Some Congruences for Andrews’ Partition Function $EO(n)$

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Abstract. Recently, Andrews introduced partition functions $EO(n)$ and $EO(n)$ where the function $EO(n)$ denotes the number of partitions of $n$ in which every even part is less than each odd part and the function $EO(n)$ denotes the number of partitions enumerated by $EO(n)$ in which only the largest even part appears an odd number of times. In this paper we obtain some congruences modulo 2, 4, 10 and 20 for the partition function $EO(n)$. We give a simple proof of the first Ramanujan-type congruences $EO(10n + 8) \equiv 0 \pmod{5}$ given by Andrews.

1. Introduction

A partition of a positive integer $n$ is a nonincreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. Let $p(n)$ be the number of partitions of $n$. For example $p(5) = 7$. The seven partitions of 5 are 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}},$$

where throughout this paper, for any complex numbers $a$ and $|q| < 1$ we define

$$(a; q)_n = (1 - a)(1 - aq)\cdots(1 - aq^{n-1}), \quad (a; q)_{\infty} = \prod_{k=0}^{\infty}(1 - aq^k).$$

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Almost a century back Ramanujan established the following identity [7],

\[
\sum_{n=0}^{\infty} p(5n+4)q^n = 5 \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6},
\]

which in fact implies Ramanujan’s congruences for \( p(n) \) modulo 5,

\[
p(5n + 4) \equiv 0 \pmod{5}.
\]

Recently, Andrews [2] introduced the partition function \( \mathcal{EO}(n) \) which counts the number of partitions of \( n \) in which every even part is less than each odd part. For example, \( \mathcal{EO}(6) = 7 \). The seven partitions of 6 it enumerates are 6, 5 + 1, 4 + 2, 3 + 3, 3 + 1 + 1 + 1, 2 + 2 + 2, 1 + 1 + 1 + 1 + 1 + 1. In [2], Andrews shows that the generating function for \( \mathcal{EO}(n) \) is

\[
\sum_{n=0}^{\infty} \mathcal{EO}(n)q^n := \frac{1}{(1-q)(q^4;q^2)_\infty^2}.
\]

Andrews [2], also defined the partition function \( \overline{EO}(n) \) which counts the number of partitions enumerated by \( \mathcal{EO}(n) \) in which only the largest even part appears an odd number of times. For example, \( \overline{EO}(6) = 4 \). The four partitions of 6 it enumerates are 6, 3 + 3, 2 + 2 + 2, 1 + 1 + 1 + 1 + 1 + 1. In [2], Andrews shows that the generating function for \( \overline{EO}(n) \) is

\[
\sum_{n=0}^{\infty} \overline{EO}(n)q^n = \frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2}.
\]

In Section 3 of this paper, we prove some congruences modulo 2 and 4 for the partition function \( \mathcal{EO}(n) \). In Section 4, we give a simple proof of Andrews’ congruences

\[
\overline{EO}(10n + 8) \equiv 0 \pmod{5},
\]

and we prove some interesting congruences modulo 10 and 20. In the Section 5, we consider

\[
\sum_{n=0}^{\infty} \mathcal{EO}_e(n)q^n := \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^2},
\]

where the function \( \mathcal{EO}_e(n) \) counts the elements in the set of partitions which are enumerated by \( \overline{EO}(n) \) together with the partitions enumerated by \( \mathcal{EO}(n) \) where all parts are odd and number of parts is even, i.e, \( \mathcal{EO}_e(n) \) denotes the number of partitions enumerated by \( \mathcal{EO}(n) \) in which only the largest even part appears an odd number of times except when parts are odd and number of parts is even. For example, \( \mathcal{EO}_e(6) = 6 \). The six partitions of 6 it enumerates are 6, 3 + 3, 2 + 2 + 2, 1 + 1 + 1 + 1 + 1 + 1 (which are counted by \( \overline{EO}(n) \)) and 5 + 1 and 3 + 1 + 1 + 1
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(only counted by $EO(n)$ in which all parts are odd and the number of parts is even). We prove some arithmetic properties modulo 2 satisfied by $EO(n)$. All of the proofs will follow from elementary generating function considerations and $q$-series manipulations. The paper concludes with a conjecture on $EO(n)$.

2. Preliminaries

We require the following definitions and lemmas to prove the main results in the next three sections. For $|ab| < 1$, Ramanujan’s general theta function $f(a, b)$ is defined as

$$(2.1) \quad f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$ 

Using Jacobi’s triple product identity [1, Theorem 2.8], (2.1) takes the shape

$$(2.2) \quad f(a, b) = (a; ab)_{\infty} (b; ab)_{\infty} (ab; ab)_{\infty}.$$ 

The special cases of $f(a, b)$ are

$$(2.3) \quad \phi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty} = \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2},$$ 

$$(2.4) \quad \psi(q) := f(q, q^4) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = (q^2; q^2)_{\infty} (q^2; q^2)_{\infty} = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}},$$ 

$$(2.5) \quad \phi(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q; q^2)_{\infty}^2}{(q^2; q^2)_{\infty}}.$$ 

**Lemma 2.1.** (Hirschhorn [6, p. 14, Eqn. 1.9.4]) We have the following 2-dissection of $\phi(q)$,

$$(2.6) \quad \phi(q) = \phi(q^4) + 2q\phi(q^8).$$ 

**Lemma 2.2.** (Hirschhorn [5] or Hirschhorn [6, p. 36, Eqn. 3.6.4]) We have,

$$(2.7) \quad (q; q)_{\infty}^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1) q^{(n^2+n)/2} 
\equiv f(-q^{10}, -q^{15}) - 3q f(-q^5, -q^{20}) \pmod{5}.$$
Lemma 2.3. (Hirschhorn [6, p. 105, Eqn. 10.7.6]) We have the following beautiful identity due to Ramanujan,

\[(q; q^4) \frac{(q^4; q^4)^2}{(q^2; q^2)^2} = \sum_{n=-\infty}^{\infty} (3n + 1)q^{3n^2 + 2n}.\]

From the Binomial Theorem, for any positive integer, \(k\),

\[(q^k; q^k)^5 \equiv (q^{5k}; q^k)^\infty \mod 5.\]

3. Congruences Modulo 2 and 4 for \(EO(n)\)

In this section we prove some congruences modulo 2 and 4 satisfied by \(EO(n)\). We require the following generating functions to prove congruences for \(EO(n)\).

Theorem 3.1. We have,

\[
\begin{align*}
(1) & \sum_{n=0}^{\infty} EO(4n)q^n = \frac{(q^4; q^4)^5}{(q; q^2)^{10}(q^2; q^2)^5}, \\
(2) & \sum_{n=0}^{\infty} EO(4n + 2)q^n = 2 \frac{(q^2; q^2)^{10}(q^8; q^8)^2}{(q; q^2)^5(q^4; q^4)^5}, \\
(3) & \sum_{n=0}^{\infty} EO(8n)q^n = \frac{(q^4; q^8)^2(q^4; q^8)^2}{(q; q^2)^{10}(q^4; q^4)^10}, \\
(4) & \sum_{n=0}^{\infty} EO(8n + 2)q^n = 2 \frac{(q^2; q^2)^5(q^8; q^8)^2}{(q; q^2)^5(q^4; q^4)^5}, \\
(5) & \sum_{n=0}^{\infty} EO(8n + 4)q^n = 4 \frac{(q^2; q^2)^5(q^4; q^4)(q^8; q^8)^2}{(q; q^2)^5}, \\
(6) & \sum_{n=0}^{\infty} EO(8n + 6)q^n = 4 \frac{(q^2; q^2)^5(q^4; q^4)(q^8; q^8)^2}{(q; q^2)^5}. \\
\end{align*}
\]

Proof. From (1.4), we have

\[
\sum_{n=0}^{\infty} EO(n)q^n = \frac{(q^4; q^4)^{10}}{(q^2; q^2)^{10}},
\]

since there are no terms on the right in which the power of \(q\) is odd, we have

\[EO(2n + 1) = 0,\]
thus by using (2.6), we obtain

\[
\sum_{n=0}^{\infty} \mathcal{E}(2n) q^n = \frac{(q^2; q^2)_{\infty}^3}{(q; q)_{\infty}^2} = \frac{(q^2; q^2)_{\infty}^3 (q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^2} \phi(q)
\]

(3.7)

\[
= \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^2} (\phi(q^4) + 2q \psi(q^8))
\]

It follows that

\[
\sum_{n=0}^{\infty} \mathcal{E}(4n) q^n = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^2} \phi(q^2) = \frac{(q^4; q^4)_{\infty}^5}{(q; q)_{\infty}^2 (q^8; q^8)_{\infty}^2}
\]

and

\[
\sum_{n=0}^{\infty} \mathcal{E}(4n + 2) q^n = 2 \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^2} \psi(q^4) = 2 \frac{(q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2},
\]

which is our (3.1) and (3.2). We have

\[
\sum_{n=0}^{\infty} \mathcal{E}(4n) q^n = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^2} \phi(q^2)
\]

\[
= \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^2} \phi(q^2) \phi(q)
\]

\[
= \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^2} \phi(q^2) \left( \phi(q^4) + 2q \psi(q^8) \right)
\]

(3.8)

It follows that

\[
\sum_{n=0}^{\infty} \mathcal{E}(8n) q^n = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^2} \phi(q) \phi(q^2)
\]

\[
= \frac{(q^4; q^4)_{\infty}^2 (q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2 (q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}
\]

\[
= \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2 (q^8; q^8)_{\infty}^2}
\]

and

\[
\sum_{n=0}^{\infty} \mathcal{E}(8n + 4) q^n = 2 \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^2} \phi(q) \psi(q^4) = 2 \frac{(q^2; q^2)_{\infty}^7 (q^8; q^8)_{\infty}^2}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2},
\]
which is our (3.3) and (3.5). We have

\[\sum_{n=0}^{\infty} \overline{EO}(4n + 2)q^n = 2 \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty^2} \psi(q^4)\]

\[= 2(q^2; q^2)_\infty^2 \psi(q^4) \frac{(q^4; q^4)_\infty^2}{(q^2; q^2)_\infty^2} \phi(q)\]

\[= 2 \frac{(q^4; q^4)_\infty^2}{(q^2; q^2)_\infty^2} \psi(q^4) \phi(q)\]

\[(3.9)\]

\[= 2 \frac{(q^4; q^4)_\infty^2}{(q^2; q^2)_\infty^2} \psi(q^4) \left( \phi(q^4) + 2q^2 \psi(q^8) \right)\].

It follows that

\[\sum_{n=0}^{\infty} \overline{EO}(8n + 2)q^n = 2 \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty^2} \psi(q^2) \phi(q^2) = 2 \frac{(q^4; q^4)_\infty^2}{(q; q)_\infty^2} \frac{(q^2; q^2)_\infty^2}{(q^2; q^2)_\infty^2} \frac{(q^8; q^8)_\infty^2}{(q^8; q^8)_\infty^2}\]

and

\[\sum_{n=0}^{\infty} \overline{EO}(8n + 6)q^n = 4 \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty^2} \psi(q^2) \phi(q^4) = 4 \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty^2} \frac{(q^4; q^4)_\infty^2}{(q^2; q^2)_\infty^2} \frac{(q^8; q^8)_\infty^2}{(q^8; q^8)_\infty^2},\]

which is our (3.4) and (3.6).

We have the following congruences.

**Corollary 3.2.** For all \( n \geq 0 \),

\[(3.10) \quad \overline{EO}(2n + 1) = 0,\]

\[(3.11) \quad \overline{EO}(4n + 2) \equiv 0 \pmod{2},\]

\[(3.12) \quad \overline{EO}(8n + 4) \equiv 0 \pmod{2},\]

\[(3.13) \quad \overline{EO}(8n + 6) \equiv 0 \pmod{4}.\]

**Remark 3.3.** The congruences (3.11)–(3.13) were obtained earlier by Andrews et al. [4]. Andrews et al. [3] introduced a partition function \( p_v(n) \) which counts the number of partitions of \( n \) in which the parts are distinct and all odd parts are less than twice the smallest part.

\[(3.14) \quad \sum_{n=0}^{\infty} p_v(n)q^n = \nu(-q),\]

where \( \nu(q) \) is a mock theta function. Andrews [2, Corollary 5.2] noted that

\[(3.15) \quad p_v(2n) = \overline{EO}(2n).\]

He proved the congruences using the properties of mock theta function, whereas we use the q-series identities.
4. Congruences Modulo 5, 10 and 20 for \(EO(n)\)

In this section we prove some congruences modulo 5, 10 and 20 for \(EO(n)\). In the next theorem, we give a simple proof of the Andrews’ result [2, Eqn. 1.6], which can be tracked back to [3, Thrm. 6.7]. He used the properties of mock theta functions to prove the congruence, whereas we manipulate the q-series identities to get the result.

**Theorem 4.1.** For all \(n \geq 0\),

\[
EO(10n + 8) \equiv 0 \pmod{5}.
\]

**Proof.** Applying (2.9) in (1.4), we obtain

\[
\sum_{n=0}^{\infty} EO(2n)q^n = \frac{(q^2; q^2)_\infty^3}{(q; q)_\infty^2} \frac{1}{(q; q^5)_\infty} \equiv \frac{(q^2; q^2)_\infty^3(q; q^3)_\infty^3}{(q^2; q^2)_\infty^3} \pmod{5}.
\]

From (2.7), we have

\[
(q; q)_\infty^3 \equiv J_0 + J_1 \pmod{5},
\]

where \(J_i\) contains terms in which the power of \(q\) is congruent to \(i\) modulo 5, then

\[
(q^2; q^2)_\infty^3 \equiv J_0^* + J_2^* \pmod{5},
\]

where \(J_i^*\) contains terms in which the power of \(q\) is congruent to \(i\) modulo 5. Substituting (4.3) and (4.4) in (4.2), we have

\[
\sum_{n=0}^{\infty} EO(2n)q^n \equiv \frac{1}{(q^2; q^2)_\infty^2} (J_0 + J_1) (J_0^* + J_2^*) \pmod{5}.
\]

There are no terms on the right in which the power of \(q\) is 4 modulo 5, so

\[
\sum_{n=0}^{\infty} EO(2(5n + 4))q^{5n+4} \equiv 0 \pmod{5},
\]

from which we deduce (4.1). \(\square\)

In the next theorem, we derive two congruences modulo 10 from the generating functions (3.2) and (3.5).

**Theorem 4.2.** For all \(n \geq 0\),

\[
EO(20n + 18) \equiv 0 \pmod{10},
\]

\[
EO(40n + 28) \equiv 0 \pmod{10}.
\]
Proof. Using (2.9) in (3.2), we have

$$\sum_{n=0}^{\infty} \mathcal{O}(4n+2)q^n = 2 \frac{1}{(q; q)_\infty^2} \frac{(q^2; q^2)_\infty^2 (q^5; q^5)_\infty^2}{(q^4; q^4)_\infty}$$

Replacing $q$ by $q^2$ in (2.8), we have

$$
\sum_{n=0}^{\infty} \mathcal{O}(4n+2)q^n \equiv 2 \frac{1}{(q; q)_\infty^2} \frac{(q^2; q^2)_\infty^2 (q^5; q^5)_\infty^2}{(q^4; q^4)_\infty} \quad (\text{mod } 10).
$$

(4.8)

Substituting (4.9) and (4.10) in (4.11), we obtain

$$\sum_{n=0}^{\infty} \mathcal{O}(4n+2)q^n \equiv 2 \frac{1}{(q; q)_\infty^2} \frac{(q^2; q^2)_\infty^2 (q^5; q^5)_\infty^2}{(q^4; q^4)_\infty} \quad (\text{mod } 10),$$

(4.10)

where $R_i^*$ contains terms in which the power of $q$ is congruent to $i$ modulo 5.

Substituting (4.3) and (4.9) in (4.8), we obtain

$$\sum_{n=0}^{\infty} \mathcal{O}(4n+2)q^n \equiv 2 \frac{1}{(q; q)_\infty^2} (J_0 + J_1) (R_0^* + R_1^* + R_2^*) \quad (\text{mod } 10),$$

(4.12)

Substituting (4.10) into (4.11), we obtain

$$\sum_{n=0}^{\infty} \mathcal{O}(4n+2)q^n \equiv 2 \frac{1}{(q; q)_\infty^2} (J_0 + J_1) (R_0^* + R_1^* + R_2^*) \quad (\text{mod } 10),$$

(4.13)
There are no terms on the right in which the power of $q$ is 3 modulo 5, so

$$
\sum_{n=0}^{\infty} \overline{EO}(8(5n + 3) + 4)q^{5n+3} \equiv 0 \pmod{10},
$$

from which we deduce (4.7).

In the next theorem, we derive a congruences modulo 20 from the generating function (3.6).

**Theorem 4.3.** For all $n \geq 0$,

$$
\overline{EO}(40n + 38) \equiv 0 \pmod{20},
$$

**(4.14)**

**Proof.** Using (2.9) in (3.6), we have

$$
\sum_{n=0}^{\infty} \overline{EO}(8n + 6)q^n = 4 \frac{1}{(q;q)_\infty} \frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty} \frac{(q^8;q^8)_\infty}{(q^4;q^4)_\infty} = 4 \frac{1}{(q;q)_\infty} \frac{(q;q)_\infty^2 (q^2;q^4)_\infty^2}{(q^2;q^2)_\infty} \frac{(q^4;q^4)_\infty^2}{(q^4;q^4)_\infty} \left(R_0 + R_2 + R_3\right) \equiv 0 \pmod{20}.
$$

**(4.15)**

From (2.8), we have

$$
\frac{(q;q^2)_\infty^2 (q^4;q^4)_\infty^2}{(q^2;q^2)_\infty} = \sum_{n=-\infty}^{\infty} (3n + 1)q^{3n^2+2n} \equiv R_0 + R_2 + R_3 \pmod{5},
$$

where $R_i$ contains terms in which the power of $q$ is congruent to $i$ modulo 5. Substituting (4.9) and (4.16) in (4.15), we obtain

$$
\sum_{n=0}^{\infty} \overline{EO}(8n + 6)q^n \equiv 4 \frac{1}{(q^5;q^5)_\infty} (R_0 + R_2 + R_3) (R'_0 + R'_2 + R'_3) \pmod{20}.
$$

**(4.17)**

There are no terms on the right in which the power of $q$ is 4 modulo 5, so

$$
\sum_{n=0}^{\infty} \overline{EO}(8(5n + 4) + 6)q^{5n+4} \equiv 0 \pmod{20},
$$

from which we deduce (4.14).

\QED
5. Congruences for $\mathcal{EO}_e(n)$

In this section we prove some congruences modulo 2 for $\mathcal{EO}_e(n)$.

**Theorem 5.1.**

(5.1) \[
\sum_{n=0}^{\infty} \mathcal{EO}_e(4n)q^n = \frac{(q^4; q^4)_\infty^5}{(q; q)_\infty^3 (q^8; q^8)_\infty^2},
\]

(5.2) \[
\sum_{n=0}^{\infty} \mathcal{EO}_e(4n + 2)q^n = 2 \frac{(q^2; q^2)_\infty^2 (q^8; q^8)_\infty^2}{(q; q)_\infty^3 (q^4; q^4)_\infty^1}.
\]

**Proof.** From (1.5), we have

\[
\sum_{n=0}^{\infty} \mathcal{EO}_e(n)q^n = \frac{(q^4; q^4)_\infty}{(q^2; q^2)_\infty},
\]

since there are no terms on the right in which the power of $q$ is odd, we have

\[
\mathcal{EO}_e(2n + 1) = 0,
\]

by using (2.6), we obtain

\[
\sum_{n=0}^{\infty} \mathcal{EO}_e(2n)q^n = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty^3} = \frac{(q^4; q^4)_\infty^2}{(q^2; q^2)_\infty^3} \phi(q)
\]

(5.3) \[
= \frac{(q^4; q^4)_\infty^2}{(q^2; q^2)_\infty^3} \left( \phi(q^4) + 2q \psi(q^8) \right).
\]

It follows that

\[
\sum_{n=0}^{\infty} \mathcal{EO}_e(4n)q^n = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty^3} \phi(q^2) = \frac{(q^4; q^4)_\infty^5}{(q; q)_\infty^3 (q^8; q^8)_\infty^2}
\]

and

\[
\sum_{n=0}^{\infty} \mathcal{EO}_e(4n + 2)q^n = 2 \frac{(q^2; q^2)_\infty^2 (q^8; q^8)_\infty^2}{(q; q)_\infty^3 (q^4; q^4)_\infty^1},
\]

which is our (5.1) and (5.2). \qed

We have the following congruences.

**Corollary 5.2.** For all $n \geq 0$,

(5.4) \[
\mathcal{EO}_e(2n + 1) = 0,
\]

(5.5) \[
\mathcal{EO}_e(4n + 2) \equiv 0 \pmod{2}.
\]
6. Conclusion

Andrews [2, Problem 4], proposed to further investigate the properties of $EO(n)$. We conclude the paper with the following conjecture. Using maple, we found the following congruences hold up to $n = 2000$.

**Conjecture 6.1.** For all $n \geq 0$,

\[
(6.1) \quad EO(50n + 18) \equiv 0 \pmod{20}, \\
(6.2) \quad EO(50n + 28) \equiv 0 \pmod{20}, \\
(6.3) \quad EO(50n + 38) \equiv 0 \pmod{20}, \\
(6.4) \quad EO(50n + 48) \equiv 0 \pmod{20}.
\]

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