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EVALUATIONS OF THE CUBIC CONTINUED FRACTION BY SOME THETA FUNCTION IDENTITIES: REVISITED

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Abstract. In this paper, we exploit some known theta function identities involving two parameters $l_{k,n}$ and $l'_{k,n}$ for the theta function ψ to find about 54 new values of the Ramanujan's cubic continued fraction.

1. Introduction

Ramanujan's cubic continued fraction $G(q)$, for $|q| < 1$, is defined by

$$G(q) = \frac{q^{1/3}}{1} \, {}_{+} \, \frac{q+q^2}{1} \, {}_{+} \, \frac{q^2+q^4}{1} \, {}_{+} \, \frac{q^3+q^6}{1} \, {}_{+} \, \cdots .$$

As stated in [8], there has been interest by number theorists in evaluating explicit values of $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$ for some positive rational numbers n. For brevity, we write q_n for $e^{-\pi\sqrt{n}}$ throughout this paper. In 1984, Ramanathan [10] found the value of $G(q_{10})$ such as $G(q_{10}) = \frac{\sqrt{9+3\sqrt{6}}-\sqrt{7+3\sqrt{6}}}{(1+\sqrt{5})\sqrt{\sqrt{5}+\sqrt{6}}}$ by using Kronecker's limit formula. Andrews and Berndt [3] also found the value of $G(q_{10})$ by employing Ramanujan's class invariants. In 1995, Berndt, Chan, and Zhang [5] evaluated $G(q_n)$ for $n = 2,\ 10,\ 22,\ 58$ and $G(-q_n)$ for $n = 1,\ 5,\ 13,\ 37$ by using Ramanujan's class invariants. In addition, Chan [6] found explicit values of $G(q_n)$ for $n = \frac{2}{9}$, 1, 2, 4 and $G(-q_n)$ for $n = 1,\ 5$ by applying some reciprocity theorems for the cubic continued fraction.

In the 2000s, Adiga, Vasuki, and Mahadeva Naika [2] evaluated $G(q_4)$ and $G(-q_n)$ for $n = \frac{1}{3}, \frac{25}{3}, \frac{49}{3}, \frac{1}{75}, \frac{1}{147}$ by using some modular equations. Moreover, Adiga, Kim, Mahadeva Naika, and Madhusudhan [1] found explicit values of $G(-q_n)$ for $n = \frac{1}{3}$,

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$\frac{1}{5}$, $\frac{1}{9}$, $\frac{1}{27}$, 1, 3, 5. Meanwhile, Yi [11] systematically found values of $G(q_n)$ for $n = \frac{1}{2}$, $\frac{1}{3}$, $\frac{4}{3}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{4}{9}$, 3, 6, 7, 8, 10, 12, 16, 28 and $G(-q_n)$ for $n = \frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{9}$, 2, 3, 4, 7 by using modular equations, in particular some eta function identities.

In the 2010s, Yi et al. [12] evaluated $G(q_n)$ for $n = \frac{1}{3}$, 1, 4, 9 and $G(-q_n)$ for $n = 4, 9$ by employing modular equations of degrees 3 or 9. In addition, Paek and Yi [7] derived some algorithms based on modular equations of degrees 3 or 9 to evaluate $G(q_n)$ for $n = \frac{4}{3}$, $\frac{16}{3}$, $\frac{64}{3}$, 36, 81, 144, 324 and $G(-q_n)$ for $n = \frac{4}{3}$, $\frac{16}{3}$, 36, 81. Paek and Yi [8] showed how to evaluate $G(q_n)$ and $G(-q_n)$ for $n = 4^m$, $\frac{1}{4^m}$, $2 \cdot 4^m$ and $\frac{1}{2\cdot 4^m}$ with some nonnegative integer m. In particular, they evaluated $G(q_n)$ for $n = \frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{128}$, 1, 8, 16, 32, 64, 128, 256 and $G(-q_n)$ for $n = \frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{128}$, 8, 16, 32, 64 by constructing some algorithms based on modular equations of degrees 3 or 9. Moreover, Paek and Yi [9] derived some algorithms based on modular equations of degrees 3 or 9 to evaluate $G(q_n)$ and $G(-q_n)$ for $n = \frac{2\cdot 4^m}{3}$, $\frac{1}{3\cdot 4^m}$, and $\frac{2}{3\cdot 4^m}$ with $m = 1, 2, 3$, and 4. In other words, they gave specific values of $G(q_n)$ for $n = \frac{8}{3}$, $\frac{32}{3}$, $\frac{128}{3}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{12}$, $\frac{1}{24}$, $\frac{1}{48}$, $\frac{1}{96}$, $\frac{1}{192}$, $\frac{1}{384}$ and $G(-q_n)$ for $n = \frac{8}{3}$, $\frac{32}{3}$, $\frac{1}{12}$, $\frac{1}{24}$, $\frac{1}{48}$, $\frac{1}{96}$, $\frac{1}{192}$, $\frac{1}{384}$.

Table 1

	$G(q_n)$	$G(-q_n)$
Ramanathan [10]	10	
Berndt et al. [5]	2, 10, 22, 58	1, 5, 13, 37
Chan [6]	$\frac{2}{9}$, 1, 2, 4	1, 5
Yi [11]	$\frac{1}{2}$, $\frac{1}{3}$, $\frac{4}{3}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{4}{9}$, 3, 6, 7, 8, 10, 12, 16, 28	$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{9}$, 2, 3, 4, 7
Adiga et al. [2]	4	$\frac{1}{3}$, $\frac{25}{3}$, $\frac{49}{3}$, $\frac{1}{75}$, $\frac{1}{147}$
Adiga et al. [1]		$\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{9}$, $\frac{1}{27}$, 1, 3, 5
Yi et al. [12]	$\frac{1}{3}$, 1, 4, 9	4, 9
Paek and Yi [7]	$\frac{4}{3}$, $\frac{16}{3}$, $\frac{64}{3}$, 36, 81, 144, 324	$\frac{4}{3}$, $\frac{16}{3}$, 36, 81
Paek and Yi [8]	$\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{128}$, 1, 8, 16, 32, 64, 128, 256	$\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{128}$ 8, 16, 32, 64
Paek and Yi [9]	$\frac{8}{3}$, $\frac{32}{3}$, $\frac{128}{3}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{12}$, $\frac{1}{24}$, $\frac{1}{48}$, $\frac{1}{96}$, $\frac{1}{192}$, $\frac{1}{384}$	$\frac{8}{3}$, $\frac{32}{3}$, $\frac{1}{12}$, $\frac{1}{24}$, $\frac{1}{48}$, $\frac{1}{96}$ $\frac{1}{192}$, $\frac{1}{384}$
Yi and Paek [14]	$\frac{1}{5}$, $\frac{4}{5}$, $\frac{9}{5}$, $\frac{16}{5}$, $\frac{36}{5}$, $\frac{144}{5}$, $\frac{5}{9}$, $\frac{20}{9}$, $\frac{80}{9}$, $\frac{1}{27}$, $\frac{4}{27}$, $\frac{16}{27}$, $\frac{1}{45}$, $\frac{4}{45}$, $\frac{16}{45}$, 5, 20, 27, 45, 48, 80, 108, 180, 432, 720	$\frac{4}{5}$, $\frac{9}{5}$, $\frac{36}{5}$, $\frac{5}{9}$, $\frac{20}{9}$, $\frac{1}{45}$, $\frac{4}{45}$, 20, 27, 45, 180

More recently, Yi and Paek [14] used some theta function identities involving parameters $h_{n,k}$ and $h'_{n,k}$ for the theta function φ to establish evaluations of $G(q_n)$ for $n = \frac{1}{5}, \frac{4}{5}, \frac{9}{5}, \frac{16}{5}, \frac{36}{5}, \frac{144}{5}, \frac{5}{9}, \frac{20}{9}, \frac{80}{9}, \frac{1}{27}, \frac{4}{27}, \frac{16}{27}, \frac{1}{45}, \frac{4}{45}, \frac{16}{45}$, 5, 20, 27, 45, 48, 80, 108, 180, 432, 720 and $G(-q_n)$ for $n = \frac{4}{5}, \frac{9}{5}, \frac{36}{5}, \frac{5}{9}, \frac{20}{9}, \frac{1}{45}, \frac{4}{45}$, 20, 27, 45, 180. Table 1 shows a summary of some known values of n for $G(q_n)$ and $G(-q_n)$ in chronological order.

Thus $G(q_n)$ were evaluated for $n = \frac{1}{2}, \frac{1}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \frac{32}{3}, \frac{64}{3}, \frac{128}{3}, \frac{1}{4}, \frac{1}{5}, \frac{4}{5}, \frac{9}{5}, \frac{16}{5}, \frac{36}{5}, \frac{144}{5}, \frac{1}{6}, \frac{1}{8}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{20}{9}, \frac{80}{9}, \frac{1}{12}, \frac{1}{16}, \frac{1}{24}, \frac{1}{27}, \frac{4}{27}, \frac{16}{27}, \frac{1}{32}, \frac{1}{45}, \frac{4}{45}, \frac{16}{45}, \frac{1}{48}, \frac{1}{96}, \frac{1}{128}, \frac{1}{192}, \frac{1}{384}$, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, 20, 22, 27, 28, 32, 36, 45, 48, 58, 64, 80, 81, 108, 128, 144, 180, 256, 324, 432, 720.

Whereas $G(-q_n)$ were evaluated for $n = \frac{1}{2}, \frac{1}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \frac{25}{3}, \frac{32}{3}, \frac{49}{3}, \frac{1}{4}, \frac{1}{5}, \frac{4}{5}, \frac{9}{5}, \frac{36}{5}, \frac{1}{8}, \frac{1}{9}, \frac{5}{9}, \frac{20}{9}, \frac{1}{12}, \frac{1}{16}, \frac{1}{24}, \frac{1}{27}, \frac{1}{32}, \frac{1}{45}, \frac{4}{45}, \frac{1}{48}, \frac{1}{75}, \frac{1}{96}, \frac{1}{128}, \frac{1}{147}, \frac{1}{192}, \frac{1}{384}$, 1, 2, 3, 4, 5, 7, 8, 9, 13, 16, 20, 27, 32, 36, 37, 45, 64, 81, 180.

In this paper, we use some theta function identities involving parameters $l_{k,n}$ and $l'_{k,n}$ for the theta function ψ to establish about 54 new values of $G(q_n)$ and $G(-q_n)$ such as $G(-q_6)$, $G(-q_{\frac{1}{6}})$, and $G(q_n)$ and $G(-q_n)$ for $n = \frac{3}{2}, \frac{2}{3}, \frac{5}{3}, \frac{20}{3}, \frac{15}{4}, \frac{3}{5}, \frac{12}{5}, \frac{3}{8}, \frac{5}{12}, \frac{1}{15}, \frac{4}{15}, \frac{3}{20}, \frac{2}{27}, \frac{5}{27}, \frac{8}{27}, \frac{20}{27}, \frac{1}{54}, \frac{1}{60}, \frac{5}{108}, \frac{1}{135}, \frac{4}{135}, \frac{1}{216}, \frac{1}{540}$, 15, 24, and 60.

Ramanujan's theta function $\psi(q)$, for $|q| < 1$, is defined by

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

For any positive real numbers k and n, define $l_{k,n}$ and $l'_{k,n}$ by

$$l_{k,n} = \frac{\psi(-q)}{k^{1/4}\psi(-q^k)} \quad \text{and} \quad l'_{k,n} = \frac{\psi(q)}{k^{1/4}\psi(q^k)},$$

where $q = e^{-\pi\sqrt{n/k}}$ (See [13] for details). We now note that the following property of $l_{k,n}$ in [13] will be useful for evaluating the cubic continued fraction later on.

$$l_{k,\frac{1}{n}} = l_{k,n}^{-1}. \tag{1.1}$$

We also note general formulas for $G^3(q_{\frac{n}{3}})$ and $G^3(-q_{\frac{n}{3}})$ in terms of $l'_{3,n}$ and $l_{3,n}$, respectively, in [13, Theorem 6.2(ii) and (v)] such as

$$G^3(q_{\frac{n}{3}}) = \frac{1}{3\,l'^4_{3,n} - 1} \tag{1.2}$$

and

$$G^3(-q_{\frac{n}{3}}) = \frac{-1}{3\,l^4_{3,n} + 1}. \tag{1.3}$$

By taking cube root of (1.2) and (1.3), we have the values of $G(q_{\frac{n}{3}})$ and $G(-q_{\frac{n}{3}})$. Hence, in view of (1.2) and (1.3), in order to find some explicit values of $G^3(q_{\frac{n}{3}})$ and $G^3(-q_{\frac{n}{3}})$, it is sufficient to evaluate $l'_{3,n}$ and $l_{3,n}$, respectively. For brevity, we write l_n, l'_n for $l_{3,n}$, $l'_{3,n}$, respectively.

2. Evaluations of l_n and l'_n

We begin this section by recalling the values of l_2 and l_5 in [13], which play key roles in evaluating some new values of l_n.

Lemma 2.1 ([13, Theorem 4.9(iv) and (v)]). *We have*

(i) $l_2 = (\sqrt{2}+\sqrt{3}\,)^{1/4}$,

(ii) $l_5 = \left(\dfrac{1+\sqrt{5}}{2}\right)^{3/2}$.

Note that l_5 in Lemma 2.1(ii) was incorrectly recorded as $\left(\dfrac{1+\sqrt{5}}{2}\right)^{2/3}$ in [13]. We now recall a theta function identity in [4, Entry 1(ii), p. 345] such as

$$\left(1+\frac{\psi(-q^{1/3})}{q^{1/3}\psi(-q^3)}\right)^3 = 1+\frac{\psi^4(-q)}{q\psi^4(-q^3)}. \tag{2.1}$$

Rewriting (2.1) in terms of l_n and l_{9n}, we have the next result.

Lemma 2.2 ([13, Theorem 4.5(i)]). *For any positive real number n, we have*

$$(1+\sqrt{3}\,l_n l_{9n})^3 = 1+3\,l_{9n}^4. \tag{2.2}$$

We first evaluate l_n for $n = \frac{1}{2}, \frac{9}{2}, \frac{2}{9}, \frac{1}{18}$, and 18.

Theorem 2.3. *We have*

(i) $l_{\frac{1}{2}} = (\sqrt{3}-\sqrt{2}\,)^{1/4}$,

(ii) $l_{\frac{2}{9}} = \dfrac{\sqrt[3]{1+3\sqrt{2}+3\sqrt{3}}\,-1}{\sqrt{3}\,(\sqrt{2}+\sqrt{3}\,)^{1/4}}$,

(iii) $l_{\frac{1}{18}} = \dfrac{\sqrt[3]{1-3\sqrt{2}+3\sqrt{3}}\,-1}{\sqrt{3}\,(\sqrt{3}-\sqrt{2}\,)^{1/4}}$,

(iv) $l_{\frac{9}{2}} = \dfrac{\sqrt{3}\,(\sqrt{2}+\sqrt{3}\,)^{1/4}}{\sqrt[3]{1+3\sqrt{2}+3\sqrt{3}}\,-1}$,

(v) $l_{18} = \dfrac{\sqrt{3}\,(\sqrt{3}-\sqrt{2}\,)^{1/4}}{\sqrt[3]{1-3\sqrt{2}+3\sqrt{3}}\,-1}$.

Proof. Part (i) is clear by (1.1) and Lemma 2.1(i). For (ii), let $n = \frac{2}{9}$ in (2.2) and put $l_2 = (\sqrt{2}+\sqrt{3}\,)^{1/4}$ in Lemma 2.1(i), then we find that

$$\left(1+\sqrt{3}\,(\sqrt{2}+\sqrt{3}\,)^{1/4}\,l_{\frac{2}{9}}\right)^3 = 1+3(\sqrt{2}+\sqrt{3}\,).$$

Taking the cube root of both sides of the last equation and simplifying to complete the proof.

For (iii), let $n = \frac{1}{18}$ in (2.2), put the value of $l_{\frac{1}{2}}$ obtained from (i), and repeat the same argument as in the proof of (ii) to complete the proof. The proofs of (iv) and (v) follow directly from (1.1). □

We next evaluate l_n for $n = \frac{1}{5}, \frac{9}{5}, \frac{5}{9}, \frac{1}{45}$, and 45.

Theorem 2.4. *We have*

(i) $l_{\frac{1}{5}} = \sqrt{\sqrt{5}-2}\,,$

(ii) $l_{\frac{5}{9}} = \dfrac{\sqrt[3]{28+12\sqrt{5}}-1}{\sqrt{6+3\sqrt{5}}}\,,$

(iii) $l_{\frac{1}{45}} = \dfrac{\sqrt[3]{28-12\sqrt{5}}-1}{\sqrt{-6+3\sqrt{5}}}\,,$

(iv) $l_{\frac{9}{5}} = \dfrac{\sqrt{6+3\sqrt{5}}}{\sqrt[3]{28+12\sqrt{5}}-1}\,,$

(v) $l_{45} = \dfrac{\sqrt{-6+3\sqrt{5}}}{\sqrt[3]{28-12\sqrt{5}}-1}\,.$

Proof. Repeat the same argument as in the proof of Theorem 2.3. □

We now turn to evaluations of l'_n. But we need the following theta function identity with respect to l_n and l'_n.

Lemma 2.5 ([12, Corollary 3.12]). *For every positive real number n, we have*

$$(l_n^4 - l_n'^4 + 3)\left(\frac{1}{l_n^4} - \frac{1}{l_n'^4} + 3\right) = 1. \tag{2.3}$$

Note that (2.3) follows from a modular equation in [12, Theorem 3.11] such as $(P^4 - Q^4 - 9)\left(\dfrac{1}{P^4} - \dfrac{1}{Q^4} - 1\right) = 1$ with $P = \frac{\psi(q)}{q^{1/4}\psi(q^3)}$ and $Q = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)}$.

In view of (2.3), we evaluate $l_n'^4$ for $n = \frac{1}{2}, \frac{9}{2}, \frac{2}{9}, \frac{1}{18}$, 2, and 18.

Theorem 2.6. *We have*

(i) $l_2'^4 = 3+2\sqrt{2}\,,$

(ii) $l'^4_{\frac{1}{2}} = \sqrt{2} + \sqrt{3}\,,$

(iii) $l'^4_{\frac{2}{9}} = \dfrac{a+4}{3} + \dfrac{(a+1)\sqrt{a+4}}{3\sqrt{a}}\,,$

(iv) $l'^4_{\frac{1}{18}} = \dfrac{b+4}{3} + \dfrac{(b+1)\sqrt{b+4}}{3\sqrt{b}}\,,$

(v) $l'^4_{\frac{9}{2}} = \dfrac{3\sqrt{a}\,(a+1+\sqrt{a^2+4a}\,)}{(2a-1)\sqrt{a+4}}\,,$

(vi) $l'^4_{18} = \dfrac{3\sqrt{b}\,(b+1+\sqrt{b^2+4b}\,)}{(2b-1)\sqrt{b+4}}\,,$

where

$$a = \frac{1}{2} + \frac{\left(\sqrt[3]{1+3\sqrt{2}+3\sqrt{3}}\,-1\right)^4}{6\,(\sqrt{2}+\sqrt{3}\,)} \quad \textit{and } b = \frac{1}{2} + \frac{\left(\sqrt[3]{1-3\sqrt{2}+3\sqrt{3}}\,-1\right)^4}{6\,(\sqrt{3}-\sqrt{2}\,)}.$$

Proof. For (i), let $n = 2$ in (2.3) and put the value of l_2 in Lemma 2.1(i), then it follows that

$$(3-\sqrt{2}+\sqrt{3}\,)\,x^2 - 2(5+3\sqrt{3}\,)\,x + 3 + \sqrt{2} + \sqrt{3} = 0,$$

where $x = l'^4_2$. Solving the last equation for x and using $x > 1$, we have the required result.

The proofs of (ii)–(vi) are similar to that of (i). □

We now evaluate l'^4_n for $n = \frac{1}{5}, \frac{9}{5}, \frac{5}{9}, \frac{1}{45}$, 5, and 45.

Theorem 2.7. *We have*

(i) $l'^4_5 = \dfrac{8+2\sqrt{15}}{3-\sqrt{5}}\,,$

(ii) $l'^4_{\frac{1}{5}} = \dfrac{8+2\sqrt{15}}{3+\sqrt{5}}\,,$

(iii) $l'^4_{\frac{5}{9}} = \dfrac{c+4}{3} + \dfrac{(c+1)\sqrt{c+4}}{3\sqrt{c}}\,,$

(iv) $l'^4_{\frac{1}{45}} = \dfrac{d+4}{3} + \dfrac{(d+1)\sqrt{d+4}}{3\sqrt{d}}\,,$

(v) $l'^4_{\frac{9}{5}} = \dfrac{3\sqrt{c}\,(c+1+\sqrt{c^2+4c}\,)}{(2c-1)\sqrt{c+4}}\,,$

(vi) $l'^4_{45} = \dfrac{3\sqrt{d}\,(d+1+\sqrt{d^2+4d}\,)}{(2d-1)\sqrt{d+4}}\,,$

where

$$c = \frac{1}{2} + \frac{\left(\sqrt[3]{28+12\sqrt{5}}\,-1\right)^4}{6(9+4\sqrt{5}\,)} \quad \textit{and } d = \frac{1}{2} + \frac{\left(\sqrt[3]{28-12\sqrt{5}}\,-1\right)^4}{6(9-4\sqrt{5}\,)}.$$

Proof. The proof follows precisely along the same lines as that for Theorem 2.6. □

We evaluate some more values of $l_n'^4$ and l_n^4 by employing the following theta function identities involving l_n', l_{4n}', and l_n.

Lemma 2.8 ([9, Corollary 3.4])**.** *For any positive real number n, we have*

$$l_n^4(\sqrt{3}\,l_{4n}'^2 + 1) = l_{4n}'^2(l_{4n}'^2 - \sqrt{3}\,) \tag{2.4}$$

Note that (2.4) follows from a modular equation $P^4(Q^2+1) = Q^2(Q^2+3)$ with $P = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)}$ and $Q = \frac{\psi(q^2)}{q^{1/2}\psi(q^6)}$.

We also need the following theta function identity involving l_n' and l_{4n}'.

Lemma 2.9 ([9, Corollary 3.2])**.** *For any positive real number n, we have*

$$l_n'^4(\sqrt{3}\,l_{4n}'^2 - 1) = l_{4n}'^2(l_{4n}'^2 + \sqrt{3}\,) \tag{2.5}$$

Note that (2.5) follows from a modular equation $P^4(Q^2-1) = Q^2(Q^2+3)$ with $P = \frac{\psi(q)}{q^{1/4}\psi(q^3)}$ and $Q = \frac{\psi(q^2)}{q^{1/2}\psi(q^6)}$.

In view of (2.4), and (2.5), we evaluate $l_n'^4$ and l_n^4 for $n = \frac{9}{8}, \frac{8}{9}, \frac{1}{72}$, and 72.

Theorem 2.10. *Let a and b be as in Theorem 2.6. Then we have*

(i) $l_{\frac{8}{9}}'^4 = \dfrac{1}{3}(a + 1 + \sqrt{a^2+4a}\,)^2$,

(ii) $l_{\frac{1}{72}}'^4 = 1 - \dfrac{(b+7)\sqrt{b} + (b+1)\sqrt{b+4}}{3\sqrt{b} - 3\sqrt{b^2+4b+(b+1)\sqrt{b^2+4b}}}$,

(iii) $l_{\frac{9}{8}}'^4 = 1 - \dfrac{3(a+1)\sqrt{a} + (5a-1)\sqrt{a+4}}{(2a-1)\sqrt{a+4} - 3\sqrt{2a-1}\sqrt{a^2+4a+(a+1)\sqrt{a^2+4a}}}$,

(iv) $l_{72}'^4 = \dfrac{3(b+1+\sqrt{b^2+4b}\,)^2}{(2b-1)^2}$.

Proof. For (i), let $n = \frac{2}{9}$ in (2.4) and put the value of $l_{\frac{2}{9}}$ in Theorem 2.3(ii), then we deduce that

$$3x^4 - 2\sqrt{3}\,(a+1)x^2 - 2a + 1 = 0,$$

where $x = l_{\frac{8}{9}}'$. Solving the last equation for x and using $x > 1$, we complete the proof.

For (ii), let $n = \frac{1}{72}$ in (2.5), put the value of $l_{\frac{1}{18}}'$ in Theorem 2.6(iv), and simplify the equation to complete the proof

The proofs of (iii) and (iv) are similar to those of (i) or (ii). □

Theorem 2.11. *Let a and b be as in Theorem 2.6. Then we have*

(i) $$l_{\frac{8}{9}}^4 = \frac{(2a-1)\sqrt{a+4}+3\sqrt{2a-1}\sqrt{a^2+4a+(a+1)\sqrt{a^2+4a}}}{3(a+1)\sqrt{a}+3a\sqrt{a+4}-3\sqrt{2a-1}\sqrt{a^2+4a+(a+1)\sqrt{a^2+4a}}},$$

(ii) $$l_{\frac{1}{72}}^4 = -1+\frac{(b+7)\sqrt{b}+(b+1)\sqrt{b+4}}{3\sqrt{b}+3\sqrt{b^2+4b+(b+1)\sqrt{b^2+4b}}},$$

(iii) $$l_{\frac{9}{8}}^4 = -1+\frac{3(a+1)\sqrt{a}+(5a-1)\sqrt{a+4}}{(2a-1)\sqrt{a+4}+3\sqrt{2a-1}\sqrt{a^2+4a+(a+1)\sqrt{a^2+4a}}},$$

(iv) $$l_{72}^4 = \frac{3\sqrt{b}+3\sqrt{b^2+4b+(b+1)\sqrt{b^2+4b}}}{(b+4)\sqrt{b}+(b+1)\sqrt{b+4}-3\sqrt{b^2+4b+(b+1)\sqrt{b^2+4b}}}.$$

Proof. For (i), let $n = \frac{8}{9}$ in (2.3) and put the value of $l'_{\frac{8}{9}}$ in Theorem 2.10(i), then we find that

$$(3\,l-1)x^2-(3l^2-10l+3)x-l^2+3l=0,$$

where $x = l_{\frac{8}{9}}^4$ and $l = l'^4_{\frac{8}{9}}$. Employing *Mathematica* to solve the last equation for x, we complete the proof. The proof of (ii) is similar to that of (i).

The proofs of (iii) and (iv) follow from (1.1). □

We evaluate l'^4_n for $n = \frac{5}{4}, \frac{45}{4}, \frac{4}{5}, \frac{36}{5}, \frac{20}{9}, \frac{1}{20}, \frac{9}{20}, \frac{5}{36}, \frac{4}{45}, \frac{1}{180}$, 20, and 180.

Theorem 2.12. *Let c and d be as in Theorem 2.7. Then we have*

(i) $$l'^4_{20} = (2+\sqrt{5}\,)^2(4+\sqrt{15}\,)^2,$$

(ii) $$l'^4_{\frac{5}{4}} = \frac{3+\sqrt{15}+(4+\sqrt{15}\,)\sqrt{3+\sqrt{5}}}{3+\sqrt{15}-\sqrt{3-\sqrt{5}}},$$

(iii) $$l'^4_{\frac{4}{5}} = (-2+\sqrt{5}\,)^2(4+\sqrt{15}\,)^2,$$

(iv) $$l'^4_{\frac{1}{20}} = \frac{3+\sqrt{15}+(4+\sqrt{15}\,)\sqrt{3-\sqrt{5}}}{3+\sqrt{15}-\sqrt{3+\sqrt{5}}},$$

(v) $$l'^4_{\frac{20}{9}} = \frac{1}{3}(c+1+\sqrt{c^2+4c}\,)^2,$$

(vi) $$l'^4_{\frac{5}{36}} = 1-\frac{(c+7)\sqrt{c}+(c+1)\sqrt{c+4}}{3\sqrt{c}-3\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}},$$

(vii) $$l'^4_{\frac{4}{45}} = \frac{1}{3}(d+1+\sqrt{d^2+4d}\,)^2,$$

(viii) $$l'^4_{\frac{1}{180}} = 1-\frac{(d+7)\sqrt{d}+(d+1)\sqrt{d+4}}{3\sqrt{d}-3\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}},$$

(ix) $l'^4_{180} = \dfrac{3(d+1+\sqrt{d^2+4d}\,)^2}{(2d-1)^2}$,

(x) $l'^4_{\frac{45}{4}} = 1 - \dfrac{3(d+1)\sqrt{d}+(5d-1)\sqrt{d+4}}{(2d-1)\sqrt{d+4}-3\sqrt{2d-1}\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}}$,

(xi) $l'^4_{\frac{36}{5}} = \dfrac{3(c+1+\sqrt{c^2+4c}\,)^2}{(2c-1)^2}$,

(xii) $l'^4_{\frac{9}{20}} = 1 - \dfrac{3(c+1)\sqrt{c}+(5c-1)\sqrt{c+4}}{(2c-1)\sqrt{c+4}-3\sqrt{2c-1}\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}$.

Proof. For (i), let $n = 5$ in (2.5) and put $l_5^4 = 9+4\sqrt{5}$ from Theorem 2.1(ii), then we find that

$$(3-\sqrt{5}\,)\,l'^4_{20} - (5\sqrt{3}+6\sqrt{5}+\sqrt{15}\,)\,l'^2_{20} + 8 + 2\sqrt{5} = 0.$$

Solve the last equation for l'_{20} and use $l'_{20} > 0$ to complete the proof. For the proofs of (iii), (v), (vii), (ix), and (xi), repeat the same argument as in the proof of (i).

For (ii), let $n = \frac{5}{4}$ in (2.4) and put the value l'_5 from Theorem 2.8(i), then we find that

$$\left(3+\sqrt{15}-\sqrt{3-\sqrt{5}}\right) l'^4_5 = 3+\sqrt{15}+(4+\sqrt{15}\,)\sqrt{3+\sqrt{5}}\,.$$

Hence we have the required result. The proofs of (iv), (vi), (viii), (x), (xii) are similar to that of (ii). □

We end this section by evaluating l_n^4 for $n = \frac{5}{4}, \frac{45}{4}, \frac{4}{5}, \frac{36}{5}, \frac{20}{9}, \frac{1}{20}, \frac{9}{20}, \frac{5}{36}, \frac{4}{45}, \frac{1}{180}$, 20, and 180.

Theorem 2.13. *Let c and d be as in Theorem 2.7. Then we have*

(i) $l^4_{20} = \dfrac{8-3\sqrt{2}+5\sqrt{6}}{6-5\sqrt{3}-4\sqrt{5}+3\sqrt{15}}$,

(ii) $l^4_{\frac{5}{4}} = \dfrac{6+5\sqrt{3}+4\sqrt{5}+3\sqrt{15}}{8+3\sqrt{2}+5\sqrt{6}}$,

(iii) $l^4_{\frac{4}{5}} = \dfrac{8+3\sqrt{2}+5\sqrt{6}}{6+5\sqrt{3}+4\sqrt{5}+3\sqrt{15}}$,

(iv) $l^4_{\frac{1}{20}} = \dfrac{6-5\sqrt{3}-4\sqrt{5}+3\sqrt{15}}{8-3\sqrt{2}+5\sqrt{6}}$,

(v) $l^4_{\frac{20}{9}} = \dfrac{(2c-1)\sqrt{c+4}+3\sqrt{2c-1}\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}{3(c+1)\sqrt{c}+3c\sqrt{c+4}-3\sqrt{2c-1}\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}$,

(vi) $l^4_{\frac{5}{36}} = -1 + \dfrac{(c+7)\sqrt{c} + (c+1)\sqrt{c+4}}{3\sqrt{c} + 3\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}$,

(vii) $l^4_{\frac{4}{45}} = \dfrac{(2d-1)\sqrt{d+4} + 3\sqrt{2d-1}\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}}{3(d+1)\sqrt{d} + 3d\sqrt{d+4} - 3\sqrt{2d-1}\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}}$,

(viii) $l^4_{\frac{1}{180}} = -1 + \dfrac{(d+7)\sqrt{d} + (d+1)\sqrt{d+4}}{3\sqrt{d} + 3\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}}$,

(ix) $l^4_{180} = \dfrac{3\sqrt{d} + 3\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}}{(d+4)\sqrt{d} + (d+1)\sqrt{d+4} - 3\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}}$,

(x) $l^4_{\frac{45}{4}} = -1 + \dfrac{3(d+1)\sqrt{d} + (5d-1)\sqrt{d+4}}{(2d-1)\sqrt{d+4} + 3\sqrt{2d-1}\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}}$

(xi) $l^4_{\frac{36}{5}} = \dfrac{3\sqrt{c} + 3\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}{(c+4)\sqrt{c} + (c+1)\sqrt{c+4} - 3\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}$,

(xii) $l^4_{\frac{9}{20}} = -1 + \dfrac{3(c+1)\sqrt{c} + (5c-1)\sqrt{c+4}}{(2c-1)\sqrt{c+4} + 3\sqrt{2c-1}\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}$.

Proof. For (i), let $n = 20$ in (2.3) and put $l^4_{20} = (2+\sqrt{5}\,)^2(4+\sqrt{15}\,)^2$ from Theorem 2.12(i), then we deduce that

$$2\,l^8_{20} - 8(69 + 40\sqrt{3} + 31\sqrt{5} + 18\sqrt{15}\,)\,l^4_{20} - 188 - 105\sqrt{3} - 84\sqrt{5} - 47\sqrt{15} = 0.$$

Using *Mathematica* to solve the last equation for l^4_{20}, we complete the proof.

For (ii), let $n = \frac{5}{4}$ in (2.5) and put the value of l'_5 in Theorem 2.8(i) to complete the proof. For (iii)– (xii), repeat the same argument as in the proofs of (i) or (ii). □

3. Evaluations of $G(q)$

In this section, we evaluate about 46 values $G(-q_n)$ and $G(q_n)$ including 36 new ones. Just for editorial convenience, we evaluate $G^3(-q_n)$ and $G^3(q_n)$. By taking cube roots of them, the required values of $G(-q_n)$ and $G(q_n)$ can easily be obtained.

We first evaluate $G^3(-q_n)$ and $G^3(q_n)$ for $n = \frac{3}{2}, \frac{2}{3}, \frac{1}{6}, \frac{2}{27}, \frac{1}{54}$, and 6.

Theorem 3.1. *We have*

(i) $G^3(-q_{\frac{2}{3}}) = \dfrac{-1}{1 + 3(\sqrt{2} + \sqrt{3}\,)}$,

(ii) $G^3(-q_{\frac{1}{6}}) = \dfrac{-1}{1-3(\sqrt{2}-\sqrt{3})}$,

(iii) $G^3(-q_{\frac{2}{27}}) = \dfrac{-3(\sqrt{2}+\sqrt{3})}{3(\sqrt{2}+\sqrt{3})+\left(1-\sqrt[3]{1+3\sqrt{2}+3\sqrt{3}}\right)^4}$,

(iv) $G^3(-q_{\frac{1}{54}}) = \dfrac{-3(\sqrt{2}-\sqrt{3})}{3(\sqrt{2}-\sqrt{3})-\left(1-\sqrt[3]{1-3\sqrt{2}+3\sqrt{3}}\right)^4}$,

(v) $G^3(-q_{\frac{3}{2}}) = \dfrac{-\left(1-\sqrt[3]{1-3\sqrt{2}+3\sqrt{3}}\right)^4}{27(\sqrt{2}+\sqrt{3})+\left(1-\sqrt[3]{1-3\sqrt{2}+3\sqrt{3}}\right)^4}$,

(vi) $G^3(-q_6) = \dfrac{\left(1-\sqrt[3]{1-3\sqrt{2}+3\sqrt{3}}\right)^4}{27(\sqrt{2}-\sqrt{3})-\left(1-\sqrt[3]{1-3\sqrt{2}+3\sqrt{3}}\right)^4}$.

Proof. The results follow from (1.3), Lemma 2.1(i), and Theorem 2.3. □

Theorem 3.2. *Let a and b be as in Theorem 2.6. Then we have*

(i) $G^3(q_{\frac{2}{3}}) = \dfrac{-4+3\sqrt{2}}{4}$,

(ii) $G^3(q_{\frac{1}{6}}) = \dfrac{1}{-1+3\sqrt{2}+3\sqrt{3}}$,

(iii) $G^3(q_{\frac{2}{27}}) = \dfrac{(a+1)\sqrt{a^2+4a}-a(a+3)}{4}$,

(iv) $G^3(q_{\frac{1}{54}}) = \dfrac{(b+1)\sqrt{b^2+4b}-b(b+3)}{4}$,

(v) $G^3(q_{\frac{3}{2}}) = \dfrac{(2a-1)\sqrt{a+4}}{9(a+1)\sqrt{a}+(7a+1)\sqrt{a+4}}$,

(vi) $G^3(q_6) = \dfrac{(2b-1)\sqrt{b+4}}{9(b+1)\sqrt{b}+(7b+1)\sqrt{b+4}}$.

Proof. The proofs are clear by (1.2) and Theorem 2.6. □

Note that an explicit value of $G(q_6)$ in [11, Theorem 6.3.3(ii)] was given by $G(q_6) = \dfrac{\sqrt[3]{3-2\sqrt{2}}}{2+2\sqrt[3]{1+\sqrt{2}}+\sqrt{2}\sqrt[3]{3+2\sqrt{2}}}$. Note also that the value of $G^3(q_{\frac{1}{6}})$ was given in [8, Theorem 5.5(i)].

We next evaluate $G(-q_n)$ and $G(q_n)$ for $n = \frac{5}{3}, \frac{3}{5}, \frac{1}{15}, \frac{5}{27}, \frac{1}{135}$, and 15.

Theorem 3.3. *We have*

(i) $G^3(-q_{\frac{5}{3}}) = \dfrac{-1}{4(7+3\sqrt{5})}$,

(ii) $G^3(-q_{\frac{1}{15}}) = \dfrac{-7-3\sqrt{5}}{16}$,

(iii) $G^3(-q_{\frac{5}{27}}) = \dfrac{-3}{3+(-2+\sqrt{5}\,)^2\left(-1+\sqrt[3]{28+12\sqrt{5}}\right)^4}$,

(iv) $G^3(-q_{\frac{1}{135}}) = \dfrac{-3}{3+(2+\sqrt{5}\,)^2\left(-1+\sqrt[3]{28-12\sqrt{5}}\right)^4}$,

(v) $G^3(-q_{\frac{3}{5}}) = \dfrac{1-\left(-1+\sqrt[3]{28+12\sqrt{5}}\right)^4}{3(6+3\sqrt{5}\,)^2+\left(-1+\sqrt[3]{28+12\sqrt{5}}\right)^4}$,

(vi) $G^3(-q_{15}) = \dfrac{1-\left(-1+\sqrt[3]{28-12\sqrt{5}}\right)^4}{3(-6+3\sqrt{5}\,)^2+\left(-1+\sqrt[3]{28-12\sqrt{5}}\right)^4}$.

Proof. The results follow from (1.3), Lemma 2.1(ii), and Theorem 2.4. □

Theorem 3.4. *Let c and d be as in Theorem 2.7. Then we have*

(i) $G^3(q_{\frac{5}{3}}) = \dfrac{3-\sqrt{5}}{21+\sqrt{5}+6\sqrt{15}}$,

(ii) $G^3(q_{\frac{1}{15}}) = \dfrac{3+\sqrt{5}}{21-\sqrt{5}+6\sqrt{15}}$,

(iii) $G^3(q_{\frac{5}{27}}) = \dfrac{(c+1)\sqrt{c^2+4c}-c(c+3)}{4}$,

(iv) $G^3(q_{\frac{1}{135}}) = \dfrac{(d+1)\sqrt{d^2+4d}-d(d+3)}{4}$,

(v) $G^3(q_{\frac{3}{5}}) = \dfrac{(2c-1)\sqrt{c+4}}{9(c+1)\sqrt{c}+(7c+1)\sqrt{c+4}}$,

(vi) $G^3(q_{15}) = \dfrac{(2d-1)\sqrt{d+4}}{9(d+1)\sqrt{d}+(7d+1)\sqrt{d+4}}$.

Proof. The results follow from (1.2) and Theorem 2.7. □

We now evaluate $G(q_n)$ and $G(-q_n)$ for $n = \frac{3}{8}, \frac{8}{27}, \frac{1}{216}$, and 24.

Theorem 3.5. *Let a and b be as in Theorem 2.6. Then we have*

(i) $G^3(q_{\frac{8}{27}}) = \dfrac{(a+1)\sqrt{a^2+4a}-a(a+3)}{8a}$,

(ii) $G^3(q_{\frac{1}{216}}) = \dfrac{-\sqrt{b}+\sqrt{b^2+4b+(b+1)\sqrt{b^2+4b}}}{(b+5)\sqrt{b}+(b+1)\sqrt{b+4}+2\sqrt{b^2+4b+(b+1)\sqrt{b^2+4b}}}$,

(iii) $G^3(q_{\frac{3}{8}})$

$$= \frac{-(2a-1)\sqrt{a+4}+3\sqrt{2a-1}\sqrt{a^2+4a+(a+1)\sqrt{a^2+4a}}}{9(a+1)\sqrt{a}+(11a-1)\sqrt{a+4}+6\sqrt{2a-1}\sqrt{a^2+4a+(a+1)\sqrt{a^2+4a}}},$$

(iv) $G^3(q_{24}) = \dfrac{5b+2-3\sqrt{b^2+4b}}{4(b+4+3\sqrt{b^2+4b}\,)}$.

Proof. The results follow from (1.2) and Theorem 2.10. □

Theorem 3.6. *Let c and d be as in Theorem 2.7. Then we have*

(i) $G^3(-q_{\frac{8}{27}})$

$$= \frac{-(a+1)\sqrt{a}-a\sqrt{a+4}+\sqrt{2a-1}\sqrt{a^2+4a+(a+1)\sqrt{a^2+4a}}}{(a+1)\sqrt{a}+(3a-1)\sqrt{a+4}+2\sqrt{2a-1}\sqrt{a^2+4a+(a+1)\sqrt{a^2+4a}}},$$

(ii) $G^3(-q_{\frac{1}{216}})$

$$= \frac{-\sqrt{b}-\sqrt{b^2+4b+(b+1)\sqrt{b^2+4b}}}{(b+5)\sqrt{b}+(b+1)\sqrt{b+4}-2\sqrt{b^2+4b+(b+1)\sqrt{b^2+4b}}},$$

(iii) $G^3(-q_{\frac{3}{8}})$

$$= \frac{-(2a-1)\sqrt{a}-3\sqrt{2a-1}\sqrt{a^2+4a+(a+1)\sqrt{a^2+4a}}}{9(a+1)\sqrt{a}+(11a-1)\sqrt{a+4}-6\sqrt{2a-1}\sqrt{a^2+4a+(a+1)\sqrt{a^2+4a}}},$$

(iv) $G^3(-q_{24})$

$$= \frac{-(b+4)\sqrt{b}-(b+1)\sqrt{b+4}+3\sqrt{b^2+4b+(b+1)\sqrt{b^2+4b}}}{(b+13)\sqrt{b}+(b+1)\sqrt{b+4}+6\sqrt{b^2+4b+(b+1)\sqrt{b^2+4b}}}.$$

Proof. The results follow from (1.3) and Theorem 2.11. □

We end this section by evaluating $G^3(q_n)$ and $G^3(-q_n)$ for $n = \frac{20}{3}, \frac{15}{4}, \frac{12}{5}, \frac{5}{12}, \frac{4}{15}, \frac{3}{20}, \frac{20}{27}, \frac{1}{60}, \frac{5}{108}, \frac{4}{135}, \frac{1}{540}$ and $n = 60$.

Theorem 3.7. *Let c and d be as in Theorem 2.7. Then we have*

(i) $G^3(q_{\frac{20}{3}}) = \dfrac{31-8\sqrt{15}}{4\left(-1+3\sqrt{5}+2\sqrt{15}\right)}$,

(ii) $G^3(q_{\frac{5}{12}}) = \dfrac{1+3\sqrt{2}-\sqrt{5}+\sqrt{30}}{11+6\sqrt{2}+15\sqrt{3}+\sqrt{5}\,(13+3\sqrt{3}+2\sqrt{6}\,)}$,

(iii) $G^3(q_{\frac{4}{15}}) = \dfrac{9+4\sqrt{5}}{4\left(21-\sqrt{5}+6\sqrt{15}\right)}$,

(iv) $G^3(q_{\frac{1}{60}}) = \dfrac{-1+3\sqrt{2}-\sqrt{5}+\sqrt{30}}{-11+6\sqrt{2}+15\sqrt{3}+\sqrt{5}\,(13-3\sqrt{3}+2\sqrt{6}\,)}$,

(v) $G^3(q_{\frac{20}{27}}) = \dfrac{(c+1)\sqrt{c^2+4c} - c(c+3)}{8c}$,

(vi) $G^3(q_{\frac{5}{108}}) = \dfrac{-\sqrt{c} + \sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}{(c+5)\sqrt{c} + (c+1)\sqrt{c+4} + 2\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}$,

(vii) $G^3(q_{\frac{4}{135}}) = \dfrac{(d+1)\sqrt{d^2+4d} - d(d+3)}{8d}$,

(viii) $G^3(q_{\frac{1}{540}}) = \dfrac{-\sqrt{d} + \sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}}{(d+5)\sqrt{d} + (d+1)\sqrt{d+4} + 2\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}}$,

(ix) $G^3(q_{60}) = \dfrac{5d+2-3\sqrt{d^2+4b}}{4(d+4+3\sqrt{d^2+4d}\,)}$,

(x) $G^3(q_{\frac{15}{4}})$

$$= \frac{-(2d-1)\sqrt{d+4} + 3\sqrt{2d-1}\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}}{9(d+1)\sqrt{d} + (11d-1)\sqrt{d+4} + 6\sqrt{2d-1}\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}},$$

(xi) $G^3(q_{\frac{12}{5}}) = \dfrac{5c+2-3\sqrt{c^2+4c}}{4(c+4+3\sqrt{c^2+4c}\,)}$,

(xii) $G^3(q_{\frac{3}{20}})$

$$= \frac{-(2c-1)\sqrt{c+4} + 3\sqrt{2c-1}\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}{9(c+1)\sqrt{c} + (11c-1)\sqrt{c+4} + 6\sqrt{2c-1}\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}.$$

Proof. The results are immediate from (1.2) and Theorem 2.12. □

Theorem 3.8. *Let c and d be as in Theorem 2.7. Then we have*

(i) $G^3(-q_{\frac{20}{3}}) = \dfrac{-6+5\sqrt{3}+4\sqrt{5}-3\sqrt{15}}{30-9\sqrt{2}-5\sqrt{3}-4\sqrt{5}+15\sqrt{6}+3\sqrt{15}}$,

(ii) $G^3(-q_{\frac{5}{12}}) = \dfrac{-16+9\sqrt{2}+8\sqrt{3}-7\sqrt{6}}{7-9\sqrt{2}+4\sqrt{3}-3\sqrt{5}+7\sqrt{6}+6\sqrt{15}}$,

(iii) $G^3(-q_{\frac{4}{15}}) = \dfrac{-6-5\sqrt{3}-4\sqrt{5}-3\sqrt{15}}{30+9\sqrt{2}+5\sqrt{3}+4\sqrt{5}+15\sqrt{6}+3\sqrt{15}}$,

(iv) $G^3(-q_{\frac{1}{60}}) = \dfrac{-16-9\sqrt{2}-8\sqrt{3}-7\sqrt{6}}{7+9\sqrt{2}-4\sqrt{3}+3\sqrt{5}+7\sqrt{6}+6\sqrt{15}}$,

(v) $G^3(-q_{\frac{20}{27}})$

$$= \frac{-(c+1)\sqrt{c} - c\sqrt{c+4} + \sqrt{2c-1}\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}{(c+1)\sqrt{c} + (3c-1)\sqrt{c+4} + 2\sqrt{2c-1}\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}},$$

(vi) $G^3(-q_{\frac{5}{108}})$

$$= \frac{-\sqrt{c} - \sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}{(c+5)\sqrt{c} + (c+1)\sqrt{c+4} - 2\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}},$$

(vii) $G^3(-q_{\frac{4}{135}})$

$$= \frac{-(d+1)\sqrt{d} - d\sqrt{d+4} + \sqrt{2d-1}\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}}{(d+1)\sqrt{d} + (3d-1)\sqrt{d+4} + 2\sqrt{2d-1}\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}},$$

(viii) $G^3(-q_{\frac{1}{540}})$

$$= \frac{-\sqrt{d} - \sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}}{(d+5)\sqrt{d} + (d+1)\sqrt{d+4} - 2\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}},$$

(ix) $G^3(-q_{60})$

$$= \frac{-(d+4)\sqrt{d} - (d+1)\sqrt{d+4} + 3\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}}{(d+13)\sqrt{d} + (d+1)\sqrt{d+4} + 6\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}},$$

(x) $G^3(-q_{\frac{15}{4}})$

$$= \frac{-(2d-1)\sqrt{d+4} - 3\sqrt{2d-1}\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}}{9(d+1)\sqrt{d} + (11d-1)\sqrt{d+4} - 6\sqrt{2d-1}\sqrt{d^2+4d+(d+1)\sqrt{d^2+4d}}},$$

(xi) $G^3(-q_{\frac{12}{5}})$

$$= \frac{-(c+4)\sqrt{c} - (c+1)\sqrt{c+4} + 3\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}{(c+13)\sqrt{c} + (c+1)\sqrt{c+4} + 6\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}},$$

(xii) $G^3(-q_{\frac{3}{20}})$

$$= \frac{-(2c-1)\sqrt{c+4} - 3\sqrt{2c-1}\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}{9(c+1)\sqrt{c} + (11c-1)\sqrt{c+4} - 6\sqrt{2c-1}\sqrt{c^2+4c+(c+1)\sqrt{c^2+4c}}}.$$

Proof. The results follow directly from (1.3) and Theorem 2.13. □

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