TWO KINDS OF CONVERGENCES IN HYPERBOLIC SPACES 
IN THREE-STEP ITERATIVE SCHEMES

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ABSTRACT. In this paper, we introduce a new three-step iterative scheme for three finite families of nonexpansive mappings in hyperbolic spaces. And, we establish a strong convergence and a ∆-convergence of a given iterative scheme to a common fixed point for three finite families of nonexpansive mappings in hyperbolic spaces. Our results generalize and unify the several main results of [1, 4, 5, 9].

1. Introduction and Preliminaries


In this paper, we consider a strong convergence and a ∆-convergence of a new three-step iterative scheme for three finite families of nonexpansive mappings in hyperbolic spaces. Our results extend and unify the corresponding ones in [1, 4, 5, 9].

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Definition 1.1 ([6]). \( (X, d, W) \) is called a \textit{hyperbolic space} if \( (X, d) \) is a metric space and \( W : X \times X \times [0, 1] \to X \) is a mapping satisfying

(i) \( d(z, W(x, y, \alpha)) \leq (1 - \alpha)d(z, x) + \alpha d(z, y) \),

(ii) \( d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|d(x, y) \),

(iii) \( W(x, y, \alpha) = W(y, x, (1 - \alpha)) \),

(iv) \( d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w) \)

for \( x, y, z, w \in X \) and \( \alpha, \beta \in [0, 1] \).

Definition 1.2 ([10]). A hyperbolic space \( (X, d, W) \) is said to be \textit{uniformly convex} if for any \( r > 0 \) and \( \varepsilon \in (0, 2] \), there exists a \( \delta \in (0, 1] \) such that for all \( x, y, z \in X \),

\[
d(x, z) \leq r, \quad d(y, z) \leq r \quad \text{and} \quad d(x, y) \geq \varepsilon r \Rightarrow d(W(x, y, \frac{1}{2})), z) \leq (1 - \delta)r.
\]

A function \( \eta : (0, \infty) \times (0, 1] \to (0, 1] \) which provides such a \( \delta = \eta(r, \varepsilon) \) for given \( r > 0 \) and \( \varepsilon \in (0, 2] \), is called a \textit{modulus} of uniform convexity. We call \( \eta \) a \textit{monotone function} if it decreases with \( r \) (for a fixed \( \varepsilon \)).

A set \( A\{x_n\} := \{x \in X : \inf_{y \in X} \limsup_{n \to \infty} d(y, x_n) = \limsup_{n \to \infty} d(x, x_n)\} \) is called the \textit{asymptotic center} of a given sequence \( \{x_n\} \) in a hyperbolic space \( X \) with an asymptotic radius \( r(\{x_n\}) := \inf_{y \in X} \limsup_{n \to \infty} d(y, x_n) \).

Definition 1.3 ([4]). A sequence \( \{x_n\} \) in \( X \) is said to be \( \Delta \)-\textit{converge} to \( x \in X \) if \( x \) is the unique asymptotic center of its all subsequences. In this case, \( x \) is called the \( \Delta \)-\textit{limit} of \( \{x_n\} \) and denoted as \( \Delta \lim_{n \to \infty} x_n = x \).

Lemma 1.4 ([4]). Let \( (X, d, W) \) be a uniformly convex hyperbolic space with a monotone modulus of uniform convexity \( \eta \). Let \( x \in X \) and \( \{a_n\} \) be a sequence in \([b, c]\) for some \( b, c \in (0, 1) \). If \( \{x_n\} \) and \( \{y_n\} \) are sequences in \( X \) such that \( \limsup_{n \to \infty} d(x_n, x) \leq t \), \( \limsup_{n \to \infty} d(y_n, x) \leq t \) and \( \limsup_{n \to \infty} d(W(x_n, y_n, a_n), x) = t \) for some \( t \geq 0 \), then \( \lim_{n \to \infty} d(x_n, y_n) = 0 \).

Lemma 1.5 ([7]). Let \( (X, d, W) \) be a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity \( \eta \). Then every bounded sequence \( \{x_n\} \) in \( X \) has a unique asymptotic center with respect to any nonempty closed convex subset \( K \) of \( X \).

Lemma 1.6 ([4]). Let \( K \) be a nonempty closed convex subset of a uniformly convex hyperbolic space and \( \{x_n\} \) a bounded sequence in \( K \) such that \( A(\{x_n\}) = \{y\} \) and
Let \( \{x_n\} \) be a sequence in \( K \). If \( \{y_m\} \) is another sequence in \( K \) such that \( \lim_{m \to \infty} r(y_m, \{x_n\}) = \rho \), where \( r(y_m, \{x_n\}) = \limsup_{n \to \infty} d(y_m, x_n) \), then \( \lim_{m \to \infty} y_m = y \).

Now, we introduce a new three-step iterative scheme for three finite families of nonexpansive self-mappings in hyperbolic spaces. A self-mapping \( T \) on \( K \) is said to be nonexpansive if \( d(Tx, Ty) \leq d(x, y) \) for \( x, y \in K \).

**Algorithm 1.1.** Let \( \{T_n\}, \{S_n\} \) and \( \{R_n\} (n \in \mathbb{N}(\equiv \{1, \cdots, N\})) \) be three finite families of nonexpansive self-mappings on \( K \), and \( \{x_n\} \) be a sequence defined by

\[
\begin{align*}
  x_1 & \in K, \\
  x_{n+1} &= W(T_n x_n, S_n y_n, \alpha_n) \\
  y_n &= W(R_n x_n, T_n z_n, \beta_n) \\
  z_n &= W(x_n, R_n x_n, \gamma_n) \quad \text{for } n \in \mathbb{N},
\end{align*}
\]

where \( T_n = T_{n(modN)} \), \( S_n = S_{n(modN)} \) and \( R_n = R_{n(modN)} \) are nonexpansive mappings, and \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are sequences in \( [0,1] \) for \( N \in \mathbb{N} \).

2. \( \Delta \)-Convergence Result

In this section, we establish a \( \Delta \)-convergence of Algorithm 1.1. Denote by \( F(T) \), the set of fixed points of \( T \). First of all, we prove the following lemmas needed in our results.

**Lemma 2.1.** Let \( K \) be a nonempty closed convex subset of a hyperbolic space \( X \) and \( \{T_i\}, \{S_i\} \) and \( \{R_i\} \) be three finite families of nonexpansive self-mappings on \( K \) with \( F := (\bigcap_{i=1}^{N} F(T_i)) \cap (\bigcap_{i=1}^{N} F(S_i)) \cap (\bigcap_{i=1}^{N} F(R_i)) \neq \emptyset \). Suppose that \( \{x_n\} \) is generated by Algorithm 1.1. Then, for any \( p \in F \), \( \lim_{n \to \infty} d(x_n, p) \) exists.

**Proof.** For any \( p \in F \),

\[
\begin{align*}
  d(x_{n+1}, p) &= d(W(T_n x_n, S_n y_n, \alpha_n), p) \\
  &\leq (1 - \alpha_n)d(T_n x_n, p) + \alpha_n d(S_n y_n, p) \\
  &= (1 - \alpha_n)d(T_n x_n, T_n p) + \alpha_n d(S_n y_n, S_n p) \\
  &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(y_n, p) \\
  &= (1 - \alpha_n)d(x_n, p) + \alpha_n d(W(R_n x_n, T_n z_n, \beta_n), p) \\
  &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n \{(1 - \beta_n)d(R_n x_n, p) + \beta_n d(T_n z_n, p)\}
\end{align*}
\]
\[ (1 - \alpha_n) d(x_n, p) + \alpha_n \{ (1 - \beta_n) d(R_n x_n, R_n p) + \beta_n d(T_n z_n, T_n p) \} \]
\[ \leq (1 - \alpha_n) d(x_n, p) + \alpha_n \{ (1 - \beta_n) d(x_n, p) + \beta_n d(z_n, p) \} \]
\[ = (1 - \alpha_n \beta_n) d(x_n, p) + \alpha_n \beta_n d(z_n, p) \]
\[ = (1 - \alpha_n \beta_n) d(x_n, p) + \alpha_n \beta_n \{ (1 - \gamma_n) d(x_n, p) + \gamma_n d(R_n x_n, p) \} \]
\[ \leq (1 - \alpha_n \beta_n) d(x_n, p) + \alpha_n \beta_n \{ (1 - \gamma_n) d(x_n, p) + \gamma_n d(x_n, p) \} \]
\[ = d(x_n, p), \]
which implies that \( \lim_{n \to \infty} d(x_n, p) \) exists for \( p \in F \).

**Lemma 2.2.** Let \( K \) be a nonempty closed convex subset of a uniformly convex hyperbolic space \( X \) with a monotone modulus of uniform convexity \( \eta \) and let \( \{T_i\}, \{S_i\}, \{R_i\}, \{x_n\} \) and \( F \) be the same as cited in Lemma 2.1. Then \( \lim_{n \to \infty} d(x_n, T_i x_n) = \lim_{n \to \infty} d(x_n, S_i x_n) = \lim_{n \to \infty} d(x_n, R_i x_n) = 0 \) for \( i \in \overline{1, N} \).

**Proof.** From Lemma 2.1, \( \lim_{n \to \infty} d(x_n, p) \) exists for \( p \in F \), say \( \lim_{n \to \infty} d(x_n, p) = c \). In the case of \( c = 0 \), the proof is trivial. Now, we deal with the case of \( c > 0 \).

\[ d(z_n, p) = d(W(x_n, R_n x_n, \gamma_n), p) \]
\[ \leq (1 - \gamma_n) d(x_n, p) + \gamma_n d(R_n x_n, p) \]
\[ \leq (1 - \gamma_n) d(x_n, p) + \gamma_n d(x_n, p) \]
\[ = d(x_n, p), \]
which implies that
\[ \limsup_{n \to \infty} d(z_n, p) \leq c \quad \text{(2.1)} \]
and
\[ d(y_n, p) = d(W(R_n x_n, T_n z_n, \beta_n), p) \]
\[ \leq (1 - \beta_n) d(R_n x_n, p) + \beta_n d(T_n z_n, p) \]
\[ \leq (1 - \beta_n) d(x_n, p) + \beta_n d(z_n, p) \]
\[ \leq (1 - \beta_n) d(x_n, p) + \beta_n d(x_n, p) \]
\[ = d(x_n, p), \]
which implies that
\[ \limsup_{n \to \infty} d(y_n, p) \leq c \quad \text{(2.2)} \]
Since \( d(T_n x_n, p) \leq d(x_n, p) \), \( d(S_n y_n, p) \leq d(y_n, p) \) and \( d(R_n x_n, p) \leq d(x_n, p) \) for \( n \in \mathbb{N} \), from (2.1) and (2.2), we get

\[
\limsup_{n \to \infty} d(T_n x_n, p) \leq c, \quad \limsup_{n \to \infty} d(S_n y_n, p) \leq c \quad \text{and} \quad \limsup_{n \to \infty} d(R_n x_n, p) \leq c.
\]

Moreover, since \( c = \lim_{n \to \infty} d(x_{n+1}, p) = \lim_{n \to \infty} d(W(T_n x_n, S_n y_n, \alpha_n), p) \), by Lemma 1.4, we have

\[
\lim_{n \to \infty} d(T_n x_n, S_n y_n) = 0.
\]

Now

\[
d(x_{n+1}, p) = d(W(T_n x_n, S_n y_n, \alpha_n), p) \\
\leq (1 - \alpha_n)d(T_n x_n, p) + \alpha_n d(S_n y_n, p) \\
\leq (1 - \alpha_n)d(T_n x_n, p) + \alpha_n d(S_n y_n, T_n x_n) + \alpha_n d(T_n x_n, p) \\
= d(T_n x_n, p) + \alpha_n d(S_n y_n, T_n x_n),
\]

which implies that

\[
c \leq \liminf_{n \to \infty} d(T_n x_n, p).
\]

From (2.3), we have

\[
\lim_{n \to \infty} d(T_n x_n, p) = c.
\]

On the other hand,

\[
d(T_n x_n, p) \leq d(T_n x_n, S_n y_n) + d(S_n y_n, p) \\
\leq d(T_n x_n, S_n y_n) + d(y_n, p),
\]

which implies that \( c \leq \liminf_{n \to \infty} d(y_n, p) \). Thus, from (2.2),

\[
c = \lim_{n \to \infty} d(y_n, p) = \lim_{n \to \infty} d(W(R_n x_n, T_n z_n, \beta_n), p). \text{ By Lemma 1.4, we have}
\]

\[
\lim_{n \to \infty} d(R_n x_n, T_n z_n) = 0.
\]

Now

\[
d(y_n, p) = d(W(R_n x_n, T_n z_n, \beta_n), p) \\
\leq (1 - \beta_n)d(R_n x_n, p) + \beta_n d(T_n z_n, p) \\
\leq (1 - \beta_n)d(R_n x_n, p) + \beta_n \{d(T_n z_n, p) + d(R_n x_n, p)\} \\
= d(R_n x_n, p) + \beta_n d(T_n z_n, R_n x_n),
\]
which implies that \( c \leq \liminf_{n \to \infty} d(R_n x_n, p) \).

From (2.3), we have \( \lim_{n \to \infty} d(R_n x_n, p) = c \). By (2.6), we obtain
\[
d(R_n x_n, p) \leq d(R_n x_n, T_n z_n) + d(T_n z_n, p)
\]
which implies that \( c \leq \liminf_{n \to \infty} d(z_n, p) \).

Thus, \( c = \lim_{n \to \infty} d(z_n, p) = \lim_{n \to \infty} d(W(x_n, R_n x_n, \gamma_n), p) \). By Lemma 1.4, we have (2.7)
\[
\lim_{n \to \infty} d(x_n, R_n x_n) = 0.
\]

Now
\[
d(z_n, x_n) = d(W(x_n, R_n x_n, \gamma_n), x_n)
\]
which implies by (2.7) that
\[
\lim_{n \to \infty} d(z_n, x_n) = 0.
\]

From (2.6), (2.7) and (2.8), we have
\[
d(x_n, T_n x_n) \leq d(x_n, R_n x_n) + d(R_n x_n, T_n z_n) + d(T_n z_n, T_n x_n)
\]
which implies that \( \lim_{n \to \infty} d(x_{n+i}, x_n) = 0 \) for \( i \in 1, N \). Further, observe that
\[
d(x_n, T_{n+i} x_n) \leq d(x_n, x_{n+i}) + d(x_{n+i}, T_{n+i} x_{n+i}) + d(T_{n+i} x_{n+i}, T_{n+i} x_n)
\]
which implies that \( \lim_{n \to \infty} d(x_{n+i}, x_n) = 0 \) for \( i \in 1, N \). Similarly, we can obtain \( \lim_{n \to \infty} d(x_n, S_i x_n) = \lim_{n \to \infty} d(x_n, R_i x_n) = 0 \) for \( i \in 1, N \).
Now, we prove the following $\Delta$-convergence result.

**Theorem 2.3.** Let $K$ be a nonempty closed convex subset of a uniformly convex hyperbolic space $X$ with a monotone modulus of uniform convexity $\eta$ and $\{T_i\}, \{S_i\}$ and $\{R_i\}$ be three finite families of nonexpansive self-mappings on $K$ with $F := (\bigcap_{i=1}^{N} F(T_i)) \cap (\bigcap_{i=1}^{N} F(S_i)) \cap (\bigcap_{i=1}^{N} F(R_i)) \neq \emptyset$. Suppose that $\{x_n\}$ is generated by Algorithm 1.1. Then $\{x_n\} \Delta$-converges to an element of $F$.

**Proof.** From Lemma 2.1, $\{x_n\}$ is bounded. Therefore by Lemma 1.5, $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x\}$ for some $x \in K$. Assume that $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$ such that $A(\{x_{n_k}\}) = \{u\}$. Then by Lemma 2.2, we get $\lim_{k \to \infty} d(x_{n_k}, T_{x_{n_k}}) = \lim_{k \to \infty} d(x_{n_k}, S_{x_{n_k}}) = \lim_{k \to \infty} d(x_{n_k}, R_{x_{n_k}}) = 0$ for $i \in \{1,N\}$. We claim that $u \in F$. Now, we define a sequence $\{v_m\}$ in $K$ by $v_m = T_{m}u$ for $m \in \mathbb{N}$, where $T_m = T_{m(modN)}$. On the other hand,

$$d(v_m, x_{n_k}) \leq d(T_m u, T_m x_{n_k}) + d(T_m x_{n_k}, T_m x_{n_k}) + \cdots + d(T_1 x_{n_k}, x_{n_k})$$

$$\leq d(u, x_{n_k}) + 2 \sum_{i=1}^{m} d(x_{n_k}, T_i x_{n_k}).$$

Therefore, we have

$$r(v_m, \{x_{n_k}\}) = \limsup_{k \to \infty} d(v_m, x_{n_k}) \leq \limsup_{k \to \infty} d(u, x_{n_k}) = r(u, \{x_{n_k}\}),$$

which implies that $|r(v_m, \{x_{n_k}\}) - r(u, \{x_{n_k}\})| \to 0$ as $k \to \infty$. By Lemma 1.6, we have $T_{m(modN)} u = u$, so $u$ is a common fixed point of $\{T_i\}$. By the same argument, we can show that $u$ is a common fixed point of $\{S_i\}$ and $\{R_i\}$. Therefore, $u \in F$. Moreover, by Lemma 2.1, $\lim_{n \to \infty} d(x_n, u)$ exists.

Assume that $x \neq u$. By the uniqueness of the asymptotic center,

$$\limsup_{k \to \infty} d(x_{n_k}, u) < \limsup_{k \to \infty} d(x_{n_k}, x) \leq \limsup_{n \to \infty} d(x_n, x) \leq \limsup_{n \to \infty} d(x_n, u) = \limsup_{k \to \infty} d(x_{n_k}, u),$$

which is a contradiction, so $x = u$. Since $\{x_{n_k}\}$ is an arbitrary subsequence of $\{x_n\}$, $A(\{x_{n_k}\}) = \{x\}$ for all subsequences $\{x_{n_k}\}$ of $\{x_n\}$. This proves that $\{x_n\} \Delta$-converges to an element of $F$. \hfill $\Box$

**Remark 2.1.** Lemma 2.1, Lemma 2.2 and Theorem 2.1 generalize Lemma 2.1, Lemma 2.2 and Theorem 2.3 in [1], respectively.
3. Strong Convergence Result

In this section, we establish a strong convergence of Algorithm 1.1.

Recall that a sequence \( \{x_n\} \) in a metric space \( X \) is said to be Fejér monotone with respect to a subset \( K \) of \( X \) if \( d(x_{n+1}, p) \leq d(x_n, p) \) for \( p \in K \) and \( n \in \mathbb{N} \).

**Lemma 3.1 ([2])**. Let \( K \) be a nonempty closed subset of a complete metric space \( (X, d) \) and let a sequence \( \{x_n\} \) in \( X \) be Fejér monotone with respect to \( K \). Then \( \{x_n\} \) converges to some \( p \in K \) if and only if \( \lim_{n \to \infty} d(x_n, K) = 0 \).

**Definition 3.2**. Three finite families \( \{T_i\}, \{S_i\}, \{R_i\} (i \in \mathbb{I}, N) \) of self-mappings on \( K \) are said to satisfy condition (A) if there exists a non-decreasing function \( f : [0, \infty) \to [0, \infty) \) with \( f(0) = 0 \) and \( f(t) > 0 \) for \( t > 0 \) such that

\[
\max_{i \in \mathbb{I}, N} \{d(x, S_i x) + d(x, T_i x) + d(x, R_i x)\} \geq f(d(x, F)) \quad \text{for} \quad x \in K,
\]

where \( d(x, F) = \inf \{d(x, p) : p \in F\} \).

Now, we prove the following strong convergence result.

**Theorem 3.3**. Let \( K \) be a nonempty closed convex subset of a uniformly convex hyperbolic space \( X \) with a monotone modulus of uniform convexity and \( \{T_i\}, \{S_i\} \) and \( \{R_i\} \) be three finite families of nonexpansive self-mappings on \( K \) satisfy the condition (A) with \( F := (\bigcap_{i=1}^{N} F(T_i)) \cap (\bigcap_{i=1}^{N} F(S_i)) \cap (\bigcap_{i=1}^{N} F(R_i)) \neq \emptyset \). Suppose that \( \{x_n\} \) is generated by Algorithm 1.1. Then a sequence \( \{x_n\} \) in \( K \) converges strongly to an element of \( F \).

**Proof**. By Lemma 2.1, \( \lim_{n \to \infty} d(x_n, F) \) exists for \( p \in F \) and from the Proof of Lemma 2.1, \( \{x_n\} \) is Fejér monotone with respect to \( F \). By Lemma 2.2, \( \lim_{n \to \infty} d(x_n, T_i x_n) = \lim_{n \to \infty} d(x_n, S_i x_n) = \lim_{n \to \infty} d(x_n, R_i x_n) = 0 \) for \( i \in \mathbb{I}, N \). From the condition (A), we have

\[
f(d(x_n, F)) \leq \max_{i \in \mathbb{I}, N} \{d(x, S_i x) + d(x, T_i x) + d(x, R_i x)\} \to 0 \quad \text{as} \quad n \to \infty.
\]

Since \( f \) is non-decreasing with \( f(0) = 0 \), we have \( \lim_{n \to \infty} d(x_n, F) = 0 \). Hence, from Lemma 3.1, \( \{x_n\} \) converges strongly to an element of \( F \).

**Remark 3.1**. Theorem 3.1 generalizes Theorem 4.5 in [5].
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