A BOUND ON HÖLDER REGULARITY FOR $\overline{\partial}$-EQUATION ON PSEUDOCONVEX DOMAINS IN $\mathbb{C}^n$ WITH SOME COMPARABLE EIGENVALUES OF THE LEVI-FORM

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Abstract. Let $\Omega$ be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and assume that the $(n-2)$-eigenvalues of the Levi-form are comparable in a neighborhood of $z_0 \in \partial \Omega$. Also, assume that there is a smooth 1-dimensional analytic variety $V$ whose order of contact with $\partial \Omega$ at $z_0$ is equal to $\eta$ and $\Delta_{n-2}(z_0) < \infty$. We show that the maximal gain in Hölder regularity for solutions of the $\overline{\partial}$-equation is at most $\frac{1}{\eta}$.

1. Introduction

Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ and assume that $z_0 \in \partial \Omega$. Suppose that there exist a neighborhood $U$ of $z_0$ and a constant $C > 0$ so that for each $v \in L^0_{\infty}(\Omega)$ with $\overline{\partial}v = 0$, there is a $u \in L^2(\Omega) \cap \Lambda_\kappa(U \cap \Omega)$ such that $\overline{\partial}u = v$ in $\Omega$ and
\[
\|u\|_{\Lambda_\kappa(U \cap \overline{\Omega})} \leq C\|v\|_{L^\infty(\Omega)},
\]
for some $\kappa > 0$, where $\Lambda_\kappa(S)$ denotes the Hölder space of order $\kappa$ on $S$. In this event, we say the Hölder estimates of order $\kappa$ hold on $U$.

When $\Omega$ is a bounded strongly pseudoconvex domain in $\mathbb{C}^n$, (1.1) holds for $\kappa = \frac{1}{2}$ [10]. For weakly pseudoconvex domain in $\mathbb{C}^n$, however, (1.1) is known only for some special cases. Namely, pseudoconvex domains of finite type in $\mathbb{C}^2$ [12, 13], convex finite type domains in $\mathbb{C}^n$ [9], etc. Therefore, the Hölder estimate for general pseudocovex domains in $\mathbb{C}^n$ is one of the big questions in several complex variables.

Meanwhile, it is of great interest to find a necessary condition or optimal possible gain of $\kappa > 0$ in (1.1). Normally this question depends on the boundary geometry of $\Omega$ near $z_0 \in \partial \Omega$. Several authors have obtained necessary conditions for Hölder regularity of $\overline{\partial}$ on restricted classes of domains [11–14].

Let $\Delta_q(z)$ denote the D’Angelo’s finite $q$-type at $z$, and let $\Delta_q^{Reg}(z)$ be the “regular finite $q$-type”, which is defined by the maximum order of contact...
of non-singular $q$-dimensional varieties [8]. Note that $\Delta_p(z) \leq \Delta_q(z)$ (and $\Delta_p^{Reg}(z) \leq \Delta_q^{Reg}(z)$) if $p \geq q$, $\Delta_q^{Reg}(z) \leq \Delta_q(z)$, and $\Delta_q^{Reg}(z)$ is a positive integer.

When $\Delta_{n-1}(z_0) := m_{n-1} < \infty$, Krantz [11] showed that $\kappa \leq \frac{1}{m_{n-1}}$. Krantz's result is sharp for $\Omega \subset \mathbb{C}^2$, and when $\alpha$ is a $(0, n - 1)$-form. In [12], McNeal proved sharp Hölder estimates for $(0, 1)$-form $\alpha$ under the condition that $\Omega$ has a holomorphic support function at $z_0 \in \Omega$. Note that the existence of holomorphic support function is satisfied for restricted domains and it is often the first step to prove the Hölder estimates for the $\overline{\partial}$-equation [13]. In the rest of this section, we let $\Omega$ be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function $r$, that is, $\Omega = \{z : r(z) < 0\} \subset \mathbb{C}^n$.

**Definition 1.1.** Let $\lambda_1(z), \ldots, \lambda_{n-1}(z)$ be the nonnegative eigenvalues of the Levi-form, $\partial \overline{\partial} r(z)$. We say the eigenvalues $\{\lambda_k : k = s, \ldots, s+l\}$ are comparable in a neighborhood of $z_0 \in \partial \Omega$ if for all constants $c, C > 0$ such that

$$c\lambda_j(z) \leq \lambda_k(z) \leq C\lambda_j(z), \quad j, k = s, \ldots, s + l, \quad z \in U.$$ 

**Definition 1.2.** We say that a 1-dimensional analytic variety $V$ has order of contact $\eta$ at $z_0 \in \partial \Omega$ if there are constants $c, C > 0$ such that

$$c|z - z_0|^\eta \leq |r(z)| \leq C|z - z_0|^\eta$$

for all $z \in V$ sufficiently close to $z_0$.

**Example.** Let $\Omega \subset \mathbb{C}^4$ be a domain defined by

$$\Omega = \{z : r(z) = 2Rez_4 + |z_1|^10 + (|z_2|^2 + |z_3|^2)^{11/3} < 0\}.$$ 

Then, $\Delta_1(0) = 10 = \Delta_2^{Reg}(0)$, $\Delta_2(0) = \frac{22}{7}$, and $V = \{(t, 0, 0, 0) : |t| \leq a\}$ is a smooth variety whose order of contact with $\partial \Omega$ at 0 is 10. Set $L_j = \frac{\partial}{\partial z_j} - (\frac{\partial r}{\partial z_4})^{-1} \frac{\partial}{\partial z_4}$, $j = 1, 2, 3$. Then, the eigenvalues $\lambda_k(z) \approx \partial \overline{\partial} r(z)(L_k, \overline{L_k})$, $k = 2, 3$, are comparable near 0.

In this paper, we want to study a necessary condition for the Hölder estimates of the $\overline{\partial}$ equation when $(n-2)$-eigenvalues of the Levi-form are comparable and $\Delta_{n-2}(z_0) < \infty$:

**Theorem 1.3.** Let $\Omega$ be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 3$, and assume that there is a smooth 1-dimensional variety whose order of contact at $z_0 \in \partial \Omega$ is $\eta < \infty$. Also, assume that the $(n-2)$-eigenvalues of the Levi-forms are comparable in a neighborhood of $z_0 \in \partial \Omega$ and $\Delta_{n-2}(z_0) < \infty$.

If there exist a neighborhood $U$ of $z_0$ and a constant $C > 0$ so that for each $v \in L^1_{\infty}(\Omega)$ with $\overline{\partial} v = 0$, there is a $u \in L^2(\Omega) \cap \Lambda_{n-2}(U \cap \Omega)$ such that $\overline{\partial} u = v$ on $\Omega$ and

$$\|u\|_{\Lambda_{n-2}(U \cap \Omega)} \leq C\|v\|_{L^\infty(\Omega)},$$

then $\kappa \leq \frac{1}{7}$.
Let $z = (z_1, \ldots, z_n)$ be local coordinates about $z_0$. In the rest of this paper, we set $z' = (z_2, \ldots, z_n)$, $z'' = (z_2, \ldots, z_{n-1})$, and the same notations will be used for other coordinates or multi-indices, $\alpha = (\alpha_1, \ldots, \alpha_n)$, that is, $\alpha' = (\alpha_2, \ldots, \alpha_n)$, and $\alpha'' = (\alpha_2, \ldots, \alpha_{n-1})$, etc.

**Remark 1.4.** (1) Since $V$ is a smooth analytic variety, we note that $\eta$ is a positive integer and $\Delta_{n-1}(z_0) := m_{n-1} \leq \eta$. Thus, we have $\kappa \leq \frac{1}{\eta} \leq \frac{1}{m_{n-1}}$ in (1.2) which improves Krantz’s result.

(2) In following, we will fix $z_1$ and consider the $z_1$ slice of $\Omega$:

$$\Omega_{z_1} := \{(z_1, z') : (z_1, z') \in \Omega\}. \tag{1.3}$$

Then, $\Omega_{z_1}$ can be regarded as a bounded pseudoconvex domain in $\mathbb{C}^{n-1}$. Since the $(n - 2)$-eigenvalues of the Levi-form are comparable, the condition $\Delta_{n-2}(z_0) < \infty$ will play as the role of the condition $\Delta_1(z_0) < \infty$ on each $\Omega_{z_1}$.

(3) If $n = 3$, the comparable eigenvalues condition of the Levi form holds vacuously. In this case, You [14] proved Theorem 1.3. Note that $\Delta_2(z_0) \leq \Delta_1^R(z_0)$ when $n = 3$. Consider the domain in $\mathbb{C}^3$ (see [8]) defined by

$$r(z) = \text{Re}z_3 + |z_1^3 - z_2^3|^2.$$ 

Then $\Delta_1^{R\Re}(0) = 6$, and $\Delta_2(0) = 4$ while $\Delta_1(0) = \infty$ as the complex analytic curve $\gamma(t) = (t, t^2, 0)$ lies in the boundary. Note that $\gamma(t)$ is not a smooth curve.

(4) Whenever we have $(n - 2)$-positive eigenvalues, these eigenvalues are comparable and hence Theorem 1.3 implies the results in [7] where we assumed that we have $(n - 2)$-positive eigenvalues and $\Delta_1(z_0) < \infty$.

In Section 2, we construct special coordinates at each reference point and show that the $z_1$-coordinate represents the given variety $V$, and the $z''$-directions represent the comparable Levi-form directions. Let $C_b(z_0, \delta_0)$ denote the curve close to the $z_1$-direction as defined in (2.8). To prove the main theorem (Theorem 1.3), for each small $\delta > 0$, we need to construct a uniformly bounded holomorphic function $f_\delta$ on $\Omega$ that satisfies

$$\frac{\partial f_\delta}{\partial z_n}(z_3) \geq \frac{1}{\delta} \tag{1.4}$$

for each $z_3 \in C_b(z_0, \delta_0)$.

In Section 2, we fix $z_1 = \tilde{z}_1$ near $z_1 = \delta_4^4$ and consider the sliced domain $\Omega_{z_1}$. Then, we construct a family of plurisubharmonic functions with maximal Hessian on each thin neighborhood of $\partial \Omega_{z_1}$ as in [1] for $n = 2$ case, and then show a bumping theorem. In Section 3, we push out the boundary of the domain $\Omega_{z_1}$ as far as possible at each reference point $\tilde{z}_3 \in \partial \Omega_{z_1}$. These are some of the main ingredients for a construction of $f_\delta$ in (1.4). Section 4 is devoted to proving Theorem 1.3.

**Remark 1.5.** Note that the bumping theorem or pushing out the domains are done for the domains with $\Delta_1(z_0) < \infty$ [2, 3, 5]. In this paper, the condition
\[ \Delta_1(z_0) < \infty \] is replaced by the conditions \( \Delta_{n-2}(z_0) < \infty \) and the compatibility of the \((n-2)\)-eigenvalues.

2. Special coordinates and polydiscs

In the sequel, we assume that \( \Omega \) is a smoothly bounded pseudoconvex domain in \( \mathbb{C}^n \), \( n \geq 3 \), with smooth defining function \( r_0 \) and that there is a smooth 1-dimensional holomorphic curve \( V \) whose order of contact with \( b\Omega \) at \( z_0 \in b\Omega \) is equal to \( \eta \) and \( \Delta_{n-2}(z_0) < \infty \). We also assume that the \((n-2)\)-eigenvalues of the Levi-form are comparable in a neighborhood \( W \) of \( z_0 \). We may assume that there are coordinate functions \( \tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_n) \) near \( z_0 \) such that \( \tilde{z}(z_0) = 0 \) and \( |\partial r_0/\partial \tilde{z}_n| \geq c_0 \) in \( W \) for some fixed constant \( c_0 > 0 \).

Using these \( \tilde{z} \)-coordinates, set

\[ L_n = \frac{\partial}{\partial \tilde{z}_n} \quad \text{and} \quad L_k = \frac{\partial}{\partial \tilde{z}_k} - \left( \frac{\partial r_0}{\partial \tilde{z}_n} \right)^{-1} \frac{\partial r_0}{\partial \tilde{z}_k}, \quad k = 1, \ldots, n, \]

set

\[ \alpha = (\alpha_{ij})_{2 \leq i, j \leq n} \text{ are comparable.} \]

and assume that the eigenvalues of the matrix \( A := (\alpha_{ij})_{2 \leq i, j \leq n} \) are comparable. Let \( m \) be the smallest integer bigger than or equal to \( \Delta_{n-2}(z_0) \) (\( \Delta_{n-2}(z_0) \) could be a rational number). Here we may also assume that \( \eta \geq m \). As in Proposition 2.3 in [6], we can prove that there are coordinate functions \( z = (z_1, \ldots, z_n) \) near \( z_0 = 0 \) such that the given smooth one dimensional variety \( V \) can be regarded as the \( z_1 \)-axis:

**Proposition 2.1.** Let \( \Omega, r_0, z_0 \in b\Omega \) and \( W \ni z_0 \) be as above. There is a biholomorphism \( \Phi_0 : \mathbb{C}^n \rightarrow \mathbb{C}^n \), \( \Phi_0(z) = \tilde{z} \), \( \Phi_0(0) = z_0 \) such that in terms of \( z \)-coordinates, \( r(z) := r_0 \circ \Phi_0(z) \) can be written as

\[
\begin{align*}
\quad r(z) &= \Re z_n + \sum_{j+k = \eta, j,k>0} a_{j,k} z_1^j \tilde{z}_1^k + \sum_{\alpha'' + \beta'' \leq m, \alpha''' + \beta''' > 0} b_{\alpha'' \beta''} z_1^{\alpha''} \tilde{z}_1^{\beta''} \\
&\quad + \sum_{\alpha'' + \beta'' \leq m, 1 \leq |\alpha'''| + |\beta'''|} c_{\alpha'' \beta''} z_1^{\alpha''} \tilde{z}_1^{\beta''} + O(E_{m, \eta}(z)),
\end{align*}
\]

where \( E_{m, \eta}(z) = |z||z_n| + |z_1|^\eta + |z''|^m \), and \( r(z) \) satisfies

\[
|t|^\eta \leq |r(t, 0, \ldots, 0, 0)| \leq C|t|^\eta
\]

for some constants \( c, C > 0 \).

**Remark 2.2.** (1) Let \( d_0(z_1) := \sum_{j+k = \eta} a_{j,k} z_1^j \tilde{z}_1^k \) be the first sum in (2.1). Then it follows from (2.1) and (2.2) that

\[
|d_0(z_1)| \approx |r(z_1, 0')| \approx |z_1|^\eta.
\]
(2) The coordinate change in Proposition 2.1 is about $z_0 = 0 \in b\Omega$, but not about arbitrary point $\tilde{z} \in W$.

In the rest of this section, we fix $\delta > 0$ and assume that $\tilde{z} = (\tilde{z}_1, \tilde{z}''_1, \tilde{z}_n) \in W$ satisfies

$$|\tilde{z}_1 - \tilde{z}_1| < \gamma \delta^\frac{1}{2}$$

for a sufficiently small $\gamma > 0$. Let us fix $\hat{z}_1$ satisfying (2.4) and consider the $\hat{z}_1$-slice defined in (1.3). Then for each $z'$ with $(\hat{z}_1, \hat{z}') \in W$, we can remove the pure terms in the $z''$ (or $\hat{z}''$) variables inductively in the Taylor series expansion of $r_{z_2} = r(\hat{z}_1, \cdot)$ as in the proof of Proposition 1.1 in [1]:

**Proposition 2.3.** For each fixed $\hat{z} = (\hat{z}_1, \hat{z}') \in W$, where $\hat{z}_1$ satisfies (2.4), there exist numbers $d_\alpha^\nu(\hat{z})$, depending smoothly on $\hat{z}$, such that in the new coordinates $\zeta = (\hat{z}_1, \zeta')$ defined by

$$z = (z_1, \Phi_2(\zeta')) = (\tilde{z}_1, \tilde{z}''_1, \hat{z}_n + \Phi_n(\zeta')),$$

where

$$\Phi_n(\zeta') = \left(\frac{\partial r}{\partial z_n}(\hat{z})\right)^{-1} \left(\frac{\zeta_n}{2} - \sum_{l=1}^m \sum_{|\alpha^\nu| = 1} d_{\alpha^\nu}(\hat{z})\zeta^{\nu}\right),$$

and the function $\rho(\hat{z}_1, \zeta') := r \circ (\hat{z}_1, \Phi_2(\zeta'))$ satisfies

$$\rho(\hat{z}_1, \zeta') = r(\hat{z}) + R \zeta_n + \sum_{|\alpha^\nu + \beta'| \leq m} c_{\alpha^\nu, \beta'}(\hat{z})\zeta^{\nu}\zeta' + O(E(\hat{z}_1, \zeta')),$$

where $E(\hat{z}_1, \zeta') = |\zeta_n| + |\hat{z}_1|^m + |\zeta'|^{m+1}$.

**Remark 2.4.** (1) Set $2\kappa_0 := \max_{|\alpha^\nu + \beta'| \leq m} |c_{\alpha^\nu, \beta'}(\hat{z}_0)|$. Since $\Delta_{\nu-2}(z_0) \leq m$, we have $\kappa_0 > 0$. Therefore it follows that

$$\max_{|\alpha^\nu + \beta'| \leq m} |c_{\alpha^\nu, \beta'}(\hat{z})| \geq \kappa_0 > 0,$$

independent of $\hat{z}$ provided $W$ is sufficiently small because $c_{\alpha^\nu, \beta'}(\hat{z})$ are smooth in $\hat{z}$.

(2) By setting $\zeta_1 = \tilde{z}_1$ and $\zeta = (\hat{z}_1, \zeta')$, we may regard that $\Phi_2 : C^\nu \rightarrow C^n$, that is,

$$\Phi_2(\zeta) = (\hat{z}_1, \zeta').$$

(3) For each $z = (z_1, \tilde{z}'_1, z_n) \in W$, define $\pi(z) := (z_1, \tilde{z}'_1, \pi_n(z)) \in b\Omega$, where $\pi_n(z)$ is the projection onto $b\Omega$ along the $z_n$ direction. For each $\hat{z}_1$ satisfying (2.4), set $\tilde{z} = (\hat{z}_1, 0')$ and set $\hat{z}_1 = \pi(\hat{z}) = (\tilde{z}_1, 0', \pi_n(\tilde{z})) \in b\Omega$. Using a Taylor series in the variable $z_n$ about $\pi_n(\tilde{z})$, we see that

$$r(\hat{z}_1, 0') = 2Re\left[\frac{\partial r}{\partial z_n}(\pi_n(\hat{z}_1))\right] + O(\pi_n(\hat{z}_1)^2).$$

Since $|\pi_n(\tilde{z})| \ll 1$ and $2|\frac{\partial r}{\partial z_n}| = 1 + O(|\tilde{z}|) \geq \frac{1}{2}$ on $W$, it follows from (2.3) that

$$|\pi_n(\tilde{z})| \approx |r(\hat{z}_1, 0')| \approx |d_0(\tilde{z}_1)| \approx |\tilde{z}_1|^n.$$
For each small $\delta > 0$, set $\tilde{z}_\delta = (\delta^{1/2}, 0')$ (i.e., $\tilde{z}_1 = \delta^{1/2}$) and set

$$z_\delta := \pi(\tilde{z}_\delta) := (\delta^{1/2}, 0', \pi_n(\tilde{z}_\delta)) \in b\Omega.$$  

(2.7)

For a sufficiently small $b > 0$, set $z_b := (\delta^{1/2}, 0', \pi_n(\tilde{z}_\delta) - b\delta) \in \Omega$, and set

$$C_b(z_0, \delta_0) := \{z_b : 0 \leq \delta \leq \delta_0\} \cup \{z_0\} \subset \Omega \cup \{z_0\},$$

where $\delta_0 > 0$ is a sufficiently small number such that $z_\delta \in W$ for all $0 \leq \delta \leq \delta_0$.

We will use the methods developed in [4–6] on each domain $\Omega_{2}z_1$, keeping track of the dependence of $\tilde{z}_1$ variable. For each $z = (\tilde{z}_1, \tilde{z}') \in W$, set

$$C_{s_2}(\tilde{z}) := \max\{|e_n,\beta''(\tilde{z})| : |\alpha'' + \beta''| = s_2\},$$

where $e_n,\beta''(\tilde{z})$ is defined in (2.5), and for each $\epsilon > 0$, define

$$\tau(\tilde{z}, \epsilon) = \min_{2 \leq s_2 \leq m} \{(\epsilon/C_{s_2}(\tilde{z}))^{1/s_2}\}.$$  

(2.10)

Note that $\tau(\tilde{z}, \epsilon)$ is well defined by (2.6) and it follows from (2.9) and (2.10) that

$$\epsilon^{1/2} \leq \tau(\tilde{z}, \epsilon) \leq \epsilon^{1/m}, \text{ and}$$

$$(\epsilon'/\epsilon)^{1/2} \tau(\tilde{z}, \epsilon') \leq \tau(\tilde{z}, \epsilon') \leq (\epsilon'/\epsilon)^{1/m} \tau(\tilde{z}, \epsilon), \text{ if } \epsilon' < \epsilon.$$

In the sequel, set $\tilde{z}_1 = (\tilde{z}_1, 0')$. Note that $\Phi_{\tilde{z}}(\tilde{z}_1) = \tilde{z}$. For each $c > 0$ and $\epsilon > 0$, define

$$R_{c\epsilon}^z(\tilde{z}) = \{|\zeta| : |\zeta_1 - \tilde{z}_1| < c\delta^{1/2}, |\zeta_k| < c\tau(\tilde{z}, \epsilon), k = 2, \ldots, n - 1, |\zeta_n| < c\epsilon\},$$

and set

$$Q_{c\epsilon}^z(\tilde{z}) = \{(\zeta_1, \Phi_{\tilde{z}}(\zeta')) : (\zeta_1, \zeta') \in R_{c\epsilon}^z(\tilde{z})\}.$$  

(2.11)

Also, we set

$$Q'_{c\epsilon}^z(\tilde{z}) = \{(\tilde{z}_1, \zeta_2, \ldots, \zeta_n) : |\zeta_k| < c\tau(\tilde{z}, \epsilon), k = 2, \ldots, n - 1, |\zeta_n| < c\epsilon\},$$

a polydisc in the $\zeta'$ variables, and

$$Q''_{c\epsilon}^z(\tilde{z}) = \{(\tilde{z}_1, \Phi_{\tilde{z}}(\zeta')) : (\tilde{z}_1, \zeta') \in Q'_{c\epsilon}^z(\tilde{z})\}.$$  

As in Proposition 1.7 in [1], there exists an independent constant $C > 0$ such that if $z = (\tilde{z}_1, z') \in Q''_{c\epsilon}^z(\tilde{z})$, then

$$Q''_{c\epsilon}^z(\tilde{z}) \subset Q''_{c\epsilon}^z(\tilde{z}), \text{ and } Q''_{c\epsilon}^z(\tilde{z}) \subset Q''_{c\epsilon}^z(\tilde{z}).$$

In view of (2.6), we note that the same inclusion relations hold if we fix $\tilde{z}'$ and vary $\tilde{z}_1$. Thus, we obtain that

$$Q''_{c\epsilon}^z(\tilde{z}) \subset Q''_{c\epsilon}^z(\tilde{z}), \text{ and } Q''_{c\epsilon}^z(\tilde{z}) \subset Q''_{c\epsilon}^z(\tilde{z}), \text{ if } z \in Q''_{c\epsilon}^z(\tilde{z}).$$

Again, by (2.6), we also have the following equivalence relations for $\tau(\tilde{z}, \epsilon)$ (Proposition 2.14 in [6]).

**Proposition 2.5.** Assume $z = (\tilde{z}_1, z') \in Q''_{c\epsilon}^z(\tilde{z})$. Then

$$\tau(\tilde{z}, \epsilon) \approx \tau(\tilde{z}, \epsilon)$$

(2.12)

for all sufficiently small $c > 0$, independent of $\delta > 0$ and $\epsilon > 0$. 

In the sequel, set $D_k = \frac{\partial}{\partial x_k}$ or $\frac{\partial}{\partial \bar{x}_k}$, $1 \leq k \leq n$, and set $\tau_1 = \delta^{1/\nu}$. Recall that $\zeta = (\hat{\zeta}, 0')$. Combining (2.4), (2.9) and (2.10), the error term $E(\hat{\zeta}, \zeta')$ in (2.5) satisfies

\begin{equation}
|D^{i_1}_1E(\zeta)| \lesssim \tau_1^{n+1-l_1} = \delta^{l_1+1}, \quad \text{and} \quad D^{i_1}_1D^{\nu''}E(\zeta) = 0, \text{ if } 0 < |\nu''| \leq m.
\end{equation}

**Proposition 2.6.** Assume $\hat{\zeta} = (\hat{\zeta}, 0') \in W$ satisfies (2.4) and assume that $|r(\hat{\zeta})| \lesssim \delta$. For each $l_1$, and for each multi index $\nu'' = (\nu_2, \ldots, \nu_{n-1})$ with $0 < |\nu''| \leq m$, we have

\begin{equation}
|D^{i_1}_1\rho(\zeta)| \lesssim \delta \tau_1^{-l_1}, \quad \text{and} \quad |D^{\nu''}_1\rho(\zeta)| \lesssim \epsilon r(\hat{\zeta}, \epsilon)^{-|\nu''|}.
\end{equation}

**Proof.** From (2.1), (2.2) and (2.13), it follows that

\begin{equation}
|D^{i_1}_1\rho(\zeta)| = |D^{i_1}_1r(\zeta)| \lesssim \delta \tau_1^{-l_1},
\end{equation}

and the second estimates follows from (2.5), (2.9), (2.10) and (2.13). \qed

For each fixed $\delta > 0$, set $\hat{\zeta}_1 = \delta^{1/\nu}$ and consider $\delta^{1/\nu}$-slice of $\Omega$, $\Omega_{\delta^{1/\nu}}$. For convenience of notation, set $\Omega_{\delta} = \Omega_{\delta^{1/\nu}}$. Then $\Omega_{\delta}$ is a smoothly bounded pseudoconvex domain in $\mathbb{C}^{n-1}$ with comparable Levi-form near $\hat{z}_{\delta} \in b\Omega_{\delta}$ where $\hat{z}_{\delta} = \pi(\delta^{1/\nu}, 0')$ is defined in (2.7). Since $\Delta_{n-2}(\hat{z}_{\delta}) \leq m$, and the Levi-forms are comparable, it follows that $\Delta_1(\hat{z}_{\delta}) \leq m$ (Proposition 2.12 in [6]).

To push out the domain $\Omega_{\delta}$ as far as possible at the reference point $\hat{z}_{\delta} \in b\Omega_{\delta} \cap W$, we need to construct bounded plurisubharmonic functions with maximal Hessian in a thin strip neighborhood of $b\Omega_{\delta}$ as in Theorem 3.1 in [1]. Set $r_{\delta}(z') = r(\delta^{1/\nu}, z')$, and for each small $\epsilon > 0$, define

\begin{align*}
\Omega_{\delta}^\epsilon &= \{ (\delta^{1/\nu}, z') : r_{\delta}(z') < \epsilon \}, \\
S_{\delta}(\epsilon) &= \{ (\delta^{1/\nu}, z') : -\epsilon < r_{\delta}(z') < \epsilon \}, \quad \text{and} \\
S_{\delta}^-(\epsilon) &= \{ (\delta^{1/\nu}, z') : -\epsilon < r_{\delta}(z') \leq 0 \}.
\end{align*}

Using the estimates (2.12) and (2.14), we can prove the following theorem as in the proof of Theorem 3.1 in [5]:

**Proposition 2.7.** For all small $\epsilon > 0$, there is a plurisubharmonic function $\lambda_{\delta} \in C^\infty(W \cap \Omega_{\delta})$ with the following properties:

(i) $|\lambda_{\delta}'(z)| \leq 1$, $z = (\delta^{1/\nu}, z') \in \Omega_{\delta} \cap W$;

(ii) for all $L' = \sum_{k=2}^n a_k L_k$ at $z = (\delta^{1/\nu}, z') \in S_{\delta}^-(\epsilon) \cap W$,

\[ \partial \bar{\partial} \lambda_{\delta}'(L', \bar{L})'(z) \approx \tau(z, \epsilon)^{-2} \sum_{k=2}^{n-1} |a_k|^2 + \epsilon^{-2} |a_n|^2, \quad \text{and} \]
(iii) if \( \Phi_2(\zeta') \) is the map associated with a given \( \tilde{z} = (\delta \tilde{z}, z') \in S_3(\epsilon) \cap W \), then
\[
|D^{\alpha'} \Phi_2(\zeta')| \leq C_\alpha \epsilon^{-\alpha \cdot \tau} |\zeta'|^{-|\alpha'|}
\]
holds for all \( \zeta' \in R_\epsilon(z) \) where \( \alpha' = (\alpha_2, \ldots, \alpha_n) \), and \( \alpha'' = (\alpha_2, \ldots, \alpha_{n-1}) \), and \( R_\epsilon(z) \) is defined in (2.11).

Remark 2.8. In Theorem 2.3 of [2], the author proved a bumping theorem near a point \( z_0 \in \Omega \) of finite 1-type. All we need for that theorem is the existence of a family of plurisubharmonic functions with maximal Hessian on each thin strip \( S_\epsilon(\epsilon) \) as stated above in Proposition 2.7. Since \( \Delta_{n-2}(\tilde{z}) \leq m \) and the Levi-form is comparable, it follows that \( \Delta_1(\tilde{z}) \leq m \) (Proposition 2.12 in [6]).

Recall that \( \tilde{z}_t = \pi(\delta \tilde{z}, \theta, t) \in B \Omega \) defined in (2.7). In the sequel, for each \( \tilde{z} = (\tilde{z}_1, \tilde{z}') \), set \( B'_{2r}(\tilde{z}) = \{(\tilde{z}_1, \tilde{z}') : |\tilde{z}' - \tilde{z}_1'| < c \} \), \( c > 0 \). Using the family of plurisubharmonic functions \( \lambda_\alpha \) in Proposition 2.7, we have the following bumping theorem for each \( \Omega_3 \) as in [2]:

**Theorem 2.9.** Let \( V \subset \subset W \) be a small neighborhood of \( z_0 \in B \Omega \). There exists an independent constant \( r_0 > 0 \) such that for each \( \tilde{z}_t \in V \cap \partial \Omega_3 \), we have \( B'_{2r_0}(\tilde{z}) \subset \subset W \cap \{ (\delta \tilde{z}, z') : b \supset \subset C^\infty \} \), and there is a smooth 1-parameter family of pseudoconvex domains \( \Omega_\delta^t \), \( 0 \leq t < t_0 \), called the bumping family of \( \Omega_3 \) with front \( B'_{2r_0}(\tilde{z}) \), each defined by \( \Omega_\delta^t = \{ (\delta \tilde{z}, z') : r_\delta^t(z') < 0 \} \) where \( r_\delta^t(z') = r^t(\delta, z') \) has the following properties:

1. \( r_\delta^t(z') \) is smooth in \( z = (\delta, z') \in W \) and in \( t \) for \( 0 \leq t < t_0 \).
2. \( r_\delta^t(z') = r_\delta(z') \) for \( (\delta \tilde{z}, z') \notin B_{2r_0}(\tilde{z}) \).
3. \( \partial r_\delta^t(z') \leq 0 \).
4. \( r\partial r_\delta^t(z') = r\partial z_\delta(z') \).
5. For \( z = (\delta \tilde{z}, z') \in B_{2r_0}(\tilde{z}) \), \( \partial z_\delta^t(z') \) \( < 0 \).

3. A construction of special functions

In this section, we construct a family of uniformly bounded holomorphic functions \( \{ f_\delta \}_{\delta > 0} \) with large derivatives in the \( z_n \)-direction along the curve \( C_\delta(z_0, \delta_0) \subset \Omega \) defined in (2.8). Let us fix \( \delta > 0 \) for a while and concentrate on the point \( \tilde{z}_\delta \in \bar{M}_\delta \) defined in (2.7) where \( \Omega_\delta := \Omega_{\delta n} = \{ (\delta \tilde{z}, z') : (\delta \tilde{z}, z') \in \Omega \} \).

For a construction of \( \{ f_\delta \}_{\delta > 0} \), we use “Bumping theorem” in Theorem 2.9 as well as pushing out \( \partial \Omega_\delta \) as far as possible at each reference point \( \tilde{z}_\delta \).

Recall the function \( \Phi_2(\zeta) = (\delta \tilde{z}, \Phi_{\tilde{z}n}(\zeta')) \) defined in Proposition 2.3. Set \( \tilde{W}_\delta = W \cap \{(\delta^{1/n}, z') : z' \in \mathbb{C}^{n-1} \} \), \( \Omega_\delta' = (\Phi_{\tilde{z}n})^{-1}(\Omega_\delta) \) and set
\[
W_\delta' = (\Phi_{\tilde{z}n})^{-1}(\tilde{W}_\delta).
\]
Then \( \Omega_\delta' \) is a smoothly bounded pseudoconvex domain in \( \mathbb{C}^{n-1} \) and the \( n-2 \)-eigenvalues are uniformly comparable, and the estimate (2.6) holds uniformly, independent of \( \delta > 0 \). We want to construct a domain \( D_\delta' \subset \mathbb{C}^{n-1} \) which
contains $\Omega'_\delta$ such that the boundary of $D'_\delta$ is pushed out essentially as far as possible near $\zeta^\delta = (\delta^\frac{1}{2}, 0') = (\Phi_{z_\delta})^{-1}(\tilde{z}_\delta) \in b\Omega'_\delta$, so that $bD'_\delta$ is pseudoconvex.

Set

$$J_\delta(\zeta') = \left( \delta^2 + |\zeta'|^2 + \sum_{2 \leq k_2 < m} C_{s_2}(\tilde{z}_\delta)|\zeta''|^2 \right)^\frac{1}{2},$$

where $C_{s_2}(\tilde{z}_\delta)$ is defined in (2.9), and let $r_0 > 0$ be the constant in Theorem 2.9. Note that $B'_{2r_0} \subset W'_{\delta}$. For each small $\epsilon > 0$, set

$$W'_{\delta,\epsilon} = \{ (\delta^{1/n}, \zeta') \in W'_\delta : \rho(\delta^\frac{1}{2}, \zeta') < \epsilon J_\delta(\zeta') \} \cap B'_{r_0}(\tilde{z}_\delta).$$

If we use the family $\{ \lambda_\delta \}$ constructed in Proposition 2.7, and follow the methods in Section 4 of [1], we can show that $W'_{\delta,\epsilon}$ is the maximally pushed out domain of $\Omega'_\delta$ near $\zeta^\delta$ such that

$$bW'_{\delta,\epsilon} := \{ (\delta^{1/n}, \zeta') \in W'_\delta : \rho(\delta^\frac{1}{2}, \zeta') = \epsilon J_\delta(\zeta') \} \cap B'_{r_0}(\tilde{z}_\delta)$$

is pseudoconvex for all sufficiently small $\epsilon > 0$.

To connect the pushed out part $W'_{\delta,\epsilon}$ and $\Omega'_\delta$, we use the bumping family $\{ \Omega'_{\delta} \}$ with front $D'_{2r_0}(\tilde{z}_\delta)$ as in Theorem 2.9. Set

$$D'_{t,\delta,e} := (\Omega'_{\delta} \setminus B'_{r_0}(\tilde{z}_\delta)) \cup (W'_{\delta,\epsilon} \cap \Omega'_{\delta}).$$

Then $D'_{t,\delta,e}$ becomes a pseudoconvex domain which is pushed out near $\zeta^\delta = (\Phi_{z_\delta})^{-1}(\tilde{z}_\delta)$ provided $t > 0$ and $\epsilon > 0$ are sufficiently small. In the sequel, we fix these $t = t_0$ and $\epsilon = \epsilon_0$ and set $D'_\delta := D'_{t_0,\delta,\epsilon_0}$. Note that these choices of $t_0$ and $\epsilon_0 > 0$ are independent of $\delta > 0$. If we use the methods in Section 6 of [1] (or Section 3 of [4]), we see that there exists a $L^2(D'_\delta)$ holomorphic function $f_\delta$ satisfying

$$\left| \frac{\partial f_\delta}{\partial n_{z_\delta}}(\tilde{z}_\delta) \right| \geq \frac{1}{\delta},$$

independent of $\delta$, where $z_\delta = (\delta^\frac{1}{2}, 0', \pi_{z_\delta}(\tilde{z}_\delta) - b\delta) \in C(z_0, \delta_0)$, and where $b > 0$ is taken so that $C_b(z_0, \delta_0) \subset \Omega$. Note that $f_\delta$ is independent of $z_1$ variable. We will show that $f_\delta$ is holomorphic in a domain including the $z_1$ direction near $z_1 = \delta^\frac{1}{2}$.

Recall that $\Omega'_\delta$ or $D'_\delta$ can be regarded as domains in $\mathbb{C}^{n-1}$ by fixing $\zeta_1 = \delta^\frac{1}{2}$. In terms of the special coordinates $\zeta = (\bar{\zeta}, \zeta')$ defined in Proposition 2.3, set

$$P_{c_1,\delta}(\tilde{z}_\delta) := \{ \zeta : |\zeta_1 - \delta^\frac{1}{2}| < \epsilon_0 c_1 \delta^\frac{1}{2}, \; |\zeta_k| < \frac{r_0}{2n^{k-1}} \; k = 2, \ldots, n \},$$

where $r_0$ is the constant fixed in Theorem 2.9, and set

$$\Omega_{c_1,\delta}(\tilde{z}_\delta) = P_{c_1,\delta}(\tilde{z}_\delta) \cap \{ \zeta : \rho(\zeta) < 0 \} \subset \Omega.$$

Also, for each $\delta > 0$, $\epsilon > 0$, and $c_1 > 0$, set

$$\Omega'_{c_1,\delta}(\tilde{z}_\delta) = P_{c_1,\delta}(\tilde{z}_\delta) \cap \{ (\zeta_1, \zeta') : \rho(\delta^\frac{1}{2}, \zeta') < \epsilon J_\delta(\zeta') \} \subset \mathbb{C}^n.$$
Then $\Omega_{c_1, \delta}^\xi (\tilde{z}_\delta)$ is obtained by moving $W'_{\delta, e}$ along the $\zeta_1$ direction.

**Lemma 3.1.** For sufficiently small $c_1 > 0$, we have $\Omega_{c_1, \delta} (\tilde{z}_\delta) \subset \Omega_{c_1, \delta}^{1/2} (\tilde{z}_\delta)$, or equivalently,

\[(3.3) \quad \rho(\delta^{1/2}, \zeta') - \rho(\zeta) < \varepsilon \frac{\varepsilon}{2} J_\delta (\zeta') \quad \text{for} \quad \zeta = (\zeta_1, \zeta'') \in (\Omega_{c_1, \delta} (\tilde{z}_\delta)).\]

**Proof.** Assume $\zeta = (\zeta_1, \zeta'') \in \Omega_{c_1, \delta} (\tilde{z}_\delta)$. Then

\[(3.4) \quad |\rho(\zeta) - \rho(\delta^{1/2}, \zeta'')| \leq c_1 \delta^{1/2} \max_{|\zeta_1 - \delta^{1/2}| \leq c_1} |D_1 \rho(\zeta_1, \zeta'')|.\]

Note that $\Phi_{\delta, \xi}$ is independent of $\zeta_1 = z_1$. Since $\rho(\zeta) = \tau \o (\zeta_1, \Phi_{\delta, \xi} (\zeta''))$, it follows from (2.5), (2.14), (3.1), and a Taylor series that

\[(3.5) \quad |D_1 \rho(\zeta_1, \zeta'')| \lesssim \delta^{1/2} \lesssim \delta^{1/2} J_\delta (\zeta').\]

Combining (3.4) and (3.5), we obtain (3.3) provided $c_1 > 0$ is sufficiently small. \qed

If we use the standard inequality:

\[ab \leq \theta a^p + b^{a/p} \theta^q, \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{for all} \quad \theta, a, b > 0,
\]

one obtains that

\[(3.6) \quad (a + b)^s \leq 2a^s + (s!)^{s-1} b^s, \quad s \geq 1.
\]

Since $f_\delta$ is independent of $\zeta_1$, we see that $f_\delta$ is holomorphic on $\Omega_{c_1, \delta}^\xi (\tilde{z}_\delta)$. We will show that $f_\delta$ is bounded uniformly on $\Omega_{c_1, \delta}^{1/4}$ for some $a_1, 0 < a_1 < c_1 \leq \frac{2}{2\pi}$, to be determined. For each $q = (q_1, q') \in \Omega_{a_1, \delta}^{1/4}$, set $\tau_1 = \delta^{1/2}, \tau_k = \tau (z_k, J_\delta (q'))$, $2 \leq k \leq n - 1, \tau_n = J_\delta (q')$, and define a non-isotropic polydisc $Q_{a_1}^\delta (q)$ by

\[Q_{a_1}^\delta (q) := \{ \zeta : |\zeta - z_k| < a_1 \tau_k, 1 \leq k \leq n \}.
\]

**Lemma 3.2.** There is an independent constant $0 < a_1 < c_1$ such that

\[(3.7) \quad Q_{a_1}^\delta (q) \subset \Omega_{a_1, \delta}^{1/4} \quad \text{for} \quad q = (q_1, q') \in \Omega_{a_1, \delta}^{1/4}.
\]

**Proof.** Assume $\zeta \in Q_{a_1}^\delta (q)$. Then, it follows from (2.9), and (2.10) that

\[(3.8) \quad C_{a_1} (\tilde{z}_\delta)^2 |\zeta'' - q''|^{2s_2} \leq (n - 2)^{s_2} a_1^{2s_2} C_{a_1} (\tilde{z}_\delta)^2 \tau (z_k, J_\delta (q'))^{2s_2}
\]

\[\leq (n - 2)^{s_2} a_1^{2s_2} J_\delta (q').
\]
and $|c_n - q_n|^2 \leq a_1^2 J_\delta(q')^2$. Thus, it follows from (3.1), (3.6), and (3.8) that

$$J_\delta(q')^2 = \delta^2 + |q_n|^2 + \sum_{s_2=2}^m C_{s_2}(\tilde{z}_\delta)^2|q''|^{2s_2}$$

$$\leq \delta^2 + 2|\zeta_n|^2 + 2|c_n - q_n|^2$$

$$+ \sum_{s_2=2}^m C_{s_2}(\tilde{z}_\delta)^2(2|\zeta''|^{2s_2} + ((2s_2)!)^{2s_2-1}|\zeta''|^{2s_2})$$

$$\leq 2J_\delta(\zeta')^2 + [2mn^m((2m)!)^{2m-1}a_1^2] J_\delta(q')^2.$$  

If we take $a_1 > 0$ so that $2mn^m((2m)!)^{2m-1}a_1^2 \leq \frac{c}{\delta}$, we obtain that $J_\delta(q') \leq 2J_\delta(\zeta')$. By the same argument, we have $J_\delta(\zeta') \leq 2J_\delta(q')$. Therefore we obtain that

$$\frac{1}{2} J_\delta(q') \leq J_\delta(\zeta') \leq 2J_\delta(q') \quad \text{for } \zeta \in Q_{a_1}^\delta(q).$$

Assume $q = (q_1, q') \in \Omega_{a_1}^{1/4}$ and $\zeta \in Q_{a_1}^\delta(q)$. Then, $\rho(\delta^{\frac{\dagger}{2}}, \zeta') \leq \frac{c}{4} J_\delta(q')$. Thus, we have

$$\rho(\delta^{\frac{\dagger}{2}}, \zeta') \leq \frac{c}{4} J_\delta(q') + |\nabla' \rho(\delta^{\frac{\dagger}{2}}, \zeta') \cdot (\zeta' - q')|$$

for some $(\delta^{\frac{\dagger}{2}}, \zeta') \in Q_{a_1}^\delta(q)$ where $\nabla'$ denotes the gradient of the $\zeta'$ variables. From (2.9), (2.10), (2.14) (with $\epsilon$ replaced by $J_\delta(q')$), we obtain that

$$|D_k \rho(\delta^{\frac{\dagger}{2}}, \zeta')| \leq J_\delta(q')\tau(\tilde{z}_\delta, J_\delta(q')^{-1}), \quad (\delta^{\frac{\dagger}{2}}, \zeta') \in Q_{a_1}^\delta(q),$$

for $2 \leq k \leq n - 1$, and $|D_n \rho| \lesssim 1$. Combining (3.9)–(3.11), we obtain that

$$\rho(\delta^{\frac{\dagger}{2}}, \zeta') \leq \frac{c}{2} J_\delta(\zeta') + C_2a_1 J_\delta(q') < cJ_\delta(\zeta'),$$

if we take $a_1 > 0$ so that $4C_2a_1 < c$. Therefore, $\zeta \in \Omega_{a_1}^{1/4}$ proving (3.7). \qed

**Remark 3.3.** In the above discussion, $\epsilon > 0$ is any number such that $0 < \epsilon \leq \epsilon_0$. Thus, in particular, we can fix $\epsilon = \epsilon_0$ where $\epsilon_0$ is fixed before (3.2).

**Theorem 3.4.** $f_\delta$ is a bounded holomorphic function in $\Omega_{a_1, \delta}^{1/4}$ and satisfies

$$\left| \frac{\partial f_\delta}{\partial \zeta_n}(z_\delta) \right| \geq \frac{1}{\delta}, \quad z_\delta \in C_{\delta}(z_0, \delta_0),$$

independent of $\delta$.

**Proof.** By (3.2) and (3.3), we already know that there is a $L^2$ holomorphic function $f_\delta$ on $\Omega_{a_1, \delta}^\delta(z_\delta)$ satisfying the estimate (3.12). We only need to show that $f_\delta$ is bounded in $\Omega_{a_1, \delta}^{1/4}$. Assume $q \in \Omega_{a_1, \delta}^{1/4} \subset \Omega_{c_1, \delta}$, where $0 < a_1 < c_1$. Then $Q_{a_1}^\delta(q) \subset \Omega_{c_1, \delta}^\delta \subset \Omega_{c_1, \delta}$ by Lemma 3.2. Now if we use the mean value theorem on polydisc $Q_{a_1}^\delta(q) \subset \Omega_{c_1, \delta}$ and the fact that $f_\delta \in L^2(\Omega_{c_1, \delta}^\delta)$ is holomorphic, we will get the boundedness of $f_\delta$ on $\Omega_{a_1, \delta}^{1/4}$. \qed
4. Proof of Theorem 1.3

The proof is similar to that in [7]. We will sketch the proof briefly here. Let $c_1 > 0$, and $a_1 > 0$ be the constants fixed in Lemma 3.1 and Lemma 3.2 respectively. We may assume that $0 < 2b < a_1 \leq c_1$. For each $\delta > 0$, let $f_\delta$ be the function defined in Theorem 3.4. Therefore, $f_\delta$ is $L^2$ holomorphic on $\Omega_{c_1, \delta}^e(\xi_0)$, bounded on $\overline{\Omega}_{a_1, \delta}$, independent of $\zeta_1$ variable, and satisfies the estimates in (3.12). Set

$$g_\delta = \phi \left( \frac{|\zeta_1 - \delta^\frac{1}{\eta}|}{c_1 \delta^\eta} \right) \phi \left( \frac{|\zeta_1|}{a_1} \right) \cdots \phi \left( \frac{|\zeta_n|}{a_1} \right) f_\delta(0, \zeta'),$$

where

$$\phi(t) = \left\{ \begin{array}{ll} 1, & |t| \leq \frac{1}{3} \\ 0, & |t| \geq \frac{1}{3} \end{array} \right.$$

Note that

$$\|\overline{\mathcal{D}} g_\delta\|_{L^\infty(\Omega)} \lesssim \delta^{-\frac{1}{2}}.$$

Assume that $u_\delta \in L^2(\Omega) \cap \Lambda_n(U \cap \Omega)$ solves $\overline{\mathcal{D}} u_\delta = \overline{\mathcal{D}} g_\delta$ on $\Omega$ as in Theorem 1.3. Then we have

$$\|u\|_{\Lambda_n(U \cap \Omega)} \leq C \|\overline{\mathcal{D}} g_\delta\|_{L^\infty(\Omega)} \lesssim \delta^{-\frac{1}{2}}.$$ 

Set $h_\delta = u_\delta - g_\delta$. Then $h_\delta$ is holomorphic in $\Omega$. Set

$$q_1^\delta(\theta) = (\delta^{1/\eta} + \frac{4}{5} \zeta_1 \delta^{1/\eta} e^{i\theta}, 0, \ldots, 0, -\frac{b\delta}{2}), \quad \text{and}$$

$$q_2^\delta(\theta) = (\delta^{1/\eta} + \frac{4}{5} \zeta_1 \delta^{1/\eta} e^{i\theta}, 0, \ldots, 0, -b\delta), \quad \theta \in \mathbb{R}.$$

Note that $g_\delta(q_1^\delta(\theta)) = g_\delta(q_2^\delta(\theta)) = 0$. From (1.2) and (4.1) we obtain that

$$H_\delta := \left| \frac{1}{2\pi} \int_0^{2\pi} [u_\delta(q_1^\delta(\theta)) - u_\delta(q_2^\delta(\theta))] d\theta \right| \lesssim \delta^{\alpha} \|\overline{\mathcal{D}} g_\delta\|_{L^\infty} \lesssim \delta^{\alpha - \frac{1}{2}}.$$

For the lower bounds of $H_\delta$, set $\zeta_1^\delta = (0^\nu, -\frac{b\delta}{2}), \zeta_2^\delta = (0^\nu, -b\delta), \zeta_3 = (\delta^\frac{1}{\eta}, \zeta_4^\delta), \zeta_5 = (\delta^\frac{1}{\eta}, \zeta_6^\delta)$, and $\zeta_7 = (\delta^\frac{1}{\eta}, \zeta_7^\delta)$. Then a Taylor’s series of $f_\delta$ in $\zeta_\nu$ variable shows that

$$f_\delta(0^\nu, \zeta_n) = f_\delta(\zeta_3^\delta) + \frac{\partial f_\delta}{\partial \zeta_n}(\zeta_3^\delta)(\zeta_n + \frac{b\delta}{2}) + O \left( \left| \zeta_n + \frac{b\delta}{2} \right|^2 \right).$$

Especially, when $\zeta_n = -b\delta$, we have

$$f_\delta(\zeta_3^\delta) - f_\delta(\zeta_4^\delta) = \left| \frac{\partial f_\delta}{\partial \zeta_n}(\zeta_3^\delta)(-\frac{b\delta}{2}) + O(|\delta^2|) \right| \geq 1,$$ 

because $|\frac{\partial f_\delta}{\partial \zeta_n}(\zeta_3^\delta)| \geq \frac{1}{\delta}$ by (3.12).
Note that $g_\delta(\zeta_\delta) = f(\zeta_\delta')$ and $g_\delta(\tilde{\zeta}_\delta) = f(\tilde{\zeta}_\delta')$ because $0 < b < a_1 \leq c_1$. Therefore, it follows from (1.2), (4.1), (4.3), and the Mean Value Property that
\begin{equation}
H_\delta = \frac{1}{2\pi} \int_0^{2\pi} \left| h_\delta(q_\delta^1(\theta)) - h_\delta(q_\delta^2(\theta)) \right| d\theta \geq \left| h_\delta(\zeta_\delta) - h_\delta(\tilde{\zeta}_\delta) \right| \\
\geq \left| f_\delta(\zeta_\delta') - f_\delta(\zeta_\delta') \right| - \left| u_\delta(\zeta_\delta) - u_\delta(\tilde{\zeta}_\delta) \right| \geq c_0 - C_0 \delta^{\kappa - \frac{1}{\eta}}
\end{equation}
for some constants $0 < c_0 < 1 < C_0$. If we combine (4.2) and (4.4), we obtain that
\begin{equation}
1 \lesssim \delta^{\kappa - \frac{1}{\eta}}.
\end{equation}
Now, if we assume $\kappa > \frac{1}{\eta}$ and take $\delta \to 0$, then (4.5) will be a contradiction. Therefore, $\kappa \leq \frac{1}{\eta}$.

References

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