EVALUATIONS OF THE CUBIC CONTINUED FRACTION
BY SOME THETA FUNCTION IDENTITIES: REVISITED

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Abstract. In this paper, we exploit some known theta function identities involving
two parameters $l_{k,n}$ and $l'_{k,n}$ for the theta function $\psi$ to find about 54 new values
of the Ramanujan’s cubic continued fraction.

1. Introduction

Ramanujan’s cubic continued fraction $G(q)$, for $|q| < 1$, is defined by

$$G(q) = \frac{q^{1/3}}{1 + \frac{q + q^2}{1 + \frac{q^2 + q^4}{1 + \ldots}}}.$$

As stated in [8], there has been interest by number theorists in evaluating explicit
values of $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$ for some positive rational numbers $n$. For
brevity, we write $q_n$ for $e^{-\pi\sqrt{n}}$ throughout this paper. In 1984, Ramanathan [10]
found the value of $G(q_{10})$ such as $G(q_{10}) = \frac{\sqrt{9+3\sqrt{6}-\sqrt{7+3\sqrt{6}}}}{(1+\sqrt{5})\sqrt{\sqrt{3}+\sqrt{6}}}$ by using Kronecker’s
limit formula. Andrews and Berndt [3] also found the value of $G(q_{10})$ by employing
$G(q_n)$ for $n = 2, 10, 22, 58$ and $G(-q_n)$ for $n = 1, 5, 13, 37$ by using Ramanujan’s class
invariants. In addition, Chan [6] found explicit values of $G(q_n)$ for $n = \frac{2}{7}, 1, 2, 4$ and
$G(-q_n)$ for $n = 1, 5$ by applying some reciprocity theorems for the cubic continued
fraction.

In the 2000s, Adiga, Vasuki, and Mahadeva Naika [2] evaluated $G(q_4)$ and $G(-q_n)$
for $n = \frac{1}{3}, \frac{25}{3}, \frac{49}{3}, \frac{1}{75}, \frac{1}{147}$ by using some modular equations. Moreover, Adiga, Kim,
Mahadeva Naika, and Madhusudhan [1] found explicit values of $G(-q_n)$ for $n = \frac{1}{3},$
\( \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 1, 3, 5 \). Meanwhile, Yi [11] systematically found values of \( G(q_n) \) for \( n = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{4}{5}, 3, 6, 7, 8, 10, 12, 16, 28 \) and \( G(-q_n) \) for \( n = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, 2, 3, 4, 7 \) by using modular equations, in particular some eta function identities.

In the 2010s, Yi et al. [12] evaluated \( G(q_n) \) for \( n = \frac{1}{3}, 1, 4, 9 \) and \( G(-q_n) \) for \( n = 4, 9 \) by employing modular equations of degrees 3 or 9. In addition, Paek and Yi [7] derived some algorithms based on modular equations of degrees 3 or 9 to evaluate \( G(q_n) \) for \( n = \frac{4}{3}, \frac{16}{9}, \frac{64}{27}, 36, 81, 144, 324 \) and \( G(-q_n) \) for \( n = \frac{4}{3}, \frac{16}{9}, 36, 81 \). Paek and Yi [8] showed how to evaluate \( G(q_n) \) and \( G(-q_n) \) for \( n = 4^m, \frac{1}{4^m}, 2 \cdot 4^m \) and \( \frac{1}{2 \cdot 4^m} \) with some nonnegative integer \( m \). In particular, they evaluated \( G(q_n) \) for \( n = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, 1, 8, 16, 32, 64, 128, 256 \) and \( G(-q_n) \) for \( n = \frac{4}{3}, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{128}, 8, 16, 32, 64 \) by constructing some algorithms based on modular equations of degrees 3 or 9. Moreover, Paek and Yi [9] derived some algorithms based on modular equations of degrees 3 or 9 to evaluate \( G(q_n) \) and \( G(-q_n) \) for \( n = \frac{2 \cdot 4^m}{3}, \frac{1}{2 \cdot 4^m}, \frac{1}{3 \cdot 4^m} \) with \( m = 1, 2, 3, 4 \). In other words, they gave specific values of \( G(q_n) \) for \( n = \frac{8}{3}, \frac{32}{3}, \frac{128}{3}, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \frac{1}{393}, \frac{1}{786}, \frac{1}{1572}, \frac{1}{3936} \) and \( G(-q_n) \) for \( n = \frac{3}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{128}, \frac{1}{393}, \frac{1}{786}, \frac{1}{1572}, \frac{1}{3936} \).

| \[ \text{Ramanathan [10]} \] | 10 | \[ 1, 5, 13, 37 \] |
| \[ \text{Berndt et al. [5]} \] | 2, 10, 22, 58 | \[ 1, 5 \] |
| \[ \text{Chan [6]} \] | \[ \frac{5}{2}, \frac{1}{2}, 1, 2, 4 \] | \[ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, 3, 6, 7, 8, 10, 12, 16, 28 \] |
| \[ \text{Yi [11]} \] | \[ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, 3, 6, 7, 8, 10, 12, 16, 28 \] | \[ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, 2, 3, 4, 7 \] |
| \[ \text{Adiga et al. [2]} \] | 4 | \[ \frac{1}{3}, \frac{25}{3}, \frac{49}{3}, \frac{175}{3}, \frac{1117}{3} \] |
| \[ \text{Adiga et al. [1]} \] | \[ \frac{1}{3}, \frac{1}{5}, \frac{1}{9}, \frac{1}{27}, 1, 3, 5 \] | \[ \frac{1}{3}, \frac{1}{5}, \frac{1}{9}, \frac{1}{27}, 1, 3, 5 \] |
| \[ \text{Yi et al. [12]} \] | \[ \frac{1}{3}, 1, 4, 9 \] | \[ 4, 9 \] |
| \[ \text{Paek and Yi [7]} \] | \[ \frac{4}{3}, \frac{16}{3}, \frac{64}{3}, 36, 81, 144, 324 \] | \[ \frac{4}{3}, \frac{16}{3}, 36, 81 \] |
| \[ \text{Paek and Yi [8]} \] | \[ \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{16}, \frac{1}{36}, \frac{1}{72}, 1, 8, 16, 32, 64, 128, 256 \] | \[ \frac{1}{3}, \frac{1}{2}, \frac{1}{16}, \frac{1}{36}, \frac{1}{72}, 8, 16, 32, 64 \] |
| \[ \text{Paek and Yi [9]} \] | \[ \frac{8}{3}, \frac{32}{3}, \frac{128}{3}, \frac{1}{3}, \frac{1}{16}, \frac{1}{36}, \frac{1}{72}, \frac{1}{144}, \frac{1}{288}, \frac{1}{576}, \frac{1}{128}, \frac{1}{393}, \frac{1}{786}, \frac{1}{1572}, \frac{1}{3936} \] | \[ \frac{8}{3}, \frac{32}{3}, \frac{1}{3}, \frac{1}{16}, \frac{1}{36}, \frac{1}{72}, \frac{1}{144}, \frac{1}{288}, \frac{1}{576}, \frac{1}{128}, \frac{1}{393}, \frac{1}{786}, \frac{1}{1572}, \frac{1}{3936} \] |
| \[ \text{Yi and Paek [14]} \] | \[ \frac{50}{9}, \frac{250}{9}, \frac{500}{9}, \frac{1000}{9}, \frac{2000}{9}, \frac{1}{18}, \frac{1}{36}, \frac{1}{72}, \frac{1}{144}, \frac{1}{288}, \frac{1}{576}, \frac{1}{128}, \frac{1}{393}, \frac{1}{786}, \frac{1}{1572}, \frac{1}{3936} \] | \[ 20, 27, 45, 48, 80, 108, 180, 432, 720 \] |
EVALUATIONS OF $G(q)$ BY SOME THETA FUNCTION IDENTITIES: REVISITED

More recently, Yi and Paek [14] used some theta function identities involving parameters $h_{n,k}$ and $h'_{n,k}$ for the theta function $\varphi$ to establish evaluations of $G(q_n)$ for $n = \frac{1}{2}, \frac{4}{2}, \frac{5}{2}, \frac{9}{2}, \frac{16}{2}, \frac{22}{2}, \frac{26}{2}, \frac{36}{2}, \frac{50}{2}, \frac{80}{2}, \frac{16}{2}, \frac{27}{2}, \frac{41}{2}, \frac{47}{2}, 5, 20, 27, 45, 48, 80, 108, 180, 432, 720$ and $G(-q_n)$ for $n = \frac{4}{5}, \frac{5}{5}, \frac{36}{5}, \frac{5}{5}, \frac{20}{5}, \frac{1}{35}, \frac{4}{35}, 20, 27, 45, 180$. Table 1 shows a summary of some known values of $n$ for $G(q_n)$ and $G(-q_n)$ in chronological order.

Thus $G(q_n)$ were evaluated for $n = \frac{1}{2}, \frac{1}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \frac{32}{3}, \frac{64}{3}, \frac{128}{3}, \frac{1}{4}, \frac{1}{5}, \frac{4}{5}, \frac{9}{5}, \frac{16}{5}, \frac{36}{5}, \frac{72}{5}, \frac{1}{6}, \frac{1}{7}, \frac{2}{7}, \frac{4}{7}, \frac{9}{7}, \frac{20}{7}, \frac{80}{7}, \frac{1}{12}, \frac{1}{15}, \frac{1}{21}, \frac{1}{27}, \frac{4}{27}, \frac{16}{27}, \frac{1}{37}, \frac{4}{37}, \frac{16}{37}, \frac{1}{45}, \frac{1}{35}, \frac{4}{35}, \frac{16}{35}, \frac{5}{36}, \frac{1}{56}, \frac{1}{128}, \frac{1}{92}, \frac{1}{361}$, $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, 20, 22, 27, 28, 32, 36, 45, 48, 58, 64, 80, 81, 108, 128, 144, 180, 256, 324, 432, 720.

Whereas $G(-q_n)$ were evaluated for $n = \frac{1}{2}, \frac{1}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \frac{32}{3}, \frac{64}{3}, \frac{128}{3}, \frac{1}{4}, \frac{1}{5}, \frac{4}{5}, \frac{9}{5}, \frac{16}{5}, \frac{36}{5}, \frac{72}{5}, \frac{1}{6}, \frac{1}{7}, \frac{2}{7}, \frac{4}{7}, \frac{9}{7}, \frac{20}{7}, \frac{80}{7}, \frac{1}{12}, \frac{1}{15}, \frac{1}{21}, \frac{1}{27}, \frac{4}{27}, \frac{16}{27}, \frac{1}{37}, \frac{4}{37}, \frac{16}{37}, \frac{1}{45}, \frac{1}{35}, \frac{4}{35}, \frac{16}{35}, \frac{5}{36}, \frac{1}{56}, \frac{1}{128}, \frac{1}{92}, \frac{1}{361}$, $1, 2, 3, 4, 5, 7, 8, 9, 13, 16, 20, 27, 32, 36, 45, 64, 81, 180.

In this paper, we use some theta function identities involving parameters $l_{k,n}$ and $l'_{k,n}$ for the theta function $\psi$ to establish about 54 new values of $G(q_n)$ and $G(-q_n)$ such as $G(-q_6), G(-q_9)$, and $G(q_n)$ and $G(-q_n)$ for $n = \frac{3}{2}, \frac{2}{3}, \frac{5}{3}, \frac{20}{3}, \frac{15}{4}, \frac{3}{5}, \frac{12}{5}, \frac{8}{5}, \frac{5}{12}, \frac{1}{15}, \frac{3}{20}, \frac{2}{27}, \frac{5}{27}, \frac{20}{27}, \frac{1}{64}, \frac{1}{65}, \frac{5}{168}, \frac{1}{135}, \frac{1}{135}, \frac{1}{168}, \frac{1}{340}, \frac{1}{15}, 24$, and 60.

Ramanujan’s theta function $\psi(q)$, for $|q| < 1$, is defined by

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$ 

For any positive real numbers $k$ and $n$, define $l_{k,n}$ and $l'_{k,n}$ by

$$l_{k,n} = \frac{\psi(-q)}{k^{1/4} \psi(-q^k)}$$

and

$$l'_{k,n} = \frac{\psi(q)}{k^{1/4} \psi(q^k)},$$

where $q = e^{-\pi \sqrt{n/k}}$ (See [13] for details). We now note that the following property of $l_{k,n}$ in [13] will be useful for evaluating the cubic continued fraction later on.

$$l_{k,n} = l_{k,n}^{-1}.$$  \hspace{1cm} (1.1)

We also note general formulas for $G^3(q_2)$ and $G^3(-q_2)$ in terms of $l'_{3,n}$ and $l_{3,n}$, respectively, in [13, Theorem 6.2(ii) and (v)] such as

$$G^3(q_2) = \frac{1}{3 l'_{3,n}^3 - 1}$$  \hspace{1cm} (1.2)

and

$$G^3(-q_2) = \frac{-1}{3 l_{3,n}^3 + 1}.$$  \hspace{1cm} (1.3)
By taking cube root of (1.2) and (1.3), we have the values of $G(q^n_3)$ and $G(q^n_3)$. Hence, in view of (1.2) and (1.3), in order to find some explicit values of $G^3(q^n_3)$ and $G^3(q^n_3)$, it is sufficient to evaluate $l'_{3,n}$ and $l_{3,n}$, respectively. For brevity, we write $l_n, l'_{n}$ for $l_{3,n}, l'_{3,n}$, respectively.

2. Evaluations of $l_n$ and $l'_{n}$

We begin this section by recalling the values of $l_2$ and $l_5$ in [13], which play key roles in evaluating some new values of $l_n$.

**Lemma 2.1 ([13, Theorem 4.9(iv) and (v)].** We have

(i) $l_2 = (\sqrt{2} + \sqrt{3})^{1/4},$

(ii) $l_5 = \left(\frac{1 + \sqrt{5}}{2}\right)^{3/2}.$

Note that $l_5$ in Lemma 2.1(ii) was incorrectly recorded as $\left(\frac{1 + \sqrt{5}}{2}\right)^{2/3}$ in [13].

We now recall a theta function identity in [4, Entry 1(ii), p. 345] such as

$$ (2.1) \left(1 + \frac{\psi(-q^{1/3})}{q^{1/3}\psi(-q^3)}\right)^3 = 1 + \frac{\psi(-q)}{q^{4/3}(-q^3)}. $$

Rewriting (2.1) in terms of $l_n$ and $l_{9n}$, we have the next result.

**Lemma 2.2 ([13, Theorem 4.5(i)].** For any positive real number $n$, we have

$$ (2.2) (1 + \sqrt{3} l_n l_{9n})^3 = 1 + 3 l_{9n}^4. $$

We first evaluate $l_n$ for $n = \frac{1}{2}, \frac{9}{2}, \frac{2}{5}, \frac{1}{18}$, and 18.

**Theorem 2.3.** We have

(i) $l_\frac{1}{2} = (\sqrt{3} - \sqrt{2})^{1/4},$

(ii) $l_\frac{9}{2} = \frac{\sqrt{3} + 3\sqrt{2} + 3\sqrt{3} - 1}{\sqrt{3}(\sqrt{2} + \sqrt{3})^{1/4}},$

(iii) $l_\frac{2}{5} = \frac{\sqrt{3} - 3\sqrt{2} + 3\sqrt{3} - 1}{\sqrt{3}(\sqrt{3} - \sqrt{2})^{1/4}},$

(iv) $l_\frac{1}{18} = \frac{\sqrt{3} + 3\sqrt{2} + 3\sqrt{3} - 1}{\sqrt{3}(\sqrt{3} - \sqrt{2})^{1/4}},$

(v) $l_{18} = \frac{\sqrt{3} - 3\sqrt{2} + 3\sqrt{3} - 1}{\sqrt{3}(\sqrt{3} - \sqrt{2})^{1/4}}.$
Proof. Part (i) is clear by (1.1) and Lemma 2.1(i). For (ii), let \( n = \frac{9}{2} \) in (2.2) and put \( l_2 = (\sqrt{2} + \sqrt{3})^{1/4} \) in Lemma 2.1(i), then we find that
\[
\left(1 + \sqrt{3} (\sqrt{2} + \sqrt{3})^{1/4} l_2^{1/3}\right)^3 = 1 + 3(\sqrt{2} + \sqrt{3}).
\]
Taking the cube root of both sides of the last equation and simplifying to complete the proof.

For (iii), let \( n = \frac{1}{18} \) in (2.2), put the value of \( l_2 \) obtained from (i), and repeat the same argument as in the proof of (ii) to complete the proof. The proofs of (iv) and (v) follow directly from (1.1).

We next evaluate \( l_n \) for \( n = \frac{1}{5}, \frac{9}{5}, \frac{5}{9}, \frac{1}{45}, \) and 45.

**Theorem 2.4.** We have

(i) \( l \frac{1}{5} = \sqrt{\frac{5}{2}} - 2 \),

(ii) \( l \frac{9}{5} = \frac{3\sqrt{28 + 12\sqrt{5}} - 1}{\sqrt{6} + 3\sqrt{5}} \),

(iii) \( l \frac{5}{9} = \frac{\sqrt{28 - 12\sqrt{5}} - 1}{\sqrt{6} + 3\sqrt{5}} \),

(iv) \( l \frac{5}{9} = \frac{\sqrt{28 + 12\sqrt{5}} - 1}{\sqrt{6} + 3\sqrt{5}} \),

(v) \( l \frac{45}{9} = \frac{\sqrt{28 - 12\sqrt{5}} - 1}{3\sqrt{28 + 12\sqrt{5}} - 1} \).

**Proof.** Repeat the same argument as in the proof of Theorem 2.3.

We now turn to evaluations of \( l'_n \). But we need the following theta function identity with respect to \( l_n \) and \( l'_n \).

**Lemma 2.5** ([12, Corollary 3.12]). For every positive real number \( n \), we have
\[
(l_n^4 - l'_n^4 + 3) \left( \frac{1}{l_n^4} - \frac{1}{l_n^4} + 3 \right) = 1.
\]

Note that (2.3) follows from a modular equation in [12, Theorem 3.11] such as
\[
(P^4 - Q^4 - 9) \left( \frac{1}{P^4} - \frac{1}{Q^4} - 1 \right) = 1 \quad \text{with} \quad P = \frac{\psi(q)}{q^{1/4} \psi(q^{1/2})} \quad \text{and} \quad Q = \frac{\psi(-q)}{q^{1/4} \psi(-q^{1/2})}.
\]

In view of (2.3), we evaluate \( l'_n^4 \) for \( n = \frac{1}{2}, \frac{9}{2}, \frac{2}{5}, \frac{1}{18}, 2, \) and 18.

**Theorem 2.6.** We have

(i) \( l'_2^4 = 3 + 2\sqrt{2} \),
\[ l'_{\frac{4}{2}} = \sqrt{2} + \sqrt{3}, \]
\[ l'_{\frac{4}{5}} = \frac{a + 4}{3} + \frac{(a + 1)\sqrt{a + 4}}{3\sqrt{a}}, \]
\[ l'_{\frac{4}{10}} = \frac{b + 4}{3} + \frac{(b + 1)\sqrt{b + 4}}{3\sqrt{b}}, \]
\[ l'_{\frac{4}{25}} = \frac{3\sqrt{a}(a + 1 + \sqrt{a^2 + 4a})}{(2a - 1)\sqrt{a + 4}}, \]
\[ l'_{\frac{4}{45}} = \frac{3\sqrt{b}(b + 1 + \sqrt{b^2 + 4b})}{(2b - 1)\sqrt{b + 4}}, \]
\[ a = \frac{1}{2} + \frac{\left(\sqrt[3]{2} + 3\sqrt{2} + 3\sqrt{3} - 1\right)}{6(\sqrt{2} + \sqrt{3})} \quad \text{and} \quad b = \frac{1}{2} + \frac{\left(\sqrt[3]{3} - 3\sqrt{2} + 3\sqrt{3} - 1\right)}{6(\sqrt{3} - \sqrt{2})}. \]

**Proof.** For (i), let \( n = 2 \) in (2.3) and put the value of \( l_2 \) in Lemma 2.1(i), then it follows that
\[ (3 - \sqrt{2} + \sqrt{3})x^2 - 2(5 + 3\sqrt{3})x + 3 + \sqrt{2} + \sqrt{3} = 0, \]
where \( x = l'_{\frac{4}{2}} \). Solving the last equation for \( x \) and using \( x > 1 \), we have the required result.

The proofs of (ii)–(vi) are similar to that of (i). \( \square \)

We now evaluate \( l'_{\frac{4}{n}} \) for \( n = \frac{1}{5}, \frac{9}{5}, \frac{5}{15}, 5, \text{ and } 45. \)

**Theorem 2.7.** We have
\[ (i) \quad l'_{\frac{4}{5}} = \frac{8 + 2\sqrt{15}}{3 - \sqrt{5}}, \]
\[ (ii) \quad l'_{\frac{4}{10}} = \frac{8 + 2\sqrt{15}}{3 + \sqrt{5}}, \]
\[ (iii) \quad l'_{\frac{4}{25}} = \frac{c + 4}{3} + \frac{(c + 1)\sqrt{c + 4}}{3\sqrt{c}}, \]
\[ (iv) \quad l'_{\frac{4}{45}} = \frac{d + 4}{3} + \frac{(d + 1)\sqrt{d + 4}}{3\sqrt{d}}, \]
\[ (v) \quad l'_{\frac{4}{25}} = \frac{3\sqrt{c}(c + 1 + \sqrt{c^2 + 4c})}{(2c - 1)\sqrt{c + 4}}, \]
\[ (vi) \quad l'_{\frac{4}{45}} = \frac{3\sqrt{d}(d + 1 + \sqrt{d^2 + 4d})}{(2d - 1)\sqrt{d + 4}}, \]
\[ c = \frac{1}{2} + \frac{\left(\sqrt[3]{28 + 12\sqrt{5}} - 1\right)}{6(9 + 4\sqrt{5})} \quad \text{and} \quad d = \frac{1}{2} + \frac{\left(\sqrt[3]{28 - 12\sqrt{5}} - 1\right)}{6(9 - 4\sqrt{5})}. \]
Proof. The proof follows precisely along the same lines as that for Theorem 2.6. □

We evaluate some more values of \( l'_n \) and \( l'_4n \) by employing the following theta function identities involving \( l'_n \), \( l'_4n \), and \( l_n \).

**Lemma 2.8** ([9, Corollary 3.4]). For any positive real number \( n \), we have

\[
(2.4) \quad l'_n (\sqrt{3} l'^2_4 + 1) = l'^2_4n (l'^2_4n - \sqrt{3})
\]

Note that (2.4) follows from a modular equation \( P^4(Q^2 + 1) = Q^2(Q^2 + 3) \) with

\[
P = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \quad \text{and} \quad Q = \frac{\psi(q^3)}{q^{1/2}\psi(q^3)}.
\]

We also need the following theta function identity involving \( l'_n \) and \( l'_4n \).

**Lemma 2.9** ([9, Corollary 3.2]). For any positive real number \( n \), we have

\[
(2.5) \quad l'_4 (\sqrt{3} l'^2_4 + 1) = l'^2_4n (l'^2_4n + \sqrt{3})
\]

Note that (2.5) follows from a modular equation \( P^4(Q^2 + 1) = Q^2(Q^2 + 3) \) with

\[
P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \text{and} \quad Q = \frac{\psi(q^3)}{q^{1/2}\psi(q^3)}.
\]

In view of (2.4), and (2.5), we evaluate \( l'_n \) and \( l'_4n \) for \( n = \frac{9}{8}, \frac{8}{9}, \frac{1}{72}, \text{and} 72 \).

**Theorem 2.10.** Let \( a \) and \( b \) be as in Theorem 2.6. Then we have

(i) \( l'^4_{\frac{9}{8}} = \frac{1}{3} (a + 1 + \sqrt{a^2 + 4a})^2 \),

(ii) \( l'^4_{\frac{1}{72}} = 1 - \frac{(b + 7)\sqrt{b} + (b + 1)\sqrt{b} + 4}{3\sqrt{b} - 3\sqrt{b^2 + 4b} + (b + 1)\sqrt{b^2 + 4b}} \),

(iii) \( l'^4_{\frac{8}{9}} = 1 - \frac{3(a + 1)\sqrt{a} + (5a - 1)\sqrt{a} + 4}{(2a - 1)\sqrt{a} + 4 - 3\sqrt{2a - 1}\sqrt{a^2 + 4a} + (a + 1)\sqrt{a^2 + 4a}} \),

(iv) \( l'^4_{\frac{1}{72}} = \frac{3(b + 1 + \sqrt{b^2 + 4b})^2}{(2b - 1)^2} \).

Proof. For (i), let \( n = \frac{9}{8} \) in (2.4) and put the value of \( l'_2 \) in Theorem 2.3(ii), then we deduce that

\[
3x^4 - 2\sqrt{3}(a + 1)x^2 - 2a + 1 = 0,
\]

where \( x = l'_8 \). Solving the last equation for \( x \) and using \( x > 1 \), we complete the proof.

For (ii), let \( n = \frac{1}{72} \) in (2.5), put the value of \( l'_4 \) in Theorem 2.6(iv), and simplify the equation to complete the proof.

The proofs of (iii) and (iv) are similar to those of (i) or (ii). □
For (i), let we find that

\[ t_8^4 = \frac{(2a - 1)\sqrt{a + 4} + 3\sqrt{2a - 1}\sqrt{a^2 + 4a + (a + 1)\sqrt{a^2 + 4a}}}{3(a + 1)\sqrt{a} + 3a\sqrt{a + 4} - 3\sqrt{2a - 1}\sqrt{a^2 + 4a + (a + 1)\sqrt{a^2 + 4a}}} , \]

\[ t_7^4 = -1 + \frac{(b + 7)\sqrt{b} + (b + 1)\sqrt{b + 4}}{3\sqrt{b} + 3\sqrt{b^2 + 4b + (b + 1)b^2 + 4b}} , \]

\[ t_8^5 = -1 + \frac{(b + 7)\sqrt{b} + (b + 1)\sqrt{b + 4}}{3\sqrt{b} + 3\sqrt{b^2 + 4b + (b + 1)b^2 + 4b}} , \]

\[ t_2^2 = \frac{3\sqrt{b} + 3\sqrt{b^2 + 4b + (b + 1)b^2 + 4b}}{(b + 4)b + (b + 1)b + 3\sqrt{b^2 + 4b + (b + 1)b^2 + 4b}} . \]

Proof. For (i), let \( n = \frac{8}{9} \) in (2.3) and put the value of \( t_8^4 \) in Theorem 2.10(i), then we find that

\[ (3l - 1)x^2 - (3l^2 - 10l + 3)x - l^2 + 3l = 0 , \]

where \( x = t_8^4 \) and \( l = t_8^4 \). Employing Mathematica to solve the last equation for \( x \), we complete the proof. The proof of (ii) is similar to that of (i).

The proofs of (iii) and (iv) follow from (1.1). \( \square \)

We evaluate \( t_n^4 \) for \( n = \frac{5}{4}, \frac{45}{4}, \frac{3}{5}, \frac{20}{9}, \frac{1}{20}, \frac{9}{20}, \frac{5}{36}, \frac{4}{15}, \frac{1}{180}, \frac{1}{20}, \text{and } 180. \)

Theorem 2.12. Let \( c \) and \( d \) be as in Theorem 2.7. Then we have

\[ t_8^5 = \frac{(2 + \sqrt{5})^2(4 + \sqrt{15})^2}{3 + \sqrt{15} + (4 + \sqrt{15})\sqrt{3 + \sqrt{5}}} , \]

\[ t_8^4 = \frac{3 + \sqrt{15} + (4 + \sqrt{15})\sqrt{3 - \sqrt{5}}}{3 + \sqrt{15} + (4 + \sqrt{15})\sqrt{3 - \sqrt{5}}} , \]

\[ t_8^4 = \frac{3 + \sqrt{15} + (4 + \sqrt{15})\sqrt{3 - \sqrt{5}}}{3 + \sqrt{15} + (4 + \sqrt{15})\sqrt{3 - \sqrt{5}}} , \]

\[ t_8^5 = \frac{1}{3}(c + 1 + \sqrt{c^2 + 4c})^2 , \]

\[ t_8^5 = 1 - \frac{(c + 7)\sqrt{c} + (c + 1)\sqrt{c + 4}}{3\sqrt{c} + 3\sqrt{c^2 + 4c + (c + 1)\sqrt{c^2 + 4c}} , \]

\[ t_8^5 = \frac{1}{3}(d + 1 + \sqrt{d^2 + 4d})^2 , \]

\[ t_8^5 = 1 - \frac{(d + 7)\sqrt{d} + (d + 1)\sqrt{d + 4}}{3\sqrt{d} + 3\sqrt{d^2 + 4d + (d + 1)\sqrt{d^2 + 4d}} , \]

\[ t_8^5 = \frac{1}{3}(d + 7)\sqrt{d} + (d + 1)\sqrt{d} , \]

\[ t_8^5 = 1 - \frac{(d + 7)\sqrt{d} + (d + 1)\sqrt{d} + 4}{3\sqrt{d} + 3\sqrt{d^2 + 4d + (d + 1)\sqrt{d^2 + 4d}} , \]
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(ix) $l'_{180} = \frac{3(d+1+\sqrt{d^2+4d})^2}{(2d-1)^2}$,

(x) $l'_{45} = 1 - \frac{3(d+1)\sqrt{d}+(5d-1)\sqrt{d+4}}{(2d-1)\sqrt{d+4} - 3\sqrt{2d-1}\sqrt{d^2+4d} + (d+1)\sqrt{d^2+4d}}$,

(xi) $l'_{4} = \frac{3(c+1+\sqrt{c^2+4c})^2}{(2c-1)^2}$,

(xii) $l'_{20} = 1 - \frac{3(c+1)\sqrt{c}+(5c-1)\sqrt{c+4}}{(2c-1)\sqrt{c+4} - 3\sqrt{2c-1}\sqrt{c^2+4c} + (c+1)\sqrt{c^2+4c}}$.

Proof. For (i), let $n = 5$ in (2.5) and put $l'_5 = 9 + 4\sqrt{5}$ from Theorem 2.1(ii), then we find that

$$(3 - \sqrt{5}) l'_5 - (5\sqrt{3} + 6\sqrt{5} + \sqrt{15}) l'_2 + 8 + 2\sqrt{5} = 0.$$ 

Solve the last equation for $l'_2$ and use $l'_2 > 0$ to complete the proof. For the proofs of (iii), (v), (vii), (ix), and (xi), repeat the same argument as in the proof of (i).

For (ii), let $n = \frac{5}{4}$ in (2.4) and put the value $l'_5$ from Theorem 2.8(i), then we find that

$$\left(3 + \sqrt{15} - \sqrt{3 - \sqrt{5}}\right) l'_5 = 3 + \sqrt{15} + (4 + \sqrt{15})\sqrt{3 + \sqrt{5}}.$$ 

Hence we have the required result. The proofs of (iv), (vi), (viii), (x), (xii) are similar to that of (ii). \hfill \square

We end this section by evaluating $l'_n$ for $n = \frac{5}{4}, \frac{45}{4}, \frac{4}{5}, \frac{36}{5}, \frac{20}{9}, \frac{1}{20}, \frac{9}{20}, \frac{5}{36}, \frac{4}{45}, \frac{1}{180}, 20,$ and 180.

**Theorem 2.13.** Let $c$ and $d$ be as in Theorem 2.7. Then we have

(i) $l'_{20} = \frac{8 - 3\sqrt{2} + 5\sqrt{6}}{6 - 5\sqrt{3} - 4\sqrt{5} + 3\sqrt{15}}$,

(ii) $l'_{4} = \frac{6 + 5\sqrt{3} + 4\sqrt{5} + 3\sqrt{15}}{8 + 3\sqrt{2} + 5\sqrt{6}}$,

(iii) $l'_{5} = \frac{6 + 5\sqrt{3} + 4\sqrt{5} + 3\sqrt{15}}{8 + 3\sqrt{2} + 5\sqrt{6}}$,

(iv) $l'_{20} = \frac{(2c-1)\sqrt{c+4} + 3\sqrt{2c-1}\sqrt{c^2+4c} + (c+1)\sqrt{c^2+4c}}{3(c+1)\sqrt{c} + 3c\sqrt{c+4} - 3\sqrt{2c-1}\sqrt{c^2+4c} + (c+1)\sqrt{c^2+4c}}$, 

(v) $l'_{45} = \frac{(2c-1)\sqrt{c+4} + 3\sqrt{2c-1}\sqrt{c^2+4c} + (c+1)\sqrt{c^2+4c}}{3(c+1)\sqrt{c} + 3c\sqrt{c+4} - 3\sqrt{2c-1}\sqrt{c^2+4c} + (c+1)\sqrt{c^2+4c}}$. 


For (i), let $n = 20$ in (2.3) and put $l_2^2 = (2 + \sqrt{5})^2 (4 + \sqrt{15})^2$ from Theorem 2.12(i), then we deduce that

$$2l_2^2 - 8(69 + 40\sqrt{3} + 31\sqrt{5} + 18\sqrt{15})l_2^4 - 8(188 - 105\sqrt{3} - 84\sqrt{5} - 47\sqrt{15}) = 0.$$  

Using Mathematica to solve the last equation for $l_2^2$, we complete the proof.

For (ii), let $n = \frac{3}{4}$ in (2.5) and put the value of $l_3^8$ in Theorem 2.8(i) to complete the proof. For (iii)–(xii), repeat the same argument as in the proofs of (i) or (ii).

3. Evaluations of $G(q)$

In this section, we evaluate about 46 values $G(-q_n)$ and $G(q_n)$ including 36 new ones. Just for editorial convenience, we evaluate $G^3(-q_n)$ and $G^3(q_n)$. By taking cube roots of them, the required values of $G(-q_n)$ and $G(q_n)$ can easily be obtained.

We first evaluate $G^3(-q_n)$ and $G^3(q_n)$ for $n = \frac{3}{2}, \frac{2}{3}, \frac{1}{5}, \frac{2}{7}, \frac{1}{11},$ and 6.

**Theorem 3.1.** We have

(i) $G^3(-q_4) = \frac{-1}{1 + 3(\sqrt{2} + \sqrt{3})}$,
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(ii) $G^3(-q_{15}) = \frac{-1}{1 - 3(\sqrt{2} - \sqrt{3})},$

(iii) $G^3(-q_{27}) = \frac{-3(\sqrt{2} + \sqrt{3})}{3(\sqrt{2} + \sqrt{3}) + (1 - \sqrt{1 + 3\sqrt{2} + 3\sqrt{3}})^2},$

(iv) $G^3(-q_{33}) = \frac{-3(\sqrt{2} - \sqrt{3})}{3(\sqrt{2} - \sqrt{3}) - (1 - \sqrt{1 - 3\sqrt{2} + 3\sqrt{3}})^4},$

(v) $G^3(-q_{1}) = \frac{-1}{27(\sqrt{2} + \sqrt{3}) + (1 - \sqrt{1 - 3\sqrt{2} + 3\sqrt{3}})^4},$

(vi) $G^3(-q_{6}) = \frac{1 - \sqrt{1 - 3\sqrt{2} + 3\sqrt{3}}}{27(\sqrt{2} - \sqrt{3}) - (1 - \sqrt{1 - 3\sqrt{2} + 3\sqrt{3}})^4}.$

Proof. The results follow from (1.3), Lemma 2.1(i), and Theorem 2.3.

Theorem 3.2. Let $a$ and $b$ be as in Theorem 2.6. Then we have

(i) $G^3(q_{3}) = \frac{-4 + 3\sqrt{2}}{4},$

(ii) $G^3(q_{5}) = \frac{-1 + 3\sqrt{2} + 3\sqrt{3}}{4},$

(iii) $G^3(q_{11}) = \frac{(a + 1)\sqrt{a^2 + 4a - a(a + 3)} + 4}{4},$

(iv) $G^3(q_{17}) = \frac{(b + 1)\sqrt{b^2 + 4b - b(b + 3)} + 4}{4},$

(v) $G^3(q_{23}) = \frac{(2a - 1)\sqrt{a + 4} + 4}{9(a + 1)\sqrt{a + (7a + 1)\sqrt{a + 4}}},$

(vi) $G^3(q_{29}) = \frac{(2b - 1)\sqrt{b + 4} + 4}{9(b + 1)\sqrt{b + (7b + 1)\sqrt{b + 4}}},$

Proof. The proofs are clear by (1.2) and Theorem 2.6.

Note that an explicit value of $G(q_{6})$ in [11, Theorem 6.3.3(ii)] was given by

$G(q_{6}) = \frac{\sqrt[3]{3 - 2\sqrt{2}}}{2 + 2\sqrt{1 + \sqrt{2} + \sqrt{2}\sqrt{3} + 2\sqrt{2}}}$.

Note also that the value of $G^3(q_{15})$ was given in [8, Theorem 5.5(i)].

We next evaluate $G(-q_{n})$ and $G(q_{n})$ for $n = \frac{5}{3}, \frac{3}{5}, \frac{1}{15}, \frac{5}{27}, \frac{1}{135}$, and 15.

Theorem 3.3. We have

(i) $G^3(-q_{\frac{5}{3}}) = \frac{-1}{4(7 + 3\sqrt{5})}$. 
(ii) \( \text{G}^3(-q_{\frac{3}{16}}) = \frac{-7 - 3\sqrt{5}}{16} \),

(iii) \( \text{G}^3(-q_{\frac{3}{27}}) = \frac{-3}{3 + (-2 + \sqrt{5})^2 \left(-1 + \frac{3}{2} \sqrt{28 + 12\sqrt{5}}\right)^4} \),

(iv) \( \text{G}^3(-q_{\frac{1}{13})} = \frac{-3}{3 + (2 + \sqrt{5})^2 \left(-1 + \frac{3}{2} \sqrt{28 - 12\sqrt{5}}\right)^4} \),

(v) \( \text{G}^3(-q_{\frac{1}{3}}) = \frac{1 - \left(-1 + \frac{3}{2} \sqrt{28 + 12\sqrt{5}}\right)^4}{3(6 + 3\sqrt{5})^2 + \left(-1 + \frac{3}{2} \sqrt{28 + 12\sqrt{5}}\right)^4} \),

(vi) \( \text{G}^3(-q_{15}) = \frac{1 - \left(-1 + \frac{3}{2} \sqrt{28 - 12\sqrt{5}}\right)^4}{3(6 + 3\sqrt{5})^2 + \left(-1 + \frac{3}{2} \sqrt{28 - 12\sqrt{5}}\right)^4} \).

\textbf{Proof.} The results follow from (1.3), Lemma 2.1(ii), and Theorem 2.4. \hfill \Box

\textbf{Theorem 3.4.} Let \( c \) and \( d \) be as in Theorem 2.7. Then we have

(i) \( \text{G}^3(q_{\frac{1}{3}}) = \frac{3 - \sqrt{5}}{21 + \sqrt{5} + 6\sqrt{15}} \),

(ii) \( \text{G}^3(q_{\frac{1}{15}}) = \frac{21 - \sqrt{5} + 6\sqrt{15}}{3 + \sqrt{5}} \),

(iii) \( \text{G}^3(q_{\frac{1}{27}}) = \frac{(c + 1)\sqrt{c^2 + 4c - c(c + 3)}}{4} \),

(iv) \( \text{G}^3(q_{\frac{1}{135}}) = \frac{(d + 1)\sqrt{d^2 + 4d - d(d + 3)}}{4} \),

(v) \( \text{G}^3(q_{\frac{1}{3}}) = \frac{(2c - 1)\sqrt{c + 4}}{9(c + 1)\sqrt{c} + (7c + 1)\sqrt{c + 4}} \),

(vi) \( \text{G}^3(q_{15}) = \frac{(2d - 1)\sqrt{d + 4}}{9(d + 1)\sqrt{d} + (7d + 1)\sqrt{d + 4}} \).

\textbf{Proof.} The results follow from (1.2) and Theorem 2.7. \hfill \Box

We now evaluate \( \text{G}(q_n) \) and \( \text{G}(-q_n) \) for \( n = \frac{3}{8}, \frac{8}{27}, \frac{1}{216}, \) and 24.

\textbf{Theorem 3.5.} Let \( a \) and \( b \) be as in Theorem 2.6. Then we have

(i) \( \text{G}^3(q_{\frac{8}{27}}) = \frac{(a + 1)\sqrt{a^2 + 4a - a(a + 3)}}{8a} \),

(ii) \( \text{G}^3(q_{\frac{1}{216}}) = \frac{-\sqrt{b} + \sqrt{b^2 + 4b + (b + 1)\sqrt{b^2 + 4b}}}{(b + 5)\sqrt{b} + (b + 1)\sqrt{b + 4} + 2\sqrt{b^2 + 4b + (b + 1)\sqrt{b^2 + 4b}}} \),

(iii) \( \text{G}^3(q_{\frac{1}{3}}) \)
The results follow from (1.3) and Theorem 2.11.

The results follow from (1.2) and Theorem 2.10.

**Proof.** The results follow from (1.2) and Theorem 2.10. \(\blacksquare\)

**Theorem 3.6.** Let \(c\) and \(d\) be as in Theorem 2.7. Then we have

\[
\begin{align*}
(i) \quad G^3(-q_{27}^2) &= - (a + 1) \sqrt{a} - a \sqrt{a + 4} + \sqrt{2a - 1} \sqrt{a^2 + 4a + (a + 1) \sqrt{a^2 + 4a}} \\
&= (a + 1) \sqrt{a} + (3a - 1) \sqrt{a + 4} + 2 \sqrt{2a - 1} \sqrt{a^2 + 4a + (a + 1) \sqrt{a^2 + 4a}}
\end{align*}
\]

\[
\begin{align*}
(ii) \quad G^3(-q_{16}) &= \frac{\sqrt{b} + (b + 1) \sqrt{b^2 + 4b}}{(b + 5) \sqrt{b} + (b + 1) \sqrt{b^2 + 4b - 2 \sqrt{b^2 + 4b + (b + 1) \sqrt{b^2 + 4b}}}}
\end{align*}
\]

\[
\begin{align*}
(iii) \quad G^3(-q_{18}) &= \frac{\sqrt{b} - (b + 1) \sqrt{b^2 + 4b + (b + 1) \sqrt{b^2 + 4b}}}{(b + 13) \sqrt{b} + (b + 1) \sqrt{b^2 + 4b + 6 \sqrt{b^2 + 4b + (b + 1) \sqrt{b^2 + 4b}}}}
\end{align*}
\]

\[
\begin{align*}
(iv) \quad G^3(q_{24}) &= \frac{- (2a + 1) \sqrt{a} - 3 \sqrt{2a - 1} \sqrt{a^2 + 4a + (a + 1) \sqrt{a^2 + 4a}}}{9(a + 1) \sqrt{a} + (11a - 1) \sqrt{a + 4} - 6 \sqrt{2a - 1} \sqrt{a^2 + 4a + (a + 1) \sqrt{a^2 + 4a}}}
\end{align*}
\]

**Proof.** The results follow from (1.3) and Theorem 2.11. \(\blacksquare\)

We end this section by evaluating \(G^3(q_n)\) and \(G^3(-q_n)\) for \(n = 20/3, 15/7, 12/5, 5/12, 4/15, 3/20, 20/7, 1/190, 4/135, 1/140\) and \(n = 60\).

**Theorem 3.7.** Let \(c\) and \(d\) be as in Theorem 2.7. Then we have

\[
\begin{align*}
(i) \quad G^3(q_{20/3}) &= \frac{31 - 8 \sqrt{15}}{4 (-1 + 3 \sqrt{5} + 2 \sqrt{15})}, \\
(ii) \quad G^3(q_{15/7}) &= \frac{1 + 3 \sqrt{2} - \sqrt{5} + \sqrt{30}}{11 + 6 \sqrt{2} + 15 \sqrt{3} + \sqrt{5} (13 + 3 \sqrt{3} + 2 \sqrt{6})}, \\
(iii) \quad G^3(q_{12/5}) &= \frac{9 + 4 \sqrt{5}}{4 (21 - \sqrt{5} + 6 \sqrt{15})}, \\
(iv) \quad G^3(q_{5/12}) &= \frac{-1 + 3 \sqrt{2} - \sqrt{5} + \sqrt{30}}{-11 + 6 \sqrt{2} + 15 \sqrt{3} + \sqrt{5} (13 - 3 \sqrt{3} + 2 \sqrt{6})},
\end{align*}
\]
The results are immediate from (1.2) and Theorem 2.12.

Proof. The results are immediate from (1.2) and Theorem 2.12.

**Theorem 3.8.** Let $c$ and $d$ be as in Theorem 2.7. Then we have

(i) \[ G^3(q_{20}^{17}) = \frac{(c+1)\sqrt{c^2 + 4c} - c(c+3)}{8c}, \]

(ii) \[ G^3(q_{18}^{11}) = \frac{(c+1)\sqrt{c^2 + 4c} + (c+1)\sqrt{c^2 + 4c}}{8d}, \]

(iii) \[ G^3(q_{4}^{135}) = \frac{(d+1)\sqrt{d^2 + 4d} - d(d+3)}{8d}, \]

(iv) \[ G^3(q_{14}^{135}) = \frac{-\sqrt{d + 4d + (d+1)\sqrt{d^2 + 4d}}}{9(d+1)\sqrt{d + 4d + 2\sqrt{d^2 + 4d} + (d+1)\sqrt{d^2 + 4d}}}, \]

(v) \[ G^3(q_{15}^{40}) = \frac{-\sqrt{c + 4d - 4d + (d+1)\sqrt{c^2 + 4c}}}{9(c+1)\sqrt{c + 4d + 6\sqrt{c^2 + 4c} + (c+1)\sqrt{c^2 + 4c}}}, \]

Proof. The results are immediate from (1.2) and Theorem 2.12.

Theorem 3.8. Let $c$ and $d$ be as in Theorem 2.7. Then we have

(i) \[ G^3(-q_{20}^{17}) = \frac{-6 + 5\sqrt{3} + 4\sqrt{3} - 3\sqrt{15}}{30 - 9\sqrt{2} - 5\sqrt{3} - 4\sqrt{5} + 15\sqrt{6} + 3\sqrt{15}}, \]

(ii) \[ G^3(-q_{12}^{17}) = \frac{-6 + 9\sqrt{2} + 8\sqrt{3} - 7\sqrt{6}}{7 - 9\sqrt{2} + 4\sqrt{3} - 3\sqrt{5} + 7\sqrt{6} + 6\sqrt{15}}, \]

(iii) \[ G^3(-q_{4}^{13}) = \frac{-6 - 5\sqrt{3} - 4\sqrt{5} - 3\sqrt{15}}{30 + 9\sqrt{2} + 5\sqrt{3} + 4\sqrt{5} + 15\sqrt{6} + 3\sqrt{15}}, \]

(iv) \[ G^3(-q_{60}^{17}) = \frac{-16 - 9\sqrt{2} - 8\sqrt{3} - 7\sqrt{6}}{7 + 9\sqrt{2} - 4\sqrt{3} + 3\sqrt{5} + 7\sqrt{6} + 6\sqrt{15}}, \]

(v) \[ G^3(-q_{20}^{27}) = \frac{(c+1)\sqrt{c} - c\sqrt{c} + 4 + \sqrt{2c} - 1\sqrt{c^2 + 4c} + (c+1)\sqrt{c^2 + 4c}}{(c+1)\sqrt{c} + (c+1)\sqrt{c^2 + 4c}} \]

(vi) \[ G^3(-q_{15}^{135}) \]
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\[ -\sqrt{c} - \sqrt{c^2 + 4c + (c + 1)\sqrt{c^2 + 4c}} \]

\[ \frac{1}{(c + 5)\sqrt{c} + (c + 1)\sqrt{c + 4} - 2\sqrt{c^2 + 4c + (c + 1)\sqrt{c^2 + 4c}}} \]

(vii) $G^3(-q^{\frac{1}{15}})$

\[ -(d + 1)\sqrt{d} - d\sqrt{d + 4} + \sqrt{2d - 1}\sqrt{d^2 + 4d + (d + 1)\sqrt{d^2 + 4d}} \]

\[ \frac{1}{(d + 1)\sqrt{d} + (3d - 1)\sqrt{d + 4} + 2\sqrt{2d - 1}\sqrt{d^2 + 4d + (d + 1)\sqrt{d^2 + 4d}}} \]

(viii) $G^3(-q^{\frac{1}{15}})$

\[ -\sqrt{d} - \sqrt{d^2 + 4d + (d + 1)\sqrt{d^2 + 4d}} \]

\[ \frac{1}{(d + 5)\sqrt{d} + (d + 1)\sqrt{d + 4} - 2\sqrt{d^2 + 4d + (d + 1)\sqrt{d^2 + 4d}}} \]

(ix) $G^3(-q^{\frac{1}{30}})$

\[ -(d + 4)\sqrt{d} - (d + 1)\sqrt{d + 4} + 3\sqrt{d^2 + 4d + (d + 1)\sqrt{d^2 + 4d}} \]

\[ \frac{1}{(d + 13)\sqrt{d} + (d + 1)\sqrt{d + 4} + 6\sqrt{d^2 + 4d + (d + 1)\sqrt{d^2 + 4d}}} \]

(x) $G^3(-q^{\frac{1}{30}})$

\[ -(2d - 1)\sqrt{d + 4} - 3\sqrt{2d - 1}\sqrt{d^2 + 4d + (d + 1)\sqrt{d^2 + 4d}} \]

\[ \frac{1}{9(d + 1)\sqrt{d} + (11d - 1)\sqrt{d + 4} - 6\sqrt{2d - 1}\sqrt{d^2 + 4d + (d + 1)\sqrt{d^2 + 4d}}} \]

(xi) $G^3(-q^{\frac{1}{30}})$

\[ -(c + 4)\sqrt{c} - (c + 1)\sqrt{c + 4} + 3\sqrt{c^2 + 4c + (c + 1)\sqrt{c^2 + 4c}} \]

\[ \frac{1}{(c + 13)\sqrt{c} + (c + 1)\sqrt{c + 4} + 6\sqrt{c^2 + 4c + (c + 1)\sqrt{c^2 + 4c}}} \]

(xii) $G^3(-q^{\frac{1}{30}})$

\[ -(2c - 1)\sqrt{c + 4} - 3\sqrt{2c - 1}\sqrt{c^2 + 4c + (c + 1)\sqrt{c^2 + 4c}} \]

\[ \frac{1}{9(c + 1)\sqrt{c} + (11c - 1)\sqrt{c + 4} - 6\sqrt{2c - 1}\sqrt{c^2 + 4c + (c + 1)\sqrt{c^2 + 4c}}} \]

Proof. The results follow directly from (1.3) and Theorem 2.13.

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