CORRIGENDUM TO “A DUAL ITERATIVE SUBSTRUCTURING METHOD WITH A SMALL PENALTY PARAMETER”, [J. KOREAN MATH. SOC. 54 (2017), NO. 2, 461–477]

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Abstract. In this corrigendum, we offer a correction to [J. Korean Math. Soc. 54 (2017), No. 2, 461–477]. We construct a counterexample for the strengthened Cauchy–Schwarz inequality used in the original paper. In addition, we provide a new proof for Lemma 5 of the original paper, an estimate for the extremal eigenvalues of the standard unpreconditioned FETI-DP dual operator.

In the first and second authors’ previous work [4], the strengthened Cauchy–Schwarz inequality used for [4, Eq. (3.8)] is incorrect and consequently, the statement of [4, Lemma 4] needs to be corrected. We present a new proof for [4, Lemma 5], that does not use [4, Lemma 4]. All notations are adopted from the original paper [4].

In the paragraph containing [4, Eq. (3.8)], it was claimed that by deriving a strengthened Cauchy-Schwarz inequality in a similar way to Lemma 4.3 in [3], it is shown that there exists a constant γ such that

$$2\tilde{a}(v_I + v_\Delta, v_c) \geq -\gamma(\tilde{a}(v_I + v_\Delta, v_I + v_\Delta) + \tilde{a}(v_c, v_c)),$$

where $0 < \gamma < 1$ is independent of $H$ and $h$. That is, the above inequality is true when there exists a constant γ such that

$$|\tilde{a}(v_I + v_\Delta, v_c)| \leq \gamma(\tilde{a}(v_I + v_\Delta, v_I + v_\Delta))^{1/2}(\tilde{a}(v_c, v_c))^{1/2},$$

where $0 < \gamma < 1$ is independent of $h$ and $H$.

On the other hand, a specific function $w = w_I + w_c + w_\Delta$ can be constructed, for which γ approaches 1 as $H$ decreases. In fact, it suffices to characterize such $w_\Delta$ because $w_I$ and $w_c$ in (1) are determined by $w_\Delta$ in terms of the discrete $\tilde{a}$-harmonic extension $H^c(w_\Delta)$.

**Proposition 1.** There is no γ ($0 < \gamma < 1$), independent of $h$ and $H$, satisfying (1).

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Proof. Noting that $H^c(v_{\Delta})$ in $X^c_h$ is $\tilde{a}(\cdot,\cdot)$-orthogonal to all the functions which vanish at the interface nodes except for the subdomain corners, we have that
\[
\tilde{a}(v_I + v_{\Delta}, v_c) = \tilde{a}(H^c(v_{\Delta}) - v_c, v_c) \\
= \tilde{a}(H^c(v_{\Delta}), v_c) - \tilde{a}(v_c, v_c) \\
= -\tilde{a}(v_c, v_c),
\]
which implies that for $\tilde{a}(v_I + v_{\Delta}, v_I + v_{\Delta}) \neq 0$, the estimate (1) is equivalent to
\[
(2) \quad \frac{\tilde{a}(v_c, v_c)}{\tilde{a}(v_I + v_{\Delta}, v_I + v_{\Delta})} \leq \gamma^2,
\]
where $\gamma < 1$ is independent of $h$ and $H$.

Next, let us divide $\Omega = (0,1)^2$ into $1/H \times 1/H$ square subdomains with a side length $H$. Each subdomain is partitioned into $2 \times H/h \times H/h$ uniform right triangles. Associated with such a triangulation, we select the function $w$ in $X^c_h$ such that $w$ is a conforming $P_1$ element function in each subdomain, and $w_{\Delta} = 1$ at all the nodes on the interface except for the subdomain corners. Then it is noted that $w$ in $X^c_h$ vanishes on $\partial \Omega$. Let us denote by $\{x_k\}$ the subdomain corners that are not on $\partial \Omega$. Hence, for $w_c$ and $w_I$ that are computed by the discrete harmonic extension of $w_{\Delta}$, it is observed that
\[
(3a) \quad w_c = 1 \text{ at all } x_k, \\
(3b) \quad w_I = 1 \text{ in } \Omega_j \text{ for } \partial \Omega_j \cap \partial \Omega = \emptyset,
\]
which imply that
\[
(4) \quad w \equiv 1 \text{ in all subdomains whose boundary does not touch } \partial \Omega.
\]

Let us first estimate $\tilde{a}(w_c, w_c)$ in (2). Using (3a), we have that
\[
\tilde{a}(w_c, w_c) = (1/H - 1)^2 \sum_{k=1}^{(1/H - 1)^2} \tilde{a}(\phi_{c,k}, \phi_{c,k}) = 4 \left(\frac{1}{H} - 1\right)^2,
\]
where $\phi_{c,k}$ is the nodal basis function associated with $x_k$. We next look over $\tilde{a}(w_I + w_{\Delta}, w_I + w_{\Delta})$ based on the fact that, for $\partial \Omega_j \cap \partial \Omega = \emptyset$
\[
(5) \quad \tilde{a}_{\Omega_j}(w_I + w_{\Delta}, w_I + w_{\Delta}) = \int_{\Omega_j} |\nabla(w_I + w_{\Delta})|^2 dx = \int_{\Omega_j} |\nabla w_c|^2 dx = 4,
\]
which follows from (4). Hence it suffices to estimate $\tilde{a}_{\Omega_j}(w_I + w_{\Delta}, w_I + w_{\Delta})$ for the following two cases:
(i) only one of the edges of the subdomain $\Omega_j$ is on $\partial \Omega$.
(ii) two edges of the subdomain $\Omega_j$ are on $\partial \Omega$.

Here, the number of subdomains corresponding to the cases (i) and (ii) is $4 \left(\frac{1}{H} - 2\right)$ and 4, respectively. Let us take $H/h = 3$ to focus only on the
dependence of $\gamma$ on either $H$ or $h$. By finding the discrete local harmonic extensions for the cases (i) and (ii), it is computed directly that

\begin{equation}
\tilde{a}_{\Omega_j}(w_I + w_\Delta, w_I + w_\Delta) = \begin{cases} 
\frac{17}{4} & \text{for the case (i)}, \\
\frac{14}{4} & \text{for the case (ii)}.
\end{cases}
\end{equation}

Then by using (5) and (6), it follows that

\begin{equation}
\tilde{a}(w_I + w_\Delta, w_I + w_\Delta) = \left( \sum_{j \text{ for } \partial \Omega_j \cap \partial \Omega = \emptyset} + \sum_{j \text{ for } \partial \Omega_j \cap \partial \Omega \neq \emptyset} \right) \tilde{a}_{\Omega_j}(w_I + w_\Delta, w_I + w_\Delta)
= 4 \left( \frac{1}{H} - 2 \right)^2 + 17 \left( \frac{1}{H} - 2 \right) + 14.
\end{equation}

Finally, from (3a) and (7), it is confirmed that for a function $w$ given above,

\begin{equation}
\lim_{H \to 0} \frac{\tilde{a}(w_c, w_c)}{\tilde{a}(w_I + w_\Delta, w_I + w_\Delta)} = 1,
\end{equation}

which implies that (2) does not hold. Therefore, the proof is completed. \(\square\)

In [4, Lemma 5], the extremal eigenvalues of the FETI-DP dual operator $F = B_\Delta S^{-1} B_\Delta^T$ were estimated using [4, Lemma 4]. Since [4, Lemma 4] is incorrect, we provide a new estimate for $F$ that does not utilize [4, Lemma 4].

We assume that each subdomain $\Omega_j$ is the union of elements in a conforming coarse mesh $T_H$ of $\Omega$. First, we consider the following Poincaré-type inequality that generalizes [4, Proposition 3].

**Lemma 2.** For any $v_j \in X^j_h$, let $I^H_j v_j$ be the linear coarse interpolation of $v_j$ such that $I^H_j v_j = v_j$ at vertices of a subdomain $\Omega_j \subset \mathbb{R}^d$. Then we have

\begin{equation}
|v_j|^2_{H^1(\Omega_j)} \lesssim \begin{cases} 
H^{-1} \left( 1 + \ln \frac{H}{h} \right)^{-1} \|v_j - I^H_j v_j\|_{L^2(\partial \Omega_j)}^2 & \text{for } d = 2, \\
h^{-1} \left( \frac{H}{h} \right)^{-2} \|v_j - I^H_j v_j\|_{L^2(\partial \Omega_j)}^2 & \text{for } d = 3.
\end{cases}
\end{equation}

**Proof.** Note that both sides of the above inequality do not change if a constant is added to $v_j$. Without loss of generality, we assume that $v_j$ has the zero average, so that the following Poincaré inequality holds:

\begin{equation}
\|v_j\|_{H^1(\Omega_j)} \lesssim |v_j|_{H^1(\Omega_j)},
\end{equation}

where $\| \cdot \|_{H^1(\Omega_j)}$ is the weighted $H^1$-norm on $\Omega_j$ given by

$$
\|v_j\|^2_{H^1(\Omega_j)} = |v_j|^2_{H^1(\Omega_j)} + \frac{1}{H^2} \|v_j\|^2_{L^2(\Omega_j)}.
$$
Since $I_H^j v_j$ attains its extremum at vertices, we have
\[
\|v_j - I_H^j v_j\|_{L^2(\partial \Omega_j)} \lesssim H^{d-1} \|v_j - I_H^j v_j\|_{L^\infty(\partial \Omega_j)} \leq H^{d-1} \left( \|v_j\|_{L^\infty(\partial \Omega_j)} + \|I_H^j v_j\|_{L^\infty(\partial \Omega_j)} \right) \lesssim H^{d-1} \|v_j\|_{L^\infty(\partial \Omega_j)}.
\]
(9)

Let $H_j v_j$ be the generalized harmonic extension of $v_j|_{\partial \Omega_j}$ introduced in [7] such that
\[
(H_j v_j) = v_j \text{ on } \partial \Omega_j
\]
and
\[
\|H_j v_j\|_{H^1(\Omega_j)} = \min_{w_j \in H^1(\Omega_j)} \|w_j\|_{H^1(\Omega_j)} \text{ with } w_j = v_j \text{ on } \partial \Omega_j
\]
(10)

Then it follows that
\[
H^{d-1} \|v_j\|_{L^\infty(\partial \Omega_j)} \leq H^{d-1} \|H_j v_j\|_{L^\infty(\partial \Omega_j)} \lesssim C_d(H, h) \|v_j\|_{L^\infty(\partial \Omega_j)}
\]
(11a)
\[
\leq C_d(H, h) \|H_j v_j\|_{L^\infty(\partial \Omega_j)} \lesssim C_d(H, h) \|v_j\|_{H^1(\Omega_j)}
\]
(11b)
\[
\lesssim C_d(H, h) \|v_j\|_{H^1(\Omega_j)} \lesssim C_d(H, h) \|v_j\|_{H^1(\Omega_j)}
\]
(11c)

where
\[
C_d(H, h) = \begin{cases} 
H \left(1 + \ln \frac{H}{h}\right) & \text{for } d = 2, \\
h \left(\frac{H}{h}\right)^2 & \text{for } d = 3,
\end{cases}
\]
and (11a) is due to the discrete Sobolev inequality [2, Lemma 2.3]. Also (10) and (8) are used in (11b) and (11c), respectively. Combination of (9) and (11) completes the proof. \qed

Note that Lemma 2 reduces to [4, Proposition 3] when $v_j$ vanishes at vertices of $\Omega_j$ so that $I_H^j v_j = 0$. Using Lemma 2, we obtain the following estimate for $F$.

**Proposition 3.** For $F = B_\Delta S^{-1} B_\Delta^T$, we have
\[
C_F \lambda^T \lambda \lesssim \lambda^T F \lambda \lesssim \overline{C}_F \lambda^T \lambda \quad \forall \lambda,
\]
where
\[
C_F = h^{2-d} \text{ for } d = 2, 3,
\]
and
\[
\overline{C}_F = \begin{cases} 
\left(\frac{H}{h}\right)^2 \left(1 + \ln \frac{H}{h}\right) & \text{for } d = 2, \\
h^{-1} \left(\frac{H}{h}\right)^2 & \text{for } d = 3.
\end{cases}
\]
Consequently, the condition number of $F$ satisfies the following bound:
\[
\kappa(F) \lesssim \begin{cases} 
\left(\frac{H}{h}\right)^2 \left(1 + \ln \frac{H}{h}\right) & \text{for } d = 2, \\
\left(\frac{H}{h}\right)^2 & \text{for } d = 3.
\end{cases}
\]
Proof. As the derivation of the maximum eigenvalue of $S$ in the original paper [4] is correct, the derivation of $C_F$ is also correct. Thus, we only estimate $C_F$ in the following.

We first prove that

$$\left( B_\Delta v_\Delta \right)^T B_\Delta v_\Delta \lesssim C_F v_\Delta^T S v_\Delta \quad \forall v_\Delta.$$  \hspace{1cm} (12)

For $v_\Delta$, we consider the discrete $a$-harmonic extension $v = H^c(v_\Delta)$. Let $w = v - I^H v$, where $I^H v$ is the linear coarse interpolation of $v$ onto $T^H$ such that $I^H v = v$ at the subdomain vertices. We write $w = w_I + w_\Delta$. Since $I^H v$ is continuous along $\Gamma$, we have $B_\Delta w_\Delta = B_\Delta v_\Delta$. Then it follows that

$$\left( B_\Delta v_\Delta \right)^T B_\Delta v_\Delta = \left( B_\Delta w_\Delta \right)^T (B_\Delta w_\Delta) \lesssim \sum_{j<k} \left\| w_j \left\| \Gamma_{jk} - w_k \right\| \Gamma_{jk} \right\|^2$$

where the last inequality is due to Lemma 2.

Then similar to [5, Theorem 4.4], we get the desired result as follows:

$$\lambda^T F \lambda = \max_{v_\Delta \neq 0} \frac{\left( (B_\Delta v_\Delta)^T \lambda \right)^2}{v_\Delta^T S v_\Delta} \lesssim C_F \max_{\mu \neq 0} \frac{\left( (B_\Delta v_\Delta)^T \lambda \right)^2}{\mu^T B_\Delta v_\Delta} \leq C_F \max_{\mu \neq 0} \frac{\left( \mu^T \lambda \right)^2}{\mu^T \mu} = C_F \lambda^T \lambda,$$

where we used [5, Lemma 4.3] in the first equality. Consequently, this completes the proof. \qed

It must be mentioned that the conclusion of Proposition 3 agrees with Lemma 5 of the original paper [4]. Since the conclusion of [4, Lemma 5] is true, it requires no additional correction in the remaining part of that paper.

For the sake of completeness, we present a correct estimate for the extremal eigenvalues of $S$ that replaces [4, Lemma 4].
Proposition 4. For $S = A_{\Delta} - A_{\Delta}^T A_{\Delta}^{-1} A_{\Delta}$, we have
\[
C_S v_{\Delta}^T v_{\Delta} \lesssim v_{\Delta}^T S v_{\Delta} \lesssim C_S v_{\Delta}^T v_{\Delta}, \quad \forall v_{\Delta},
\]
where
\[
C_S = \begin{cases} 
H h (1 + \ln \frac{H}{h})^{-1} & \text{for } d = 2, \\
h^3 & \text{for } d = 3,
\end{cases}
\]
and
\[
C_S = h^{d-2} \text{ for } d = 2, 3.
\]

Proof. Since the derivation of $C_S$ in the original paper [4] is correct, we only consider an estimate for $C_S$. Take any $v_{\Delta}$ and its corresponding finite element function $v_{\Delta}$. Let $v = \mathcal{H}(v_{\Delta})$ be the discrete $\bar{a}$-harmonic extension of $v_{\Delta}$. Proceeding as in [6, Lemma 4.11], we get
\[
v_{\Delta}^T v_{\Delta} \lesssim h^{1-d} \sum_{j=1}^{N_{\Delta}} \|v_{\Omega_j}\|^2_{L^2(\Omega_j)}
\lesssim H h^{1-d} \sum_{j=1}^{N_{\Delta}} \left(\|v\|^2_{H^1(\Omega_j)} + H^{-2} \|v\|^2_{L^2(\Omega_j)}\right)
= H h^{1-d} v_{\Delta}^T S v_{\Delta} + H^{-1} h^{1-d} \|v\|^2_{L^2(\Omega)}.
\]
Note that we cannot apply the discrete Poincaré inequality [1, Lemma 5.1] in each subdomain $\Omega_j$ since $\mathcal{H} v_{\Delta}$ does not vanish at the subdomain vertices in general.

It remains to show that
\[
\|v\|^2_{L^2(\Omega)} \lesssim \begin{cases} 
(1 + \ln \frac{H}{h}) v_{\Delta}^T S v_{\Delta} & \text{for } d = 2, \\
h^3 v_{\Delta}^T S v_{\Delta} & \text{for } d = 3,
\end{cases}
\]
for $d = 2$ and [6, Lemma 4.12] for $d = 3$, respectively. This completes the proof. \(\square\)
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