J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2021.28.1.71 ISSN(Online) 2287-6081
Volume 28, Number 1 (February 2021), Pages 71–89

CONSTANT RATIO CURVES IN THE ISOTROPIC PLANE AND THEIR DEFLECTION PROPERTIES

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Abstract. We define the constant ratio curves in the isotropic plane and investigate their deflection properties.

1. Introduction

One of the non-Euclidean geometries which is of uttermost importance is Lorentzian geometry, which provides a mathematical formulation of the relativistic mechanics of Einstein. On the other hand, the classical mechanics, which is often called Galilean mechanics, can be formulated in terms of isotropic geometry, which is often called Galilean geometry in many literatures. One of the standard references for this subject is [8].

It is a very interesting task to compare the Euclidean plane $\mathbb{E}^2$, the Lorentzian plane $\mathbb{L}^2$ and the isotropic plane $\mathbb{I}^2$. Many concepts and propositions from Euclidean geometry have corresponding counterparts in Lorentzian and isotropic geometries. Our main goal in this article is to examine one such proposition, which is about constant ratio curves in the isotropic plane $\mathbb{I}^2$ and their deflection properties. In any of the Euclidean, Lorentzian, or isotropic plane with a point F and a line ℓ preassigned, a constant ratio curve is the set of points P such that the ratio of the distance between P and F to the distance between P and ℓ is constant. In the Euclidean plane $\mathbb{E}^2$, they turn out to be ellipses, parabolas, or hyperbolas. They have so many interesting properties. In particular, the deflection property says that if a particle is emanated from a focus, then after hitting the curve the particle proceeds toward or away from the other focus. See Figure 1.

Received by the editors October 28, 2020. Accepted December 22, 2020.
2010 *Mathematics Subject Classification.* 39B22.
Key words and phrases. isotropic plane, constant ratio curve, deflection property.
The second author was supported in part by NRF2017R1E1A1A03070929 and NRF 2020 R1F1A1A01074585 from the National Research Foundation of Korea.
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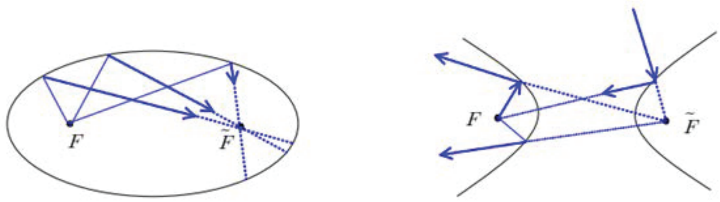

Figure 1. The deflection property of ellipses and hyperbolas in $\mathbb{E}^2$

We investigate what the constant ratio curves in the isotropic plane look like and whether or not they also have the deflection properties. Constant ratio curves in the Lorentz plane $\mathbb{L}^2$ and their deflection properties are studied in [5].

This article is organized as follows: In Section 2, we collect some known facts. In Section 3, we derive the isotropic transformations of $\mathbb{I}^2$ using the isometries of the Lorentzian three-space. In Section 4, we look at the angles and perpendicularity in $\mathbb{I}^2$. In Section 5, we study the distance between a point and a line in $\mathbb{I}^2$. In Section 6, we look at the constant ratio curves in $\mathbb{I}^2$ with a non-vertical directrix and its deflection property. In Section 7, we look at the constant ratio curves in $\mathbb{I}^2$ with a vertical directrix. This article is based on the first author's work for a doctoral degree [6].

2. Preliminaries

The Euclidean plane $\mathbb{E}^2$, the isotropic plane $\mathbb{I}^2$, and the Lorentzian plane $\mathbb{L}^2$ are related by the following equation for the metric:

$$ds^2 = dx^2 + \epsilon dy^2.$$

If $\epsilon = 1, 0, -1$, we obtain the metric of $\mathbb{E}^2, \mathbb{I}^2, \mathbb{L}^2$, respectively.

The descriptations about Euclidean plane $\mathbb{E}^2$ and the Lorentz plane $\mathbb{L}^2$ in this paper are based on the terminologies, notation, and contents of [5], [6], [9] and [10]. In particular, we use x, y for the standard coordinate system for $\mathbb{E}^2$, x, t for $\mathbb{L}^2$, and x, l for $\mathbb{I}^2$.

2.1. A metrical description of $\mathbb{I}^2$ In terms of the x, l coordinates, the metric of the isotropic plane is dx^2. Then, as in $\mathbb{E}^2$ and in $\mathbb{L}^2$, the inner product, the norm, and the distance function are naturally defined in the following way.

Definition 2.1. For two vectors $(x_1, l_1), (x_2, l_2) \in \mathbb{I}^2$, define

$$\langle (x_1, l_1), (x_2, l_2) \rangle_{\mathbb{I}} := x_1 x_2. \tag{2.1}$$

$\langle \ , \ \rangle_{\mathbb{I}}$ is called the *isotropic inner product.*

The Cauchy-Schwarz Inequality in $\mathbb{E}^2$ or in $\mathbb{L}^2$ says that

$$\langle \vec{a}, \vec{b} \rangle_{\mathbb{E}}^2 \leq \langle \vec{a}, \vec{a} \rangle_{\mathbb{E}} \langle \vec{b}, \vec{b} \rangle_{\mathbb{E}} \quad \text{or} \quad \langle \vec{a}, \vec{b} \rangle_{\mathbb{L}}^2 \geq \langle \vec{a}, \vec{a} \rangle_{\mathbb{L}} \langle \vec{b}, \vec{b} \rangle_{\mathbb{L}}, \tag{2.2}$$

respectively. In $\mathbb{I}^2$, we have an equality rather than an inequality.

Lemma 2.2 (Cauchy-Schwarz Equality)**.** *Given two arbitrary vectors $\vec{a}, \vec{b}$, we have*

$$\langle \vec{a}, \vec{b} \rangle_{\mathbb{I}}^2 = \langle \vec{a}, \vec{a} \rangle_{\mathbb{I}} \langle \vec{b}, \vec{b} \rangle_{\mathbb{I}}. \tag{2.3}$$

Proof. Let $\vec{a} = (x_1, l_1)$, $\vec{b} = (x_2, l_2)$. Then $\langle \vec{a}, \vec{b} \rangle_{\mathbb{I}} = x_1 x_2$, $\langle \vec{a}, \vec{a} \rangle_{\mathbb{I}} = x_1^2$, $\langle \vec{b}, \vec{b} \rangle_{\mathbb{I}} = x_2^2$, hence the result follows. □

Given a vector $X = (x, l) \in \mathbb{I}^2$, we define

$$||X||_{\mathbb{I}} := \sqrt{\langle X, X \rangle_{\mathbb{I}}}$$

which is called the *isotropic norm* of X.

Definition 2.3. Given two vectors $X = (x_1, l_1)$ and $Y = (x_2, l_2)$, we define

$$d(X, Y) := ||X - Y||_{\mathbb{I}} \tag{2.4}$$

which is called the *distance* between X and Y.

According to the previous definition,

$$||(x, l)||_{\mathbb{I}} = |x|, \qquad d\left((x_1, l_1), (x_2, l_2)\right) = |x_1 - x_2|.$$

If $\triangle ABC$ is a Euclidean triangle in $\mathbb{E}^2$, then $d(B, C) + d(C, A) > d(A, B)$. If $\triangle ABC$ is a Lorentzian triangle in $\mathbb{L}^2$ with $\overrightarrow{BC}, \overrightarrow{CA}$ timelike, then $d(B, C) + d(C, A) \leq d(A, B)$. See [1]. If $\triangle ABC$ is an isotropic triangle in $\mathbb{I}^2$ with $d(A, B) = max\{d(A, B), d(B, C), d(C, A)\}$, then (cf. [8, p. 202])

$$d(B, C) + d(C, A) = d(A, B).$$

Many concepts such as points, lines, rays, segments, vectors, parallel lines, a direction vector of a line, etc, from Euclidean and Lorentzian plane geometries are also defined in the isotropic plane in much the same way, which we do not elaborate in this article.

3. Derivation of the Isotropic Transformations of $\mathbb{I}^2$

A complete description of the isotropic plane would not be complete without the description of the transformations in this geometry. Recall that an isometric transformation is a map f which satisfies that

$$d(P,Q) = d(f(P), f(Q)) \quad \text{for any two points } P, Q$$

and that the isometric transformations of $\mathbb{E}^2$ or in $\mathbb{L}^2$ can be completely determined from the metric. See [5] for example for the derivation of the isometries of the Lorentzian plane from the Lorentzian metric. However, this is not the case for $\mathbb{I}^2$. It can be immediately seen that any map of the form

$$f(x,l) = (x, g(x,l))$$

for any function $g : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$\begin{aligned} d(f(x_1,l_1), f(x_2,l_2)) &= d\left((x_1, g(x_1,l_1)), (x_2, g(x_2,l_2))\right) = |x_1 - x_2| \\ &= d((x_1,l_1),(x_2,l_2)). \end{aligned}$$

In particular, such a map f needs not be affine. So in order to determine the isotropic transformation of $\mathbb{I}^2$, we need a different approach. [8] does it from the viewpoint of the classical mechanics. The most geometric approach, in our opinion, is to look at the specific model of $\mathbb{I}^2$, which we do in the next subsection. We present two derivations of the isotropic transformation, one is extrinsic, and the other is intrinsic.

3.1. An extrinsic derivation of isotropic transformations [8, §13] regards isotropic geometry as a limiting case of Euclidean and Lorentzian geometries. Isotropic plane is in fact the lightlike plane in the three-dimensional Lorentz-Minkowski space $\mathbb{L}^3$, and the isotropic plane geometry is the limiting geometry as we tilt and rescale the Euclidean planes or Lorentzian planes in $\mathbb{L}^3$ appropriately. See [3].

The coordinates y and $x+t$ in the lightlike plane correspond to x and l in $\mathbb{I}^2$. We will derive the isotropic transformations using this model. A novel feature of this approach is that we obtain the shapes of circles in this geometry in a natural manner.

Fix an arbitrary real number $k \in \mathbb{R}$ and consider the lightlike plane

$$\Pi_k := \{(x,y,t) \in \mathbb{L}^3 : x - t = k\}.$$

A good coordinate system for Π_k is $y, x+t$.

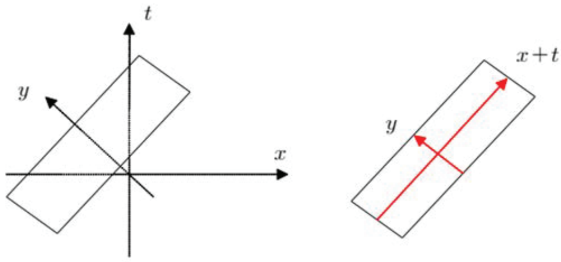

Figure 2. A lightlike plane in $\mathbb{L}^3$ and its coordinate system

Note that the metric of $\mathbb{L}^3$ restricted to Π_k becomes dy^2. This Π_k is our model of the isotropic plane. We want to see how isometries of $\mathbb{L}^3$ which preserves Π_k look like.

Firstly, we consider translations. The following is a translation of $\mathbb{L}^3$ which preserves Π_k:

$$\begin{pmatrix} x \\ y \\ t \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \\ T \end{pmatrix} = \begin{pmatrix} x \\ y \\ t \end{pmatrix} + \begin{pmatrix} d/2 \\ e \\ d/2 \end{pmatrix}.$$

We see that

$$\begin{pmatrix} Y \\ X+T \end{pmatrix} = \begin{pmatrix} y \\ x+t \end{pmatrix} + \begin{pmatrix} e \\ d \end{pmatrix}. \tag{3.1}$$

Secondly, we consider the reflections which preserve Π_k:

$$\begin{pmatrix} x \\ y \\ t \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \\ T \end{pmatrix} = \begin{pmatrix} x \\ -y \\ t \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ t \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \\ T \end{pmatrix} = \begin{pmatrix} -x+k \\ y \\ -t-k \end{pmatrix}.$$

We see that

$$\begin{pmatrix} Y \\ X+T \end{pmatrix} = \begin{pmatrix} -y \\ x+t \end{pmatrix}, \quad \begin{pmatrix} Y \\ X+T \end{pmatrix} = \begin{pmatrix} y \\ -(x+t) \end{pmatrix} \tag{3.2}$$

We remark that these two reflections are not equivalent.

Thirdly, we turn our attention to rotations. It is very well known that there are three kinds of rotations in $\mathbb{L}^3$, which are called the elliptic, hyperbolic and parabolic rotations in $\mathbb{L}^3$ depending upon the causal character of the axis of rotation. The typical rotations are

$$M_e := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_h := \begin{pmatrix} \cosh\phi & 0 & \sinh\phi \\ 0 & 1 & 0 \\ \sinh\phi & 0 & \cosh\phi \end{pmatrix},$$

$$M_p := \begin{pmatrix} 1-\psi^2/8 & \psi/2 & \psi^2/8 \\ -\psi/2 & 1 & \psi/2 \\ -\psi^2/8 & \psi/2 & 1+\psi^2/8 \end{pmatrix},$$

where θ, ϕ, ψ are arbitrary real numbers which represent the magnitude of rotation. Among them, $\begin{pmatrix} x \\ y \\ t \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \\ T \end{pmatrix} := M_p \begin{pmatrix} x \\ y \\ t \end{pmatrix}$ preserves Π_k . Now fix arbitrary $(a, b, c) \in \mathbb{L}^3$ and consider the following

$$\begin{pmatrix} x \\ y \\ t \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \\ T \end{pmatrix} := M_p \left(\begin{pmatrix} x \\ y \\ t \end{pmatrix} - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right) + \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

This is clearly an isometry of $\mathbb{L}^3$. Direct calculations show that $X - T = x - t$, hence this map preserves Π_k. Furthermore,

$$\begin{pmatrix} Y \\ X+T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \psi & 1 \end{pmatrix} \begin{pmatrix} y \\ x+t \end{pmatrix} - b \begin{pmatrix} 0 \\ \psi \end{pmatrix} + \frac{(a-c-k)}{2} \begin{pmatrix} \psi \\ \psi^2/2 \end{pmatrix}. \tag{3.3}$$

There are the rotations in Π_k. Note that there are 3, not 4, parameters here. Namely, $a - c, b$ and ψ. Here, ψ corresponds to the magnitude of the rotation.

Before we interpret these in terms of the isotropic plane, let's think about the trajectory of a point $(x_0, y_0, t_0) \in \Pi_k$ as ψ varies (with a, b, c fixed). There are two cases to consider.

Firstly, if $(a, b, c) \in \Pi_k$, i.e. if $a - c - k = 0$, then

$$Y = y_0, \qquad X + T = (y_0 - b)\psi + x_0 + t_0 \tag{3.4}$$

so the trajectory of (x_0, y_0, t_0) is a null line in Π_k. This line corresponds to a vertical line in the isotropic plane, and later we will see that it is one half of an isotropic circle.

Secondly, if $(a, b, c) \notin \Pi_k$, i.e. if $a - c - k \neq 0$, then $\psi = 2(Y - y_0)/(a - c - k)$, hence

$$\begin{aligned} X + T &= 2(x_0 + t_0) + (y_0 - b)\psi + (a - c - k)\psi^2/4 \\ &= 2(x_0 + t_0) + 2(y_0 - b)\frac{Y - y_0}{a - c - k} + \frac{(Y - y_0)^2}{a - c - k}. \end{aligned} \tag{3.5}$$

which is a parabola in Π_k. This parabola corresponds to an i-circle of parabolic type in the isotropic plane $\mathbb{I}^2$, in the language of [4].

Remark 3.1. [8] called the curves given by (3.4) as circles, and curves given by (3.5) as cycles, respectively. Our method of deriving the existence of (3.5) is quite different from the way how [8] derived it.

In order to translate the above calculations in terms of isotropic plane, let's rename the coordinates of Π_k from $y, x+t$ to x, l. Combining all the transformations (3.1), (3.2), (3.3) yields the isotropic transformations of $\mathbb{I}^2$.

Definition 3.2. The isotropic transformation of $\mathbb{I}^2$ is a map of the form

$$\begin{pmatrix} x \\ l \end{pmatrix} \mapsto \begin{pmatrix} \pm 1 & 0 \\ \psi & \pm 1 \end{pmatrix} \begin{pmatrix} x \\ l \end{pmatrix} + \begin{pmatrix} x_0 \\ l_0 \end{pmatrix}, \quad \psi, x_0, l_0 \in \mathbb{R}.$$

There are four choices of the signs.

Isotropic plane geometry is the study of the properties of objects in the plane which are invariant under this group of isotropic transformations.

The following isotropic transformations are of particular importance.

$$\begin{gathered} \begin{pmatrix} x \\ l \end{pmatrix} \mapsto \begin{pmatrix} x \\ l \end{pmatrix} + \begin{pmatrix} d \\ e \end{pmatrix}, \quad \begin{pmatrix} x \\ l \end{pmatrix} \mapsto \begin{pmatrix} -x \\ l \end{pmatrix}, \quad \begin{pmatrix} x \\ l \end{pmatrix} \mapsto \begin{pmatrix} x \\ -l \end{pmatrix}, \\ \begin{pmatrix} x \\ l \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ \psi & 1 \end{pmatrix} \begin{pmatrix} x \\ l \end{pmatrix} + \alpha \begin{pmatrix} \psi \\ \psi^2/2 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ \psi \end{pmatrix} \quad \alpha, \beta \in \mathbb{R}. \end{gathered} \tag{3.6}$$

Definition 3.3. We call the first transformation from (3.6) a *translation*, the second and the third ones *reflections*, and the last one a *rotation*, respectively.

The rotations with $\alpha = \beta = 0$ will be used in the next section to define the magnitude of angles for two non-vertical intersecting lines.

A derivation of the isotropic transformations for the isotropic three-space $\mathbb{I}^3$ as a hyperspace of the four dimensional Lorentz-Minkowski space $\mathbb{L}^3$ is discussed in [7].

3.2. An intrinsic derivation of isotropic transformations Now we present an argument on how we may intrinsically derive the isotropic transformations f. f preserving the metric is a must. Another natural requirement for f is that f is affine. We want this because we want that lines are sent to lines by f. For Euclidean or Lorentzian transformations, affinity follows from the metric, but not for isotropic transformation as we observed in the beginning of this section. So we impose it as a condition.

Lemma 3.4. *For an $f : \mathbb{I}^2 \to \mathbb{I}^2$, suppose that*

(1) f is affine,

(2) f preserves the metric.

Then f is of the following form: $f\begin{pmatrix} x \\ l \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ c & d \end{pmatrix}\begin{pmatrix} x \\ l \end{pmatrix} + \begin{pmatrix} x_0 \\ l_0 \end{pmatrix}$.

Proof. Suppose that $\begin{pmatrix} \tilde{x} \\ \tilde{l} \end{pmatrix} = f\begin{pmatrix} x \\ l \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ l \end{pmatrix} + \begin{pmatrix} x_0 \\ l_0 \end{pmatrix}$. Then $\tilde{x} = ax + bl + x_0, \tilde{l} = cx + dl + l_0$. Hence $d\tilde{x}^2 = (adx + bdl)^2$. This equals dx^2 if and only if $a^2 = 1$ and $b = 0$. □

Note that $M := \begin{pmatrix} \pm 1 & 0 \\ c & d \end{pmatrix}$ satisfies $M^T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Recall that the Euclidean orthogonal group and the Lorentzian orthogonal group are

$$O\mathbb{E}(2) := \left\{ M \in \mathcal{M}(2,2) : M^T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$
$$O\mathbb{L}(2) := \left\{ M \in \mathcal{M}(2,2) : M^T\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

So we are very tempted to define

$$\left\{ M \in \mathcal{M}(2,2) : M^T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \tag{3.7}$$

as the set of isotropic orthogonal matrices. However, while $O\mathbb{E}(2)$ and $O\mathbb{L}(2)$ are 1-dimensional this set is 2-dimensional as we see in the following.

Lemma 3.5. *M belongs to the set in* (3.7) *if and only if* $M = \begin{pmatrix} \pm 1 & 0 \\ c & d \end{pmatrix}$ *for some* $c, d \in \mathbb{R}$.

Proof. Suppose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $M^T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if $a^2 = 1, ab = 0, b^2 = 0$, from which the conclusion follows. □

For this dimensional reason, we want to have one more restriction for f, which can be done in various ways.

3.2.1. *Way 1)* One way is to impose the condition that

$$(\det M)^2 = 1. \tag{3.8}$$

Note that this condition follows from $M^T\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}M = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$, but not from $M^T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. (3.8) forces $d = \pm 1$, and we recover the matrices we want to have. Here, $d = -1$ corresponds to the reversing of the lightlike orientation.

3.2.2. *Way 2)* In the previous subsection we observed what the rotations are in the isotropic plane, hence we know what isotropic circles are. Once we know the shapes of circles, we can describe an intrinsic way of finding the formula for isotropic transformations as follows.

Lemma 3.6. *For an $f : \mathbb{I}^2 \to \mathbb{I}^2$, suppose that*

(1) f is affine,

(2) f preserves the metric,

(3) f preserves the isotropic circle $\{(x, l) \in \mathbb{I}^2 : l = Ax^2+2Bx+C\}$ with $A \neq 0$.

Then f is of the following form:

$$f\begin{pmatrix} x \\ l \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ \psi & 1 \end{pmatrix}\begin{pmatrix} x \\ l \end{pmatrix} + \frac{4B}{A}\begin{pmatrix} 0 \\ \psi \end{pmatrix} + \frac{1}{2A}\begin{pmatrix} \pm\psi \\ \psi^2/2 \end{pmatrix} + (\pm 1 - 1)\frac{B}{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(There are 2, not 4, choices of signs, i.e. $++$ or $--$.)

Proof. From (1), (2), we see that $\begin{pmatrix} \tilde{x} \\ \tilde{l} \end{pmatrix} = f\begin{pmatrix} x \\ l \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ c & d \end{pmatrix}\begin{pmatrix} x \\ l \end{pmatrix} + \begin{pmatrix} x_0 \\ l_0 \end{pmatrix}$. That is, $\tilde{x} = \pm x + x_0,\ \tilde{l} = cx + dl + l_0$. Now $\tilde{l} = A\tilde{x}^2 + 2B\tilde{x} + C$ implies that

$$cx + dl + l_0 = A(\pm x + x_0)^2 + 2B(\pm x + x_0) + C.$$

Inserting $\ell = Ax^2 + 2Bx + C$, we see that a quadratic polynomial of x is 0 for all x, which implies that the three coefficients are all 0, from which we have that

$$d = 1,\ x_0 = \frac{\pm(c + 2B) - 2B}{2A},\ \ \ell_0 = \frac{4B}{A}c + \frac{1}{4A}c^2.$$

So the conclusion follows. □

3.3. Isotropic orthogonal group For these reasons, we define

Definition 3.7.

$$O\mathbb{I}(2) := \{M \in \mathcal{M}(2,2) : M^T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, (\det M)^2 = 1\},$$

$$O\mathbb{I}(2)^+ := \{M \in O\mathbb{I}(2) : M_{22} > 0\},$$

$$SO\mathbb{I}(2) := \{M \in O\mathbb{I}(2) : \det M = 1\},$$

$$SO\mathbb{I}(2)^+ := O\mathbb{I}(2)^+ \cap SO\mathbb{I}(2).$$

Now an isotropic transformation $f : \mathbb{I}^2 \to \mathbb{I}^2$ is defined to be a map

$$f(X) = MX + T, \qquad M \in O\mathbb{I}(2),\ T \in \mathbb{I}^2.$$

The intrinsic approach is good because it is similar in nature to the derivation of Euclidean and Lorentzian transformations. But this is bad in the sense that we do

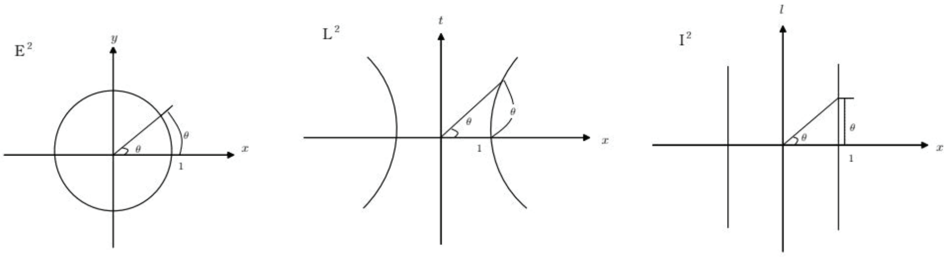

Figure 3. Angles and lengths of arcs in $\mathbb{E}^2$, $\mathbb{L}^2$, $\mathbb{I}^2$

not get that the rotations are much more than just $O\mathbb{I}(2)$ part (cf. (3.6)). In order to get the shape of the rotations, we need extra care as we do in Lemma 3.6.

For us, we always look at the lightlike plane model of the isotropic plane when we want to understand or get an intuition of what is going on.

4. Angles and Perpendicularity

4.1. Special rotations Consider the rotation (3.6) with $\alpha = \beta = 0$, that is,

$$f\begin{pmatrix} x \\ l \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \psi & 1 \end{pmatrix}\begin{pmatrix} x \\ l \end{pmatrix}.$$

It rotates the vector $\begin{pmatrix} x \\ l \end{pmatrix}$ to $\begin{pmatrix} x \\ l + \psi x \end{pmatrix}$. In particular, the x-coordinate does not change. We regard ψ as the signed magnitude of this rotation. Note that this f fixes the entire l-axis pointwise.

4.2. Oriented angles and their magnitudes An oriented angle $\angle(\vec{v}_1, \vec{v}_2)$ is an ordered pair of two vectors $\vec{v}_1$, $\vec{v}_2$ with the same base point. Given three non-collinear points A, B, C, then $\angle ABC := \angle(\overrightarrow{BA}, \overrightarrow{BC})$.

We define the magnitudes of oriented angles using the special rotation.

Definition 4.1. Given an oriented angle $\angle(\vec{v}_1, \vec{v}_2)$ with the common starting point (x_0, l_0) and ending points (x_1, l_1) and (x_2, l_2), respectively, suppose that there are $\psi \in \mathbb{R}$ and $t \in \mathbb{R}^+$ such that

$$\begin{pmatrix} x_2 - x_0 \\ l_2 - l_0 \end{pmatrix} = t\begin{pmatrix} 1 & 0 \\ \psi & 1 \end{pmatrix}\begin{pmatrix} x_1 - x_0 \\ l_1 - l_0 \end{pmatrix}.$$

Then we say that the oriented angle $\angle(\vec{v}_1, \vec{v}_2)$ is *measurable*, define ψ to be its signed magnitude, and denote it by

$$m\left(\angle(\vec{v}_1, \vec{v}_2)\right) := \psi. \tag{4.1}$$

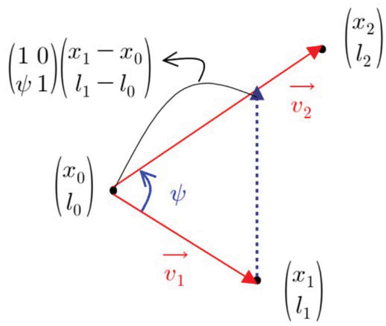

Figure 4. Signed magnitude of an oriented angle

An oriented angle is non-measurable if it is not measurable.

Note that there are oriented angles whose signed magnitudes are not defined. An example is $\angle(\vec{v}_1, \vec{v}_2)$ with $\vec{v}_1 = (1, 1)$ and $\vec{v}_2 = (-1, 1)$. See Figure 5.

Lemma 4.2. *An oriented angle* $\angle(\vec{v}_1, \vec{v}_2)$ *is measurable if and only if* $\langle \vec{v}_1, \vec{v}_2 \rangle_{\mathbb{I}} > 0$.

Proof. A proof follows by checking all possible cases. □

Lemma 4.3. *The signed magnitude of an oriented angle formed by the positive x-axis and its image by* $f\begin{pmatrix}x\\ l\end{pmatrix} := \begin{pmatrix}1 & 0\\ \psi & 1\end{pmatrix}\begin{pmatrix}x\\ l\end{pmatrix}$ *is equal to the slope (in the sense of Euclidean geometry) of the image.*

Proof. This is true because ψ is equal to the slope of the image of the positive x-axis by f. □

Corollary 4.4. *The signed magnitude of an angle between two intersecting non-vertical lines is equal to the difference of their slopes.*

In order for this to be a well defined quantity in the isotropic plane geometry, it must be invariant under isotropic transformations.

Lemma 4.5. *The signed magnitude of an oriented angle from $\vec{v}_1$ to $\vec{v}_2$, if defined, is invariant under any orientation preserving isotropic transformation.*

Proof. Clearly translations do not change the signed magnitude while reflections change its sign but not the absolute value. Finally we can check that rotations do not change the signed magnitude by working in coordinates. □

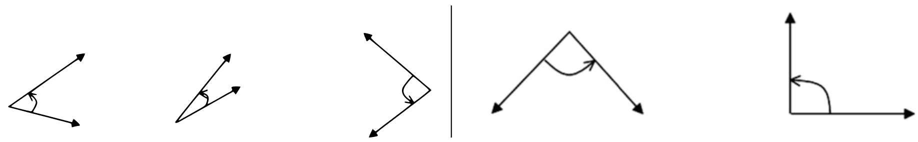

Figure 5. Measurable angles and non-measurable angles

4.3. Perpendicularity

Definition 4.6. Two vectors v and w are said to be *perpendicular* to each other if

$$\langle v, w\rangle_{\mathbb{I}} = 0. \tag{4.2}$$

Two straight lines are said to be perpendicular to each other if the direction vectors of them are perpendicular to each other.

Lemma 4.7. *Two nonzero vectors $\vec{v}, \vec{w}$ are perpendicular to each other if and only if at least one of them is of the form*

$$(0, l) \tag{4.3}$$

for some $l \in \mathbb{R} \setminus \{0\}$. A vector of the form (4.3) *is perpendicular to any vector in $\mathbb{I}^2$.*

Proof. Simple calculation, omitted. □

Definition 4.8. We call a nonzero vector of the form (4.3) a *vertical vector*. We call a straight line a *vertical line* if its direction vector is a vertical vector. We call a vector or a line which is not vertical a *non-vertical vector* or a *non-vertical line*, respectively.

In terms of Lorentzian geometry, a vector is vertical if and only if it is null, and is non-vertical if and only if it is spacelike.

Lemma 4.9. *Given two straight lines $\ell_1 : a_1x+b_1l+c_1 = 0$ and $\ell_2 : a_2x+b_2l+c_2 = 0$, the following are equivalent:*

- *ℓ_1 and ℓ_2 are perpendicular to each other.*
- *$b_1b_2 = 0$.*
- *Either ℓ_1 or ℓ_2 is vertical.*

Proof. $(b_i, -a_i)$ is a direction vector of line $a_ix + b_il + c_i = 0$. The conclusion follows from the fact that $\langle(b_1, -a_1), (b_2, -a_2)\rangle_{\mathbb{I}}$=$b_1b_2$ by (2.1). □

Lemma 4.10. *Given a point $P(x_0, l_0)$ and a non-vertical line $\ell : ax + bl + c = 0$ which does not pass through P, there exists exactly one straight line m which passes through $P(x_0, l_0)$ and is perpendicular to ℓ. Its equation is $x = x_0$.*

Proof. Since $(b, -a)$ is a direction vector of ℓ, $(0, 1)$ is a direction vector of the line perpendicular to ℓ. There is only one such line passing $P(x_0, l_0)$, which is $x = x_0$. □

5. Distance between a Point and a Line

5.1. Vertical distance If two points P, Q are on a vertical line, then $d(P, Q) = 0$. But, it is helpful to have a distance between P, Q, one of whose use is exhibited in Section 6.

[8] introduce a definition for it. The distance between $P(a, b)$ and $Q(a, c)$ is defined to be $|b-c|$. Unfortunately, it is not clear why this is chosen as the definition.

For this matter, we use angles, motivated by the facts that in $\mathbb{E}^2$ and also in $\mathbb{L}^2$ the lengths of arcs are related to angles and that the vertical line segment PQ in $\mathbb{I}^2$ is in fact a circular arc.

Definition 5.1. Given two points $P(a, b)$ and $Q(a, c)$ on a vertical line, we define the orient distance $d'(P, Q)$ as the signed measure of the oriented angle $\angle(POQ)$ where $O := (a - 1, b)$. That is,

$$d'(P, Q) := m(\overrightarrow{OP}, \overrightarrow{OQ}).$$

This will be used in the next subsection in defining the distance between a point and a non-vertical line.

Lemma 5.2. $|d'(\cdot, \cdot)|$ *is equal to the distance defined in* [8].

Proof. If $P = (a, b)$ and $Q = (a, c)$, then $d'(P, Q) = c - b$, hence the conclusion follows. □

The following lemma shows that in fact one may use $O' = (a-1, d)$ for any $d \in \mathbb{R}$ instead of O in the above definition.

Lemma 5.3. *Given two points P and Q on a vertical line ℓ and two points O and O' on another vertical line ℓ', we have*

$$m(\overrightarrow{OP}, \overrightarrow{OQ}) = m(\overrightarrow{O'P}, \overrightarrow{O'Q}).$$

Proof. Trivial. □

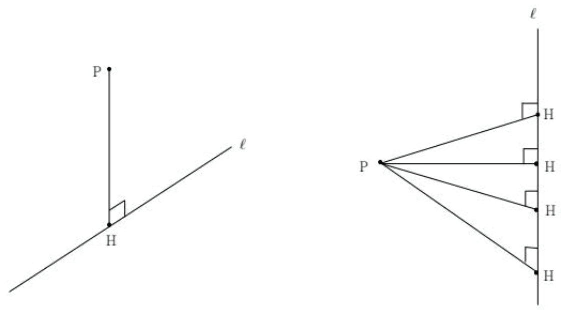

Figure 6. Foot of perpendicular

5.2. Distance between a point and a line Now we are ready to talk about the distance between a point and a line.

In $\mathbb{E}^2$ and in $\mathbb{L}^2$, the foot of perpendicular H drawn from a point P onto a line ℓ which does not go through P is used in the definition of $d(P,\ell)$. However, the situation is not so in $\mathbb{I}^2$. First of all, if ℓ is vertical, then any point H on ℓ is a foot of perpendicular drawn from P onto ℓ. That is, the foot of perpendicular is not unique.

If ℓ is non-vertical, then there is a unique foot of perpendicular H. However, $d(P,H)=0$. So, if we define the distance between a point and a non-vertical line as the distance between the point and the foot of perpendicular, then the distance is 0, which is not good.

Albeit these difficulties, we can define the distance between a point and a line as follows.

Definition 5.4. If ℓ is a non-vertical line, we define

$$d_{NV}(P,\ell) := |d'(P,H)|$$

where H is the foot of perpendicular drawn from P onto ℓ. If ℓ is a vertical line, we define

$$d_V(P,\ell) := d(P,H)$$

where H is an arbitrary point of ℓ.

d_V is well-defined because $d(P,H_1)=d(P,H_2)$ for any $H_1,H_2\in\ell$ if ℓ is vertical.

Lemma 5.5. *Given $P(x_0,l_0)$ and $\ell : ax+bl+c=0$, if ℓ is non-vertical, then $b\neq 0$ and*

$$d_{NV}(P,\ell) = \left|\frac{ax_0+bl_0+c}{b}\right|.$$

and if ℓ is non-vertical, then $b = 0$, $a \neq 0$, and

$$d_V(P, \ell) = \left| x_0 + \frac{c}{a} \right|$$

Proof. If ℓ is non-vertical, it is easy to see that the coordinates of the foot of pependicular H drawn from P onto ℓ is $H = \left(x_0, -\dfrac{ax_0 + c}{b} \right)$, hence the formula for d_{NV} follows.

The formula for d_V is trivial. □

We will use d_{NV} in Section 6 in defining the constant ratio curves in $\mathbb{I}^2$, which are the analogues of conic sections.

6. Deflection Property of Constant Ratio Curves with a Non-vertical Directrix

The conic sections in $\mathbb{E}^2$ or in $\mathbb{L}^2$ have a very interesting property: A particle emanated from one focus proceeds, after hitting the conic section and being deflected, toward or away from the other focus. See [5] for a proof in the Lorentzian case.

In this section, we investigate whether this property still holds for similar kind of curves $\mathbb{I}^2$. We first define the analogues of conic sections in $\mathbb{I}^2$.

Definition 6.1. Given a point F and a non-vertical line ℓ in $\mathbb{I}^2$ and a positive constant e, we call the following set

$$\mathcal{C}_{NV} := \left\{ P \in \mathbb{I}^2 : d(P, F) = e \cdot d_{NV}(P, \ell) \right\}$$

a *constant ratio curve*, where d_{NV} is defined in Definition 5.4. We call F, ℓ, and e the *focus*, the *directrix*, and the *eccentricity* of the constant ratio curve.

By applying isotropic transformations if necessary, we may assume that they are $F(0, l_0)$ with $l_0 > 0$ and $\ell : l = 0$ in $\mathbb{I}^2$. The following is immediate from the definitions.

Proposition 6.2. *Let $\mathcal{C}_{NV}$ be the constant ratio curve in $\mathbb{I}^2$ with the focus $F = (0, l_0)$, the directrix $\ell : l = 0$, and the eccentricity $e > 0$. Then $P(x, l) \in \mathcal{C}_{NV}$ if and only if*

$$|x| = e|l|, \quad \text{or equivalently} \quad l = \pm e^{-1} x.$$

Proof. This is trivial from the various definitions. □

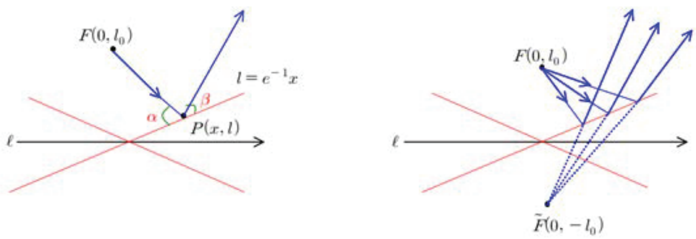

Figure 7. The deflection property of $\mathcal{C}_{NV}$

Note that there is a point $P \in \mathcal{C}_{NV}$ such that $d(P, F) = d_{NV}(P, \ell) = 0$, unlike the constant ratio curves in $\mathbb{E}^2$.

It is easy to see that the above $\mathcal{C}_{NV}$ coincides with the constant ratio curve with $F'(0, -l_0)$ as the focus, $\ell : l = 0$ as the directrix , and $e > 0$ as the eccentricity. We call F' the second focus of $\mathcal{C}_{NV}$.

Now, we want to find if $\mathcal{C}_{NV}$ has the deflection property aforementioned. To see this, let's consider a particle emanated from F. Suppose that it hits the constant ratio curve $\mathcal{C}_{NV}$ at P and gets deflected. As in the Euclidean and in the Lorentzian case we require in Figure 7 that $\alpha = \beta$.

We have the following.

Theorem 6.3. *Given a constant ratio curve curve $\mathcal{C}_{NV}$ in $\mathbb{I}^2$ with the focus F and the directrix ℓ, suppose that a particle is emanated from F, and is deflected after hitting $\mathcal{C}_{NV}$. Then, the line which contains the trajectory of the particle after the deflection passes through the second focus F' of $\mathcal{C}_{NV}$.*

Proof. Let P be the point where the particle hits $\mathcal{C}_{NV}$. Without loss of generality we may assume that $F = (0, l_0)$, $\ell : l = 0$ and $P = (x, e^{-1}x)$ with $x > 0$. The case for $x < 0$ can be proved similarly, and is omitted.

The slope of FP is
$$\frac{e^{-1}x - l_0}{x - 0} = e^{-1} - \frac{l_0}{x}.$$
So $\alpha = e^{-1} - \left(e^{-1} - \frac{l_0}{x}\right) = \frac{l_0}{x}$. (Recall that the angle is the difference of slopes.) So $\beta = \alpha = \frac{l_0}{x}$. Let $\tilde{L}$ be the line containing the trajectory of the particle after deflection. The slope of $\tilde{L}$ is $e^{-1} + \beta = e^{-1} + \frac{l_0}{x}$ and the equation of $\tilde{L}$ is $\tilde{l} = (e^{-1} + \frac{l_0}{x})(\tilde{x} - x) + e^{-1}x$. The l-intercept of $\tilde{L}$ is $(e^{-1} + \frac{l_0}{x})(0 - x) + e^{-1}x = -l_0$. Therefore $\tilde{L}$ passes through $F' := (0, -l_0)$ regardless of the position of the point P in the constant ratio curve $l = e^{-1}x$. $\square$

7. Deflection Property of Constant Ratio Curves with a Vertical Directrix

In Definition 6.1, the directrix ℓ was assumed to be non-vertical. Let's think about what happens if ℓ is vertical. In this case, an appropriate definition of a constant ratio curve $\mathcal{C}_V$ would be the following:

Definition 7.1. Given a point F and a vertical line ℓ in $\mathbb{I}^2$ and a positive constant e, we call the following set

$$\mathcal{C}_V := \left\{P \in \mathbb{I}^2 : d(P,F) = e \cdot d_V(P,\ell)\right\}$$

a *constant ratio curve*, where d_V is defined in Definition 5.4. We call F, ℓ, and e the *focus*, the *directrix*, and the *eccentricity* of the constant ratio curve.

For simplicity, we assume $0 < e < 1$ or $e > 1$. By applying a translation and a reflection if necessary, we may set $F = (0,0)$ and $\ell : x = a$ for some $a > 0$. It is clear that $(x,l) \in \mathcal{C}_V$ if and only if $x = \frac{e}{e \pm 1}a$, so $\mathcal{C}_V$ is a pair of vertical lines.

What could be the second focus? Well, before trying to answer this question, we observe that $\mathcal{C}_V = \{P \in \mathbb{I}^2 : d(P,\tilde{F}) = e \cdot d_V(P,\ell)\}$ for any point $\tilde{F}$ of the form $(0,\tilde{l})$. Therefore, $\tilde{F}(0,\tilde{l})$ for any $\tilde{l}$ is also a focus of $\mathcal{C}_V$ with respect to ℓ. But this $\tilde{F}$ is not the second focus we are looking for. Therefore, it is not clear which is the right focus to start with.

Even though there is an ambiguity for the choice of a focus, we may still proceed further as follows. A basic idea is that for any vertical line $x = x_0$, the transformation $(x,l) \mapsto (2x_0 - x, l)$ still may be thought of as a natural reflection with respect to the vertical line $x = x_0$. Then the vertical line $x = \frac{1}{2}\left(\frac{e}{e+1}a + \frac{e}{e-1}a\right) = \frac{e^2}{e^2-1}a$, which is obtained as the mid line for $\mathcal{C}_V$, is a line of symmetry for $\mathcal{C}_V$. Then the reflections F' and ℓ' of F and ℓ with respect to this midline of symmetry, that is,

$$F' = \left(2\frac{e^2}{e^2-1}a, 0\right), \qquad \ell' : x = \frac{e^2+1}{e^2-1}a,$$

turn out to be another pair of a focus and a directrix for $\mathcal{C}_V$. That is,

$$\mathcal{C}_V = \{P \in \mathbb{I}^2 : d(P,F') = e \cdot d_V(P,\ell')\}.$$

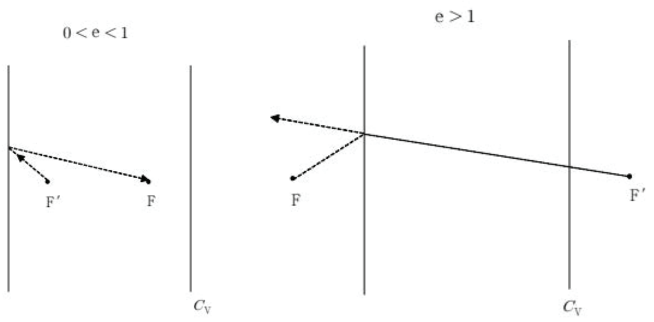

Figure 8. A proposed deflection property of $\mathcal{C}_V$

This F' may be thought as the second focus. That is, we think of F and F' as a pair of foci to think of for the deflection problem. Here we note that

$$\begin{aligned} 0 < e < 1 \qquad &\Rightarrow \qquad \frac{e}{e-1}a < 2\frac{e^2}{e^2-1}a < 0 < \frac{e}{e+1}a, \\ 1 < e < \infty \qquad &\Rightarrow \qquad 0 < \frac{e}{e+1}a < \frac{e}{e-1}a < 2\frac{e^2}{e^2-1}a. \end{aligned}$$

So the position of $F(0,0)$ and $F'\left(2\frac{e^2}{e^2-1}a, 0\right)$ with respect to the constant ratio curve $\mathcal{C}_V : x = \frac{e}{e\pm1}a$ look as in Figure 8.

Now consider a particle emanated from F. After hitting the constant ratio curve $\mathcal{C}_V$, will it proceed as is drawn in Figure 8? The difficulty in answering this question stems from the fact that the foot of perpendicular onto a vertical line is not unique. At this moment, we do not know how to answer this question.

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