \beta = 1}^d A^{\alpha\beta} \xi_\beta \cdot \xi_\alpha \geq \lambda \sum_{\alpha = 1}^d |\xi_\alpha|^2.$$

The adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ is defined by

$$\mathcal{L}^* u = D_\alpha ((A^{\beta\alpha})^T D_\beta u),$$

where $(A^{\beta\alpha})^T$ is the transpose of $A^{\beta\alpha}$.
In the definition below, $G = G(x, y)$ is a $d \times d$ matrix-valued function and $\Pi = \Pi(x, y)$ is a $1 \times d$ vector-valued function in $\Omega \times \Omega$.

**Definition 2.1.** Let $d \geq 3$ and $\Omega$ be a bounded domain in $\mathbb{R}^d$. We say that $(G, \Pi)$ is a Green function (for the flow velocity) of $\mathcal{L}$ in $\Omega$ if it satisfies the following properties.

(a) For any $y \in \Omega$ and $R > 0$,

$$G(\cdot, y) \in W^{1,1}_0(\Omega)^{d \times d} \cap W^{1,2}(\Omega \setminus B_R(y))^{d \times d}$$

and

$$\Pi(\cdot, y) \in \tilde{L}^1(\Omega)^d \cap L^2(\Omega \setminus B_R(y))^d.$$ 

(b) For any $y \in \Omega$, $(G(\cdot, y), \Pi(\cdot, y))$ satisfies

$$\begin{cases}
\mathcal{L}G(\cdot, y) + \nabla \Pi(\cdot, y) = -\delta_y I & \text{in } \Omega,

\operatorname{div} G(\cdot, y) = 0 & \text{in } \Omega.
\end{cases}$$

(c) If $(u, p) \in W^{1,2}_0(\Omega)^d \times L^2(\Omega)$ is a weak solution of

$$\begin{cases}
\mathcal{L}^* u + \nabla p = f & \text{in } \Omega,

\operatorname{div} u = g & \text{in } \Omega,
\end{cases}$$

where $f \in L^\infty(\Omega)^d$ and $g \in \tilde{L}^\infty(\Omega)$, then for a.e. $y \in \Omega$, we have

$$u(y) = -\int_\Omega G(x, y)^\top f(x) \, dx + \int_\Omega \Pi(x, y)^\top g(x) \, dx. \quad (2.1)$$

**Remark 2.2.** The property (c) in the definition above together with the solvability result in, for instance, [6, Lemma 3.2] gives the uniqueness of a Green function in the sense that if $(\tilde{G}, \tilde{\Pi})$ is another Green function satisfying the above properties, then for each $\phi \in C^\infty_0(\Omega)^d$ and $\varphi \in C^\infty_0(\Omega)$, we have

$$\int_\Omega (G(x, y)^\top - \tilde{G}(x, y)^\top) \phi(x) \, dx = \int_\Omega (\Pi(x, y)^\top - \tilde{\Pi}(x, y)^\top) \varphi(x) \, dx = 0$$

for a.e. $y \in \Omega$.

We make the following assumptions to construct the Green function of $\mathcal{L}$ in $\Omega$.

**Assumption 2.3.** There exists a constant $K_0 > 0$ such that the following holds. For any $g \in \tilde{L}^2(\Omega)$, there exists $u \in W^{1,2}_0(\Omega)^d$ satisfying

$$\operatorname{div} u = g \text{ in } \Omega, \quad \|Du\|_{L^2(\Omega)} \leq K_0 \|g\|_{L^2(\Omega)}.$$ 

**Remark 2.4.** From [1, Theorem 4.1], it follows that Assumption 2.3 holds in a bounded John domain. Moreover, by a scaling argument, we see that if $\Omega = B_R$, then the assumption holds with $q \in (1, \infty)$ in place of 2 and the constant $K_0$ depending only on $d$ and $q$.

The following assumption holds, for instance, when the coefficients $A^{\alpha\beta}$ of $\mathcal{L}$ are variably partially BMO; see [2, Theorem 6.2].
**Theorem 2.6.** Let the same statement holds true when Assumptions 2.3 and 2.5. Then the following hold.

**Assumption 2.5.** There exist constants $R_0 \in (0,1]$ and $A_0 > 0$ such that the following holds. Let $x_0 \in \Omega$ and $0 < R < \min\{R_0, \text{dist}(x_0, \partial \Omega)\}$. If $(u,p) \in W^{1,2}(B_R(x_0))^d \times L^2(B_R(x_0))$ satisfies

$$
\begin{align*}
Lu + \nabla p &= f \quad \text{in } B_R(x_0), \\
\text{div} u &= 0 \quad \text{in } B_R(x_0),
\end{align*}
$$

where $f \in L^\infty(B_R(x_0))^d$, then we have $u \in C(B_{R/2}(x_0))^d$ (in fact, a version of $u$ belongs to $C(B_{R/2}(x_0))^d$) with the estimate

$$
\|u\|_{L^\infty(B_{R/2}(x_0))} \leq A_0 (R^{-d/2} \|u\|_{L^2(B_R(x_0))} + R^2 \|f\|_{L^\infty(B_R(x_0))}).
$$

The same statement holds true when $L$ is replaced by $L^\ast$.

In the theorem below and throughout the paper, we denote

$$
d_y = \text{dist}(y, \partial \Omega), \quad d^* = \min\{R_0, d_y\}.
$$

**Theorem 2.6.** Let $\Omega$ be a bounded domain in $\mathbb{R}^d$, where $d \geq 3$. Then, under Assumptions 2.3 and 2.5, there exist Green functions $(G,\Pi)$ of $L$ and $(G^\ast,\Pi^\ast)$ of $L^\ast$ such that for any $y \in \Omega$, we have

$$
G(\cdot, y), G^\ast(\cdot, y) \in C(\Omega \setminus \{y\})^{d \times d},
$$

and there exists a measure zero set $N_y \subset \Omega$ such that

$$
G(x, y) = G^\ast(y, x)^\top, \quad G(y, x) = G^\ast(x, y)^\top \quad \text{for all } x \in \Omega \setminus N_y.
$$

Moreover, for any $x, y \in \Omega$ with $0 < |x - y| < d^*_y/2$,

$$
|G(x, y)| \leq N |x - y|^{2-d},
$$

where $N = N(d, \lambda, K_0, A_0)$.

We have the following corollary, the estimates in which can be derived from the proof of Theorem 2.6. Note that the estimates for $\Pi(\cdot, y)$ in (ii) are new even in the case when $\Omega$ is a Lipschitz domain; cf. [6, Theorem 2.3].

**Corollary 2.7.** Let $(G, \Pi)$ be the Green function of $L$ derived from 2.6 under Assumptions 2.3 and 2.5. Then the following hold.

(i) For any $y \in \Omega$ and $R \in (0, d^*_y)$, we have that

$$
\begin{align*}
\|G(\cdot, y)\|_{L^{2d/(d-2)}(\Omega \setminus \overline{B_R(y)})} + \|DG(\cdot, y)\|_{L^2(\Omega \setminus \overline{B_R(y)})} &\leq NR^{(2-d)/2}, \\
\|G(\cdot, y)\|_{L^q(B_R(y))} &\leq N_q R^{2-d+q/d}, \quad q \in [1, d/(d-2)], \\
\|DG(\cdot, y)\|_{L^q(B_R(y))} &\leq N_q R^{1-d+q/d}, \quad q \in [1, d/(d-1)].
\end{align*}
$$

Moreover,

$$
\begin{align*}
|\{x \in \Omega : |G(x, y)| > t\}| &\leq N_t^{-d/(d-2)}, \quad \forall t > (d^*_y)^{2-d}, \\
|\{x \in \Omega : |D_x G(x, y)| > t\}| &\leq N_t^{-d/(d-1)}, \quad \forall t > (d^*_y)^{1-d}.
\end{align*}
$$
For any \( y \in \Omega \) and \( R \in (0, d_y^*) \), we have that
\[
\| \Pi(\cdot, y) \|_{L^2(\Omega \setminus B_R(y))} \leq NR^{2-d}/2,
\]
(2.5)
\[
\| \Pi(\cdot, y) \|_{L^q(B_R(y))} \leq N_q R^{1-d+4/d}, \quad q \in [1, d/(d - 1)).
\]
Moreover,
\[
\| \Pi(\cdot, y) \|_{L^q(B_R(y))} \leq N_q R^{1-d+4/d}, \quad q \in [1, d/(d - 1)).
\]
In the above, the constant \( N \) depends only on \( d, \lambda, K_0, \) and \( A_0 \), and \( N_q \) depends also on \( q \).

To obtain the global pointwise bound for \( G(x, y) \), we impose the following assumption. By using the \( W^{1,q} \)-estimates in [8, Theorem 2.4] and a bootstrap argument (see, for instance, the proof of [6, Theorem 2.9]), one can check that the assumption holds when the coefficients \( A^\alpha \) of \( \mathcal{L} \) are variably partially BMO and \( \Omega \) is a Reifenberg flat domain.

**Assumption 2.8.** There exist constants \( R_0 \in (0, 1] \) and \( A_1 > 0 \) such that the following holds. Let \( x_0 \in \partial \Omega \) and \( R \in (0, R_0) \). If \((u, p) \in W^{1,2}(\Omega_R(x_0))^d \times L^2(\Omega_R(x_0))\) satisfies
\[
\begin{cases}
\mathcal{L} u + \nabla p = f & \text{in } \Omega_R(x_0), \\
\text{div } u = 0 & \text{in } \Omega_R(x_0), \\
u = 0 & \text{on } \partial \Omega \cap B_R(x_0),
\end{cases}
\]
where \( f \in L^\infty(\Omega_R(x_0))^d \), then we have
\[
\|u\|_{L^\infty(\Omega_{R/2}(x_0))} \leq A_1 \left(R^{-d/2}\|u\|_{L^2(\Omega_R(x_0))} + R^2\|f\|_{L^\infty(\Omega_R(x_0))}\right).
\]
The same statement holds true when \( \mathcal{L} \) is replaced by \( \mathcal{L}^* \).

**Theorem 2.9.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \), where \( d \geq 3 \). Let \((G, \Pi)\) be the Green function constructed in Theorem 2.6 under Assumptions 2.3 and 2.5. If we assume Assumption 2.8 (in addition to Assumptions 2.3 and 2.5), then for any \( x, y \in \Omega \) with \( 0 < |x - y| < R_0 \), we have
\[
|G(x, y)| \leq N|x - y|^{2-d},
\]
where \( N = N(d, \lambda, K_0, A_0, A_1) \).

We end this section with the following remark.

**Remark 2.10.** It is not clear us if the estimates in Corollary 2.7 still hold for \( R \in (0, R_0) \) under Assumption 2.8 in addition to Assumptions 2.3 and 2.5. In fact, the proofs of Lemmas 3.2–3.5 in Section 3 does not work for \( y \in \Omega \) and \( d_y < R < R_0 \), because the divergence equation is not necessarily solvable in \( \Omega_R(x) \).
3. Approximated Green functions

Hereafter in the paper, we use the following notation.

**Notation 3.1.** For nonnegative (variable) quantities \( A \) and \( B \), we denote \( A \lesssim B \) if there exists a generic positive constant \( N \) such that \( A \leq NB \). We add subscript letters like \( A \lesssim_{a,b} B \) to indicate the dependence of the implicit constant \( N \) on the parameters \( a \) and \( b \).

In this section, we assume that the hypotheses of Theorem 2.6 hold. Under the hypotheses, we shall construct approximated Green functions for the flow velocity of the Stokes system. We mainly follow the arguments in [12] (see also [2, 6]).

Let \( y \in \Omega, \varepsilon \in (0, 1], k \in \{1, \ldots, d\} \), and

\[
\Phi_{\varepsilon,y}(x) = -\frac{1}{|\Omega_\varepsilon(y)|} I_{\Omega_\varepsilon(y)}(x).
\]  

(3.1)

By [6, Lemma 3.2], there exists a unique \((v, \pi) = (v_{\varepsilon,y,k}, \pi_{\varepsilon,y,k}) \in W^{1,2}_0(\Omega)^d \times \tilde{L}^2(\Omega)\) satisfying

\[
\begin{aligned}
L v + \nabla \pi &= \Phi_{\varepsilon,y} e_k \quad \text{in } \Omega, \\
\text{div } v &= 0 \quad \text{in } \Omega,
\end{aligned}
\]  

(3.2)

where \( e_k \) is the \( k \)th unit vector in \( \mathbb{R}^d \). Moreover, we have

\[
\|Dv\|_{L^2(\Omega)} + \|\pi\|_{L^2(\Omega)} \lesssim_{d,\lambda,K_0} |\Omega_\varepsilon(y)|^{(2-d)/(2d)}.
\]  

(3.3)

Due to Assumption 2.5, there is a version \( \tilde{v} \) of \( v \) such that \( \tilde{v} = v \) a.e. in \( \Omega \) and \( \tilde{v} \) is continuous in \( \Omega \). We define the approximated Green function \((G_{\varepsilon}(\cdot, y), \Pi_{\varepsilon}(\cdot, y))\) (for the flow velocity) of \( L \) by

\[
G^{j}_k(\cdot, y) = \tilde{v}^j = \tilde{v}^{j}_{\varepsilon,y,k}, \quad \Pi^{k}_k(\cdot, y) = \pi = \pi_{\varepsilon,y,k}.
\]

In the lemma below, we obtain a pointwise bound for \( G_{\varepsilon}(\cdot, y) \).

**Lemma 3.1.** Let \( \varepsilon \in (0, 1] \) and \( x, y \in \Omega \) with

\[
0 < 2\varepsilon < \frac{|x - y|}{2} < \frac{d^*_y}{3}.
\]

Then we have

\[
|G_{\varepsilon}(x, y)| \lesssim_{d,\lambda,K_0,A_0} |x - y|^{2-d}.
\]

**Proof.** Denote

\[
(v, \pi) = (G^k_{\varepsilon}(\cdot, y), \Pi^k_{\varepsilon}(\cdot, y)),
\]  

(3.4)

where \( G^k_{\varepsilon}(\cdot, y) \) is the \( k \)th column of \( G_{\varepsilon}(\cdot, y) \). Let \( x \in \Omega \) and \( 0 < 2\varepsilon < R < d^*_y \).

Find \((u, p) \in W^{1,2}_0(\Omega)^d \times \tilde{L}^2(\Omega)\) satisfying

\[
\begin{aligned}
Lu + \nabla p &= -f \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega,
\end{aligned}
\]  

(3.5)
where $f = I_{\Omega_R(x)}(\text{sgn} v^1, \ldots, \text{sgn} v^d)^T$. Then by Assumption 2.5, the Sobolev inequality, and the $W^{1,2}$-estimate, we have

$$
\|u\|_{L^\infty(B_{R/2}(y))} \lesssim \| Du\|_{L^2(B_R(y))} + R^2 \\
\lesssim_d A_0 R^{1-d/2} \| Du\|_{L^2(\Omega)} + R^2
$$

(3.5)

Observe that

$$
\int_{\Omega_R(x)} f \cdot v \, dz = \int_\Omega A^{\alpha\beta} D_{\beta} v \cdot D_{\alpha} u \, dz = \int_{B_d(y)} u^k \, dz. \tag{3.6}
$$

Since $\varepsilon < R/2$, combining (3.5) and (3.6), we get

$$
\|v\|_{L^1(\Omega_R(x))} \lesssim_d A_0 R^2 \tag{3.7}
$$

for any $x \in \Omega$ and $0 < 2\varepsilon < R < d^*_{y}$.

We now ready to prove the lemma. Let

$$
0 < 2\varepsilon < R := \frac{|x - y|}{2} < \frac{d^*_y}{3}.
$$

Since $B_R(x) \subset \Omega$ and $B_R(x) \cap B_d(y) = \emptyset$, (3.2) yields that

$$
\begin{cases}
L v + \nabla \pi = 0 & \text{in } B_R(x), \\
\text{div } v = 0 & \text{in } B_R(x).
\end{cases} \tag{3.8}
$$

For any $z \in B_R(x)$ and $r \in (0, R]$ with $B_r(z) \subset B_R(x)$, by Assumption 2.5 applied to (3.8) in $B_r(z)$, we have

$$
\|v\|_{L^\infty(B_{r/2}(z))} \lesssim r^{-d/2} \|v\|_{L^2(B_r(z))},
$$

and thus, by a standard iteration argument (see, for instance, [11, pp. 80–82]), we obtain

$$
\|v\|_{L^\infty(B_{r/2}(z))} \lesssim R^{-d} \|v\|_{L^1(B_R(x))}. \tag{3.9}
$$

From this together with (3.7) and continuity of $v$, we get the desired estimate. The lemma is proved.

From Lemma 3.1, by following the steps in the proof of [6, Lemma 4.3], we obtain the following uniform estimates for $G_\varepsilon(\cdot, y)$ and $DG_\varepsilon(\cdot, y)$. The corresponding estimates for $\Pi_\varepsilon(\cdot, y)$ can be found in Lemma 3.5 below.

**Lemma 3.2.** Let $\varepsilon \in (0, 1]$, $y \in \Omega$, and $R \in (0, d^*_y)$. Then we have

$$
\|G_\varepsilon(\cdot, y)\|_{L^{2\alpha/(\alpha-2)}(\Omega \setminus \overline{B_R(y)})} + \|DG_\varepsilon(\cdot, y)\|_{L^2(\Omega \setminus \overline{B_R(y)})} \leq NR^{2-d}/2, \tag{3.10}
$$

$$
\|G_\varepsilon(\cdot, y)\|_{L^{q_1}(B_R(y))} \leq N_1 R^{2-d+d/q_1}, \quad q_1 \in [1, d/(d-2)),
$$

$$
\|DG_\varepsilon(\cdot, y)\|_{L^{q_2}(B_R(y))} \leq N_2 R^{1-d+d/q_2}, \quad q_2 \in [1, d/(d-1)). \tag{3.11}
$$

Moreover, we have

$$
\left|\left\{x \in \Omega : |G_\varepsilon(x, y)| > t\right\}\right| \leq N d^{-d/(d-2)}, \quad \forall t > (d^*_y)^{2-d},
$$
Recall the notation (3.4). Let us fix

\begin{equation}
\text{Proof.}
\end{equation}

In the above, the constant \(N\) depends only on \(d\), \(\lambda\), \(K_0\), and \(A_0\), and \(N_1\) depend also on \(q_1\).

**Lemma 3.3.** Let \(\varepsilon \in (0, 1]\), \(y \in \Omega\), and \(R \in (0, d^*_y)\). Then we have

\[ \|\Pi_\varepsilon(\cdot, y) - (\Pi_\varepsilon(\cdot, y))\|_{\dot{L}^1(B_R(y))} \lesssim_{d, \lambda, K_0, A_0} R. \]

**Proof.** Recall the notation (3.4). Let us fix \(q \in (1, d/(d-1))\) and \(q' \in (d, \infty)\) with \(\frac{1}{q} + \frac{1}{q'} = 1\). By the existence of solutions to the divergence equation in a ball (see, for instance, [1]), there exists \(\phi \in W^{1,q'}_0(B_R(y))^d\) such that

\[ \text{div } \phi = \text{sgn}(\pi - (\pi)_{B_R(y)}) - (\text{sgn}(\pi - (\pi)_{B_R(y)}))_{B_R(y)} \quad \text{in } B_R(y) \]

and

\[ \|D\phi\|_{L^{q'}(B_R(y))} \lesssim N R^{d/q'}. \]

Here, by a scaling argument, one can check that the constant \(N\) in the above inequality depends only on \(d\) and \(q'\). We extend \(\phi\) by zero in \(\Omega \setminus B_R(y)\) and apply \(\phi\) as a test function to (3.2) to get

\[ \int_\Omega \pi \text{div } \phi \, dx = -\int_\Omega A^{\alpha\beta} D_\beta v \cdot D_\alpha \phi \, dx + \int_{B_r(y)} \phi \, dx. \]

Observe that

\[ \int_\Omega \pi \text{div } \phi \, dx = \int_\Omega (\pi - (\pi)_{B_R(y)}) \text{div } \phi \, dx = \int_{B_R(y)} |\pi - (\pi)_{B_R(y)}| \, dx \]

and

\[ \|\phi\|_{L^\infty(\Omega)} = \|\phi\|_{L^\infty(B_R(y))} \lesssim_d R^{1-d/q'} \|D u\|_{L^{q'}(B_R(y))} \lesssim R. \]

Combining these together and using H"older’s inequality and (3.11), we have

\[ \|\pi - (\pi)_{B_R(y)}\|_{\dot{L}^1(B_R(y))} \lesssim \|D v\|_{L^q(B_R(y))} \|D \phi\|_{L^{q'}(B_R(y))} + R \lesssim_{d, \lambda, K_0, A_0} R. \]

The lemma is proved. \(\square\)

**Lemma 3.4.** Let \(\varepsilon \in (0, R/16]\), \(y \in \Omega\), and \(R \in (0, d^*_y/2)\). For \(k \in \{1, \ldots, d\}\), we set

\[ \tilde{\Pi}_\varepsilon^k(\cdot, y) = \Pi_\varepsilon^k(\cdot, y) - (\Pi_\varepsilon^k(\cdot, y))_{B_R(y) \setminus \overline{B_{R/2}(y)}}. \]

Then we have

\[ \|\tilde{\Pi}_\varepsilon^k(\cdot, y)\|_{\dot{L}^2(B_R(y) \setminus \overline{B_{R/2}(y)})} \lesssim_{d, \lambda, K_0, A_0} R^{2-d)/2}, \]

where \(G_\varepsilon^k(\cdot, y)\) is the \(k\)th column of \(G_\varepsilon(\cdot, y)\).
Proof. Recall the notation 3.4, and set
\[ \tilde{\pi} = \pi - (\pi)_{B_R(y) \setminus B_{R/2}(y)}. \]
It suffices to show that
\[ \|\tilde{\pi}\|_{L^2(B_R(y) \setminus B_{R/2}(y))} \lesssim R^{-1}\|v\|_{L^2(B_{5R/4}(y) \setminus B_{R/4}(y))}, \tag{3.12} \]
by Lemma 3.1, we have
\[ \|v\|_{L^2(B_{5R/4}(y) \setminus B_{R/4}(y))} \lesssim d, K_0, A_0 \frac{R^2 - d/2}{a}. \]
From the existence of solutions to the divergence equation, there exists a function \( \phi \in W_0^{1,2}(B_R(y) \setminus \overline{B_R(y)})^d \) such that
\[ \text{div } \phi = \tilde{\pi} \quad \text{in } B_R(y) \setminus \overline{B_{R/2}(y)} \]
and
\[ \|D\phi\|_{L^2(B_R(y) \setminus B_{R/2}(y))} \lesssim_d \|\tilde{\pi}\|_{L^2(B_R(y) \setminus B_{R/2}(y))}. \tag{3.13} \]
We extend \( \phi \) by zero in \( \Omega \setminus (B_R(y) \setminus \overline{B_{R/2}(y)}) \) and apply \( \phi \) as a test function to (3.2) to get
\[ \int \pi \text{div } \phi \, dx = - \int A^{\alpha \beta} D_{\beta} v \cdot D_{\alpha} \phi \, dx. \]
Thus by using Hölder’s inequality, (3.13), and the fact that
\[ \int \pi \text{div } \phi \, dx = \int \tilde{\pi} \text{div } \phi \, dx = \int_{B_R(y) \setminus \overline{B_{R/2}(y)}} |\tilde{\pi}|^2 \, dx, \]
we have
\[ \int_{B_R(y) \setminus \overline{B_{R/2}(y)}} |\tilde{\pi}|^2 \, dx \lesssim_{d, K} \int_{B_R(y) \setminus \overline{B_{R/2}(y)}} |Dv|^2 \, dx. \tag{3.14} \]
Now we let \( z \in B_R(y) \setminus \overline{B_{R/2}(y)} \). Since \( B_{R/4}(z) \cap B_\varepsilon(y) = \emptyset \), it holds that
\[ \begin{cases} \mathcal{L} v + \nabla \pi = 0 & \text{in } B_{R/4}(z), \\ \text{div } v = 0 & \text{in } B_{R/4}(z). \end{cases} \]
By the Caccioppoli inequality (see, for instance, [6, Lemma 3.3]) applied to the above system and the fact that \( B_{R/4}(z) \subset (B_{5R/4}(y) \setminus \overline{B_{R/4}(y)}) \), we have
\[ \int_{B_{R/4}(z)} |Dv|^2 \, dx \lesssim \frac{1}{R^2} \int_{B_{5R/4}(y) \setminus \overline{B_{R/4}(y)}} |v|^2 \, dx. \]
Since the above inequality holds for all \( z \in B_R(y) \setminus \overline{B_{R/2}(y)} \), by a covering argument, we obtain that
\[ \int_{B_R(y) \setminus \overline{B_{R/2}(y)}} |Dv|^2 \, dx \lesssim \frac{1}{R^2} \int_{B_{5R/4}(y) \setminus \overline{B_{R/4}(y)}} |v|^2 \, dx, \]
which together with (3.14) gives (3.12). The lemma is proved. \( \square \)

From the above two lemmas, we get the following uniform estimates for \( \Pi_\varepsilon(\cdot, y) \).
Lemma 3.5. Let \( \varepsilon \in (0, 1) \), \( y \in \Omega \), and \( R \in (0, d^*_y) \). Then we have
\[
\| \Pi_{\varepsilon}(\cdot, y) \|_{L^2(\Omega \setminus B_R(y))} \leq NR^{(2-d)/2},
\]
\[
\| \Pi_{\varepsilon}(\cdot, y) \|_{L^q(B_R(y))} \leq N_q R^{1-d/d^q}, \quad q \in [1, d/(d-1)).
\]  
Moreover, we have
\[
|\{x \in \Omega : |\Pi_{\varepsilon}(x, y)| > t\}| \leq Nt^{-d/(d-1)}, \quad \forall t > (d^*_y)^{1-d}.
\]
In the above, the constant \( N \) depends only on \( d, \lambda, K_0, \) and \( A_0 \), and \( N_q \) depends also on \( q \).

Proof. We only prove (3.15) because the others are its easy consequences; see, for instance, [2, Lemmas 4.4 and 4.5]. Recall the notation (3.4). We may assume that \( 0 < R < d^*_y/2 \). If \( R/16 \leq \varepsilon \leq 1 \), then by (3.3) we have
\[
\| \pi \|_{L^2(\Omega)} \lesssim |B_{R/2}(y)|^{(2-d)/(2d)} \lesssim R^{(2-d)/2}.
\]
Assume \( 0 < \varepsilon \leq R/16 \), and let \( \eta \) be an infinitely differentiable function in \( \mathbb{R}^d \) satisfying
\[
0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_{R/2}(y), \quad \text{supp } \eta \subset B_R(y), \quad |\nabla \eta| \lesssim_d R^{-1}.
\]
By Assumption 2.3, there exists \( \phi \in W^{1,2}_0(\Omega)^d \) such that
\[
\text{div } \phi = \pi I_{\Omega \setminus B_R(y)} - \left( \pi I_{\Omega \setminus B_{R/2}(y)} \right)_\Omega \quad \text{in } \Omega \quad (3.16)
\]
and
\[
\| \phi \|_{L^{2d/(d-2)}(\Omega)} + \|D\phi\|_{L^2(\Omega)} \lesssim_d K_0 \| \pi \|_{L^2(\Omega \setminus B_{R/2}(y))}. \quad (3.17)
\]
We apply \((1 - \eta)\phi\) as a test function to (3.2) to obtain
\[
\int_{\Omega} \pi \text{div}((1 - \eta)\phi) \, dx = -\int_{\Omega} A_{\alpha \beta} D_{\beta} v \cdot D_{\alpha}((1 - \eta)\phi) \, dx.
\]
Set \( \tilde{\pi} = \pi - (\pi)_{B_R(y) \setminus B_{R/2}(y)} \), and note that by (3.16) and \( \int_{\Omega} \pi \, dx = 0 \), we have
\[
\int_{\Omega} \pi \text{div}((1 - \eta)\phi) \, dx = \int_{\Omega} \pi \text{div}(\phi) \, dx - \int_{\Omega} \tilde{\pi} \text{div}(\eta \phi) \, dx
\]
\[
= \int_{\Omega \setminus B_R(y)} |\pi|^2 \, dx - \int_{B_R(y) \setminus B_{R/2}(y)} \tilde{\pi} \nabla \phi \cdot \eta \, dx - \left( \pi I_{\Omega \setminus B_{R/2}(y)} \right)_\Omega \int_{B_R(y) \setminus B_{R/2}(y)} \tilde{\pi} \eta \, dx.
\]
We also note that
\[
\left| \left( \pi I_{\Omega \setminus B_{R/2}(y)} \right)_\Omega \int_{B_R(y) \setminus B_{R/2}(y)} \tilde{\pi} \eta \, dx \right| \lesssim R^{-d/2} \| \pi \|_{L^2(\Omega \setminus B_{R/2}(y))} \| \pi - (\pi)_{B_{R/2}(y)} \|_{L^1(B_{R/2}(y))}.
\]
Therefore, from Hölder’s inequality, (3.17), and the fact that
\[
\int_{B_R(y)} |\tilde{\pi}| \, dx \lesssim \int_{B_R(y)} |\pi - (\pi)_{B_R(y)}| \, dx,
\]
we get
\[ \| \pi \|_{L^2(\Omega \setminus B_R(y))} \lesssim \| Dv \|_{L^2(\Omega \setminus B_R(y))} + \| \tilde{\pi} \|_{L^2(B_R(y) \setminus B_{R/2}(y))} + R^{-d/2} \| \pi - (\pi)_{B_R(y)} \|_{L^1(B_R(y))}. \]

Finally, by using (3.10) and Lemmas 3.3 and 3.4, we conclude (3.15). \[\square\]

### 4. Proofs of main theorems

Throughout this section, we denote by \((G_\varepsilon(\cdot, y), \Pi_\varepsilon(\cdot, y))\) the approximated Green function constructed in Section 3.

**Proof of Theorem 2.6.** Let \(\varepsilon \in (0, 1]\) and \(y \in \Omega\). By Lemmas 3.2 and 3.5, the weak compactness theorem, and a diagonalization process, we see that there exist a sequence \(\{\varepsilon_\rho\}_{\rho=1}^\infty\) tending to zero and a pair \((G_\varepsilon(\cdot, y), \Pi(\cdot, y))\) such that

\[ (1 - \eta_R)G_{\varepsilon_\rho}(\cdot, y) \rightharpoonup (1 - \eta_R)G(\cdot, y) \text{ weakly in } W^{1,2}_0(\Omega)^d \times d, \]

\[ \Pi_{\varepsilon_\rho}(\cdot, y) \rightharpoonup \Pi(\cdot, y) \text{ weakly in } L^2(\Omega \setminus B_R(y))^d, \]

where \(\eta_R\) is any smooth function in \(\mathbb{R}^d\) satisfying \(\eta_R = 1\) in \(B_R(y)\), and that

\[ G_{\varepsilon_\rho}(\cdot, y) \rightharpoonup G(\cdot, y) \text{ weakly in } W^{1,q}(B_{d^*}(y))^d \times d, \]

\[ \Pi_{\varepsilon_\rho}(\cdot, y) \rightharpoonup \Pi(\cdot, y) \text{ weakly in } L^q(B_{d^*}(y))^d. \]

Then one can check that \((G(\cdot, y), \Pi(\cdot, y))\) satisfies the properties (a)–(c) in Definition 2.1, which means that \((G, \Pi)\) is the Green function of \(L\) in \(\Omega\). Due to the property (b) and Assumption 2.5, the function \(G(\cdot, y)\) is continuous in \(\Omega \setminus \{y\}\). More precisely, we choose a version of \(G\) which is continuous in \(\Omega \setminus \{y\}\) and denote it again by \(G\).

Note that the identity (2.1) holds for all \(y \in \Omega\) if \(g \equiv 0\). Indeed, if \((u, p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)\) is a weak solution of (2.1) with \(g \equiv 0\), then by Assumption 2.5 there is a continuous version of \(u\) in \(\Omega\), denoted again by \(u\), satisfying

\[ \int_{\Omega_{\varepsilon_\rho}(y)} u \, dx = - \int_\Omega G_{\varepsilon_\rho}(x, y) \top f(x) \, dx \quad \text{for all } y \in \Omega. \]

By taking \(\rho \to \infty\) and using the continuity of \(u\), we have

\[ u(y) = - \int_\Omega G(x, y) \top f(x) \, dx \quad \text{for all } y \in \Omega. \]

Now we shall prove the estimates in Corollary 2.7 and the pointwise bound (2.3). The estimates (2.4) and (2.5) are simple consequences of (3.10), (3.15), and the weak semi-continuity. Then by following the same steps used in, for instance, [2, Lemmas 4.4 and 4.5], one can prove that the others in the corollary
also hold. To show (2.3), let \( x, y \in \Omega \) with \( 0 < |x - y| < d^*_{y}/2 \), and denote \( r = |x - y|/2 \). Since \((G(\cdot, y), \Pi(\cdot, y))\) satisfies
\[
\begin{cases}
\mathcal{L}G(\cdot, y) + \nabla \Pi(\cdot, y) = 0 & \text{in } B_r(x), \\
d \text{div } G(\cdot, y) = 0 & \text{in } B_r(x),
\end{cases}
\]
by Assumption 2.5, Hölder’s inequality, and the estimates in Corollary 2.7, we have
\[
|G(x, y)| \lesssim r^{(2-d)/2} \|G(\cdot, y)\|_{L^{2u/(d-2)}(B_r(x))} \\
\lesssim r^{(2-d)/2} \|G(\cdot, y)\|_{L^{2u/(d-2)}(\Omega \setminus B_r(y))} \\
\lesssim r^{2-d},
\]
which gives (2.3).

For \( \sigma \in (0, 1] \) and \( x, y \in \Omega \), let \((G^*_\sigma(\cdot, x), \Pi^*_\sigma(\cdot, x))\) be the approximated Green function for \( \mathcal{L}^* \), i.e., if we set \( w = w_{\sigma, x, \ell} \) as the \( \ell \)-th column of \( G^*_\sigma(\cdot, x) \) and \( \kappa = \kappa_{\sigma, x, \ell} \) as the \( \ell \)-th component of \( \Pi^*_\sigma(\cdot, x) \), then \((w, \kappa) \in W^{1,2}_0(\Omega) \times \tilde{L}^2(\Omega)\) satisfies
\[
\begin{cases}
\mathcal{L}^* w + \nabla \kappa = \Phi^*_{\sigma, x} e_{\ell} & \text{in } \Omega, \\
d \text{div } w = 0 & \text{in } \Omega,
\end{cases}
\]
where \( \Phi^*_{\sigma, x} \) is given in (3.1). By proceeding similarly as above, we can find a sequence \( \sigma_j \to 0 \) and a unique Green function \((G^*(\cdot, x), \Pi^*(\cdot, x))\) of \( \mathcal{L}^* \) in \( \Omega \) such that \((G^*_\sigma(\cdot, x), \Pi^*_\sigma(\cdot, x))\) and \((G^*(\cdot, x), \Pi^*(\cdot, x))\) satisfy the natural counterparts of (4.1), (4.2), and the properties of the Green function of \( \mathcal{L} \). Notice from (4.3) that
\[
G^*_\sigma(y, x) = \int_{\Omega^*_\sigma(x)} G(z, y) \mathbf{1} \, dz \quad \text{for all } x, y \in \Omega.
\]
Then by the continuity of \( G(\cdot, y) \) on \( \Omega \setminus \{y\} \), we have
\[
\lim_{\sigma \to 0} G^*_\sigma(y, x) = G(x, y) \mathbf{1} \quad \text{for all } x, y \in \Omega \text{ with } x \neq y. \tag{4.4}
\]

Now we prove (2.2). Let \( y \in \Omega \) be given. Then there exists a measure zero set \( N_y \subset \Omega \) containing \( y \) such that, by passing to a subsequence,
\[
\lim_{\rho \to 0} G^*_{\varepsilon_\rho}(x, y) = G(x, y) \mathbf{1} \quad \text{for all } x \in \Omega \setminus N_y. \tag{4.5}
\]
Indeed, since it holds that
\[
\|G^*_{\varepsilon_\rho}(\cdot, y)\|_{W^{1,1}(\Omega)} \lesssim d, \lambda, K_0, A_0, R_0, \text{dist}(y, \partial \Omega)^{-1},
\]
by the Rellich-Kondrachov compactness theorem, for a sufficiently small \( \delta > 0 \), there exists a subsequence of \( \{G^*_{\varepsilon_\rho}(\cdot, y)\} \) which converges a.e. to \( G(\cdot, y) \) on \( \Omega_\delta \), where \( \Omega_\delta \) is a smooth subdomain satisfying
\[
\{x \in \Omega : B_\delta(x) \subset \Omega\} \subset \Omega_\delta \subset \Omega.
\]
Thus by a diagonalization process, one can easily see that (4.5) holds. Combining (4.5) and the counterpart of (4.4), we have

\[ G(x, y) = G^*(y, x)^\top \]

for all \( x \in \Omega \setminus \mathcal{N}_y \). Similarly, we see that the above identity holds for all \( y \in \Omega \setminus \mathcal{N}_x \). This gives (2.2). The theorem is proved.

We now prove Theorem 2.9.

Proof of Theorem 2.9. The proof is similar to that of Lemma 3.1. Note that by utilizing Assumptions 2.5 and 2.8, and following the steps used in deriving (3.7), we have

\[ \| G_\varepsilon(\cdot, y) \|_{L^1(\Omega_R(x))} \lesssim d, \lambda, K_0, A_0, A_1 R^2 \] (4.6)

for all \( x, y \in \Omega \) and \( 0 < 2\varepsilon < R < R_0 \).

Let \( x, y \in \Omega \) with \( 0 < |x - y| < R_0 \), and set \( R = |x - y|/2 \). Then for \( \sigma \in (0, R) \), since \( (G_\sigma^*(\cdot, x), \Pi_\sigma^*(\cdot, x)) \) satisfies

\[
\begin{aligned}
\mathcal{L}^* G_\sigma^*(\cdot, x) + \nabla \Pi_\sigma^*(\cdot, x) &= 0 \quad \text{in } \Omega_R(y), \\
\operatorname{div} G_\sigma^*(\cdot, x) &= 0 \quad \text{in } \Omega_R(y),
\end{aligned}
\]

by Assumptions 2.5 and 2.8, we have

\[ \| G_\sigma^*(\cdot, x) \|_{L^\infty(\Omega_{r/2}(z))} \lesssim d, A_0, A_1 \| G_\sigma^*(\cdot, x) \|_{L^2(\Omega_r(z))} \]

for any \( z \in \Omega_R(y) \) and \( 0 < r < \operatorname{dist}(z, \partial B_R(y)) \). From this together with a standard iteration argument, we get

\[ \| G_\sigma^*(\cdot, x) \|_{L^\infty(\Omega_{r/2}(y))} \lesssim R^{-d} \| G_\sigma^*(\cdot, x) \|_{L^1(\Omega_R(y))} \lesssim R^{2-d} \]

where we used the counterpart of (4.6) in the second inequality. Finally, by using (4.4) we obtain the desired global bound for \( G(x, y) \). The theorem is proved.

References


Jongkeun Choi
Department of Mathematics Education, Pusan National University, Busan 46241, Republic of Korea

E-mail address: jongkeun_choi@pusan.ac.kr