ON ARTINIANNESS OF GENERAL LOCAL COHOMOLOGY MODULES

NGUYEN MINH TRI

ABSTRACT. In this paper, we show some results on the artinianness of local cohomology modules with respect to a system of ideals. If $M$ is a $\Phi$-minimax $\mathcal{Z}$-module, then $H_{\Phi}^{\dim M}(M)/aH_{\Phi}^{\dim M}(M)$ is artinian for all $a \in \Phi$. Moreover, if $M$ is a $\Phi$-minimax $\mathcal{Z}$-module, $t$ is a non-negative integer and $H_{\Phi}^{i}(M)$ is minimax for all $i > t$, then $H_{\Phi}^{i}(M)$ is artinian for all $i > t$.

1. Introduction

Throughout this paper, $R$ is a noetherian commutative (with non-zero identity) ring and $\Phi$ is a system of ideals of $R$. It is well-known that the local cohomology theory of Grothendieck is an important tool in commutative algebra and algebraic geometry. In [3], a non-empty set of ideals $\Phi$ of $R$ is called to be a system of ideals if whenever $a, b \in \Phi$, then there is an ideal $c \in \Phi$ such that $c \subseteq ab$. For an $R$-module $M$, the $\Phi$-torsion submodule of $M$ is $\Gamma_{\Phi}(M) = \{ x \in M \mid ax = 0 \text{ for some } a \in \Phi \}$. The authors denoted by $H_{\Phi}^{i}$ the $i$-th right derived functor of the functor $\Gamma_{\Phi}$. It is clear that when $\Phi = \{ a^{n} \mid n \in \mathbb{N} \}$, the functor $H_{\Phi}^{i}$ coincides with the usual local cohomology functor $H_{a}^{i}$. In [5, Proposition 2.3], Bijan-Zadeh showed that

$$H_{\Phi}^{i}(M) \cong \lim_{a \in \Phi} \text{Ext}^{i}_R(R/a, M)$$

for all $i \geq 0$. Moreover, [4, Lemma 2.1] gave us the isomorphism

$$H_{\Phi}^{i}(M) \cong \lim_{a \in \Phi} H_{a}^{i}(M)$$

for all $i \geq 0$.

We recall that an $R$-module $M$ is minimax if there is a finitely generated submodule $N$ of $M$ such that $M/N$ is artinian. The minimax modules were first introduced in [12] and then developed in [9, 13]. It is clear that if $M$ is a minimax $R$-module and $\text{Supp}_{R}M \subseteq \text{Max}R$, then $M$ is artinian. We see that $M$...
is minimax if and only if \( M/N \) has finite Goldie dimension for each submodule \( N \) of \( M \). Note that, an \( R \)-module \( N \) is said to have finite Goldie dimension (written \( \text{Gdim}N < \infty \)) if \( N \) does not contain an infinite direct sum of non-zero submodules, or equivalently the injective hull \( E(N) \) of \( N \) decomposes as a finite direct sum of indecomposable (injective) submodules.

For a prime ideal \( p \), let \( \mu^0(p, N) \) denote the 0-th Bass number of \( N \) with respect to the prime ideal \( p \). It is known that \( \mu^0(p, N) > 0 \) if and only if \( p \in \text{Ass}_R N \). It is clear by the definition of the Goldie dimension that

\[
\text{Gdim}N = \sum_{p \in \text{Ass}_R N} \mu^0(p, N).
\]

In [6], the authors introduced and studied the concept \( a \)-relative Goldie dimension which is a generalization of Goldie dimension. They used the \( a \)-relative Goldie dimension to investigate the artianness of local cohomology modules with respect to an ideal. Let \( a \) be an ideal of \( R \). The \( a \)-relative Goldie dimension of \( N \), denoted by \( \text{Gdim}_a N \), is defined as

\[
\text{Gdim}_a N = \sum_{p \in V(a) \cap \text{Ass}_R N} \mu^0(p, N).
\]

Since \( \text{Ass}_R N \cap V(a) = \text{Ass}_R \Gamma_a(N) \), one obtains

\[
\text{Gdim}_a N = \text{Gdim}_a \Gamma_a(N).
\]

By using \( a \)-relative Goldie dimension and ZD-modules, many results on the artianness of local cohomology modules were provided in [6].

In [5], the author showed that \( H^\text{dim}_R(R) \) is artinian provided that \( R \) is a local ring. In generally, if \( M \) is a finitely generated \( R \)-module, then \( H^\text{dim}_M(M) \) is not artinian.

The purpose of this paper is to investigate the artianness of local cohomology modules with respect to a system of ideals \( H^\Phi_i(M) \). First, we introduce the concept \( \Phi \)-relative Goldie dimension of a module which is an extension of \( a \)-relative Goldie dimension in [6]. Next, an \( R \)-module \( M \) is called \( \Phi \)-minimax if the \( \Phi \)-relative Goldie dimension of any quotient module of \( M \) is finite. The first main result is Theorem 3.1 which says that if \( M \) is a \( \Phi \)-minimax ZD-module, then \( H^\text{dim}_M(M)/aH^\text{dim}_M(M) \) is artinian for all \( a \in \Phi \). Next, we will see in Theorem 3.2 that if \( (R, m) \) is a local ring and \( M \) is a \( \Phi \)-minimax ZD-module such that \( \text{Supp}_R H^\Phi_i(M) \subseteq \{m\} \) for all \( i < t \), then \( H^\Phi_i(M) \) is artinian for all \( i < t \). Theorem 3.3 is devoted to the study the relationship on the vanishing and the finiteness of \( H^\Phi_i(M) \). The paper is closed by Theorem 3.8 which shows that if \( M \) is a \( \Phi \)-minimax ZD-module and \( H^\Phi_i(M) \) is minimax for all \( i > t \), then \( H^\Phi_i(M) \) is artinian for all \( i > t \).
2. \(\Phi\)-minimax modules

Let \(\Phi\) be a system of ideals of \(R\). An \(R\)-module \(M\) is called \(\Phi\)-torsion-free if \(\Gamma_\Phi(M) = 0\). It is worth noting that

\[
\Gamma_\Phi(M) = \bigcup_{a \in \Phi} (0 :_M a) \cong \varprojlim_{a \in \Phi} \Gamma_a(M).
\]

Therefore, if \(M\) is \(\Phi\)-torsion-free, then it is \(a\)-torsion-free for all \(a \in \Phi\). Let \(\Omega = \bigcup_{a \in \Phi} V(a)\), we introduce a new notion which is motivated by \(a\)-relative Goldie dimension.

**Definition 2.1.** Let \(\Phi\) be a system of ideals of \(R\) and \(M\) an \(R\)-module. The \(\Phi\)-relative Goldie dimension of \(M\), denoted by \(\text{Gdim}_\Phi M\), is defined as

\[
\text{Gdim}_\Phi M = \sum_{p \in \text{Ass}_R M \cap \Omega} \mu^0(p, M).
\]

**Lemma 2.2.** Let \(\Phi\) be a system of ideals of \(R\) and \(M\) an \(R\)-module. Then \(\text{Ass}_R \Gamma_\Phi(M) = \text{Ass}_R M \cap \Omega\).

**Proof.** Let \(p \in \text{Ass}_R \Gamma_\Phi(M)\). There exists a non-zero element \(x \in \Gamma_\Phi(M)\) such that \(p = \text{Ann}_R x\). Since \(x \in \Gamma_\Phi(M)\), we have an ideal \(a \in \Phi\) such that \(ax = 0\). This implies that \(a \subseteq p\) and then \(\text{Ass}_R \Gamma_\Phi(M) \subseteq \text{Ass}_R M \cap \Omega\).

Let \(p \in \text{Ass}_R M \cap \Omega\). Then we have a non-zero element \(x \in M\) and \(a \in \Phi\) such that \(a \subseteq p = \text{Ann}_R x\). Therefore, one obtains \(ax = 0\). Hence, \(x \in \Gamma_\Phi(M)\) and \(p \in \text{Ass}_R \Gamma_\Phi(M)\). \(\square\)

**Corollary 2.3.** Let \(\Phi\) be a system of ideals of \(R\) and \(M\) an \(R\)-module. Then \(\text{Gdim}_\Phi M = \text{Gdim}_\Phi \Gamma_\Phi(M)\).

In [7], an \(R\)-module \(M\) is said to be a ZD-module if for any submodule \(N\) of \(M\), the set \(Z_R(M/N)\) is a finite union of prime ideals in \(\text{Ass}_R(M/N)\). Let \(S\) be a multiplicatively closed subset of \(R\), we denote

\[
S^{-1} \Phi = \{S^{-1}a \mid a \in \Phi\}.
\]

If \(p \in \text{Spec} R\) and \(S = R \setminus p\), we rewrite \(S^{-1} \Phi\) by \(\Phi_p\).

**Proposition 2.4.** Let \(M\) be a ZD-module. The following statements are equivalent:

(i) \(\text{Gdim}_\Phi M\) is finite;

(ii) \(\text{Gdim}_{\Phi_p} M_p\) is finite for all prime ideals \(p\) of \(R\);

(iii) \(\text{Gdim}_{\Phi_p} M_p\) is finite for all prime ideals \(p\), which is maximal in \(\text{Ass}_R M\).

**Proof.** (i) \(\Rightarrow\) (ii) It follows from Corollary 2.3 that \(\text{Gdim}_{\Phi_p} M_p = \text{Gdim}_{\Gamma_\Phi_p}(M_p)\). Let \(E(\Gamma_\Phi(M))\) be the injective hull of \(\Gamma_\Phi(M)\). Note that, if \(S\) is a multiplicatively closed subset of \(R\), then

\[
S^{-1}E_R(R/p) \cong \begin{cases} 
0, & S \cap p \neq \emptyset, \\
E_R(R/p), & S \cap p = \emptyset.
\end{cases}
\]
On the other hand, if $S \cap p = \emptyset$, then
\[ S^{-1}E_R(R/p) \cong E_{S^{-1}R}(S^{-1}(R/p)). \]
Hence, we have
\[
E(\Gamma_{\Phi}(M))_p = \bigoplus_{q \subseteq p, q \in \text{Ass}_R(\Phi\Gamma(M))} \mu^0(q, M)E_R(R/q)
\cong \bigoplus_{q \subseteq p, q \in \text{Ass}_R(\Phi\Gamma(M))} \mu^0(q, M)E_{R_p}(R_p/qR_p)
= E(\Gamma_{\Phi_p}(M_p)).
\]

Therefore, we can claim that $\text{Gdim}_{\Phi_p} M_p \leq \text{Gdim}_{\Phi} M$.

(ii)$\Rightarrow$(iii) It is clear.

(iii)$\Rightarrow$(i) Since $M$ is a ZD-module, by [6, Lemma 2.3] we may assume that $\{p_1, \ldots, p_k\}$ is the set of maximal elements in $\text{Ass}_R M$. For each $p_i$, we have by Corollary 2.3 that
\[
\text{Gdim}_{\Phi_{p_i}} M_{p_i} = \sum_{pR_{p_i} \in \text{Ass}_{R_{p_i}} \Gamma_{\Phi_{p_i}}(M_{p_i})} \mu^0(pR_{p_i}, M_{p_i})
= \sum_{p \in \text{Ass}_R \Phi\Gamma(M), p \subseteq p_i} \mu^0(p, M).
\]
This implies that
\[
\text{Gdim}_{\Phi} M \leq \sum_{i=1}^k \left( \sum_{p \in \text{Ass}_R \Phi\Gamma(M), p \subseteq p_i} \mu^0(p, M) \right)
= \sum_{i=1}^k \text{Gdim}_{\Phi_{p_i}} M_{p_i},
\]
and which completes the proof.

Azami, Naghipour and Vakili [2] defined that an $R$-module $N$ is $a$-minimax if the $a$-relative Goldie dimension of any quotient module of $N$ is finite. The concept of $a$-minimax modules is a generalization of the one of miminax modules. The following definition is an extension of $a$-minimax modules.

**Definition 2.5.** An $R$-module $M$ is $\Phi$-minimax if the $\Phi$-relative Goldie dimension of any quotient module of $M$ is finite.

We have some primary properties on $\Phi$-minimax modules.

**Proposition 2.6.** Let $M$ be an $R$-module. The following statements hold:

(i) If $M$ is a $\Phi$-minimax $R$-module, then $\text{Ass}_R M \cap \Omega$ is a finite set.

(ii) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $R$-modules. Then $B$ is $\Phi$-minimax if and only if $A$ and $C$ are both $\Phi$-minimax. Thus any submodule and quotient of a $\Phi$-minimax module as well as any finite direct sum of $\Phi$-minimax modules are $\Phi$-minimax.
(iii) Let $N$ be a finitely generated $R$-module and $M$ a $\Phi$-minimax $R$-module. Then $\text{Ext}_R^i(N, M)$ and $\text{Tor}_R^i(N, M)$ are $\Phi$-minimax for all $i \geq 0$.

Proof. (i) It follows from Definition 2.5.

(ii) We can assume that $A$ is a submodule of $B$ and $C \cong B/A$.

$(\Rightarrow)$ Let $B$ be a $\Phi$-minimax $R$-module and $A'$ a submodule of $A$. The short exact sequence

$$0 \rightarrow \frac{A}{A'} \rightarrow \frac{B}{A'} \rightarrow \frac{B}{A} \rightarrow 0$$

induces the following exact sequence

$$0 \rightarrow \text{Hom}_R \left( k(p), \frac{A_p}{A'_p} \right) \rightarrow \text{Hom}_R \left( k(p), \frac{B_p}{A'_p} \right),$$

where $p \in \text{Spec} R$ and $k(p) = R_p / pR_p$. Moreover, we have

$$\text{Ass}_R \left( A/A' \right) \cap \Omega \subseteq \text{Ass}_R \left( B/B' \right) \cap \Omega.$$ 

Consequently, we can conclude that $A$ is $\Phi$-minimax. Next, let $C'$ be a submodule of $C$. There exists a submodule $D$ of $B$ containing $A$ such that $C' \cong D/A$. It follows that $C/C' \cong B/D$ and then $C$ is $\Phi$-minimax.

$(\Leftarrow)$ Assume that $A, C$ are both $\Phi$-minimax and $B'$ a submodule of $B$. The short exact sequence

$$0 \rightarrow \frac{A}{A \cap B'} \rightarrow \frac{B}{B'} \rightarrow \frac{B}{A + B'} \rightarrow 0$$

induces the following exact sequence

$$0 \rightarrow \text{Hom}_R \left( k(p), \frac{A + B'}{B'_p} \right) \rightarrow \text{Hom}_R \left( k(p), \frac{B}{B'_p} \right) \rightarrow \text{Hom}_R \left( k(p), \frac{B_p}{A_p + B'_p} \right),$$

where $p \in \text{Spec} R$ and $k(p) = R_p / pR_p$. Moreover, there is an isomorphism

$$\frac{B}{A + B'} \cong \frac{B/A}{(A + B')/A}$$

and an inclusion

$$\text{Ass}_R \left( B/B' \right) \cap \Omega \subseteq (\text{Ass}_R \left( A/A \cap B' \right) \cap \Omega) \cup (\text{Ass}_R \left( B/(A + B') \right) \cap \Omega).$$

Therefore, one can claim that $B$ is $\Phi$-minimax.

(iii) We will prove the assertion for the Ext modules, and it is similar to the case of the Tor modules. Since $N$ is a finitely generated $R$-module over a noetherian ring, $N$ has a free resolution

$$\mathbf{F} : \cdots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow 0,$$

where $F_i$ is finitely generated free for all $i \geq 0$. Consequently, for each non-negative integer $i$, there is a positive integer $t$ such that $\text{Hom}_R (F_i, M) = \oplus^t M$. We know that $\text{Ext}_R^i (N, M) = H^i (\text{Hom}_R (\mathbf{F}, M))$ which is a subquotient of the $\Phi$-minimax module $\text{Hom}_R (F_i, M)$.

Hence, the assertion follows from (ii). \qed
Lemma 2.7. Let $M$ be an $R$-module such that $\text{Ass}_R M \subseteq \Omega$. Then $M = \Gamma_\Phi(M)$.

Proof. Let $x \in M$ a non-zero element. Then $\text{Ass}_R(Rx) = \{p_1, \ldots, p_k\}$ is a finite set. There exist $a_1, \ldots, a_k \in \Phi$ such that $p_i \in V(a_i)$ for each $i = 1, 2, \ldots, k$. Note that
\[
\sqrt{\text{Ann}_R(Rx)} = \bigcap_{i=1}^k p_i \supseteq \bigcap_{i=1}^k a_i \supseteq \prod_{i=1}^k a_i.
\]
Hence, there is an ideal $a \in \Phi$ such that $ax = 0$. We assert that $M = \Gamma_\Phi(M)$, as required. \hfill \Box

The following result is an extension of [2, Proposition 2.6].

Proposition 2.8. Let $M$ be a $\Phi$-minimax $R$-module and $\text{Ass}_R M \subseteq \Omega$. Then $H^n_\Phi(M)$ is $\Phi$-minimax for all $i \geq 0$.

Proof. Since $\Gamma_\Phi(M)$ is a submodule of $M$, it follows from Proposition 2.6(ii) that $\Gamma_\Phi(M)$ is $\Phi$-minimax.

By Lemma 2.7, we have $M = \Gamma_\Phi(M)$. It follows from [10, 1.4] that $H^n_\Phi(M) = 0$ for all $i > 0$, and the proof is complete. \hfill \Box

3. On the artinianness of local cohomology modules

In this section, we will consider the artinianness of general local cohomology module $H^n_\Phi(M)$ under condition that $M$ is a $\Phi$-minimax $ZD$-module. In [6, Corollary 3.3], if $M$ is a $a$-minimax $ZD$-module of dimension $n$, then $H^n_a(M)$ is artinian. We have the first main result of this paper which is an extension of [6, Corollary 3.3].

Theorem 3.1. Let $M$ be a $ZD$-module of dimension $n$. Assume that $M$ is $\Phi$-minimax. Then $H^n_a(M)/aH^n_a(M)$ is artinian for all $a \in \Phi$.

Proof. The proof is by induction on $n$. Let $n = 0$. By Proposition 2.6(ii), $H^n_a(M)$ is $\Phi$-minimax. Hence, the set $\text{Ass}_R H^n_a(M)$ is finite. Moreover, we have $\text{Ass}_R H^n_a(M) \subseteq \text{Ass}_R M \subseteq \text{Max} R$ since $\dim M = 0$. This implies that the injective hull $E(H^n_a(M))$ is artinian. Hence, so is $H^n_a(M)$.

Let $n > 0$. It follows from [10, 1.4] that $H^n_a(M) \cong H^n_a(M/\Gamma_\Phi(M))$. We may assume that $M$ is $\Phi$-torsion-free. Therefore, $M$ is $a$-torsion-free for all $a \in \Phi$. Let $a \in \Phi$, there exists an element $x \in a$ which is $M$-regular. The short exact sequence
\[
0 \to M \xrightarrow{\bar{x}} M \to M/xM \to 0
\]
yields the following exact sequence
\[
H^{n-1}_a(M/xM) \to H^n_a(M) \xrightarrow{\bar{x}} H^n_a(M) \to H^n_a(M/xM).
\]
By [7, Proposition 4] and Proposition 2.6, $M/xM$ is a $\Phi$-minimax $ZD$-module and $\dim M/xM \leq n - 1$. In view of [3, 2.7], one obtains $H^n_a(M/xM) = 0$. 

Applying the functor $R/a\otimes_R -$ to the above exact sequence, we have a following exact sequence

$$H^{n-1}_\Phi(M/xM)/aH^{n-1}_\Phi(M) \to H^n_\Phi(M)/aH^n_\Phi(M) \to 0.$$ 

The inductive hypothesis shows that $H^{n-1}_\Phi(M/xM)/aH^{n-1}_\Phi(M)$ is artinian. This leads the artinianness of $0 : H^n_\Phi(M)/aH^n_\Phi(M)$. Hence, one gets that $0 : H^n_\Phi(M)$ is artinian. This leads the following exact sequence

$$0 \to \mathcal{M} \to \mathcal{M}/x\mathcal{M} \to 0.$$

By the assumption, $\text{Supp}_R H^1_\Phi(M) \subseteq \{m\}$ for all $i < t$. Then $H^1_\Phi(M)$ is artinian for all $i < t$.

**Theorem 3.2.** Let $(R, m)$ be a local ring, $M$ a ZD-module and $t$ a non-negative integer. Assume that $M$ is $\Phi$-minimax and $\text{Supp}_R H^1_\Phi(M) \subseteq \{m\}$ for all $i < t$. Then $H^1_\Phi(M)$ is artinian for all $i < t$.

**Proof.** The proof is by induction on $i$. It is similar to the argument of the proof of Theorem 3.1, we see that $H^i_\Phi(M)$ is artinian. Let $i > 0$ and assume that $H^{i-1}_\Phi(M)$ is artinian. Let $\mathcal{M} = M/\Gamma_\Phi(M)$, it follows from [10, 1.4] that $H^1_\Phi(M) \cong H^1_\Phi(\mathcal{M})$. Since $\mathcal{M}$ is $\Phi$-torsion-free, it is $a$-torsion-free for any $a \in \Phi$. Let $a \in \Phi$, then $a$ contains an element $x$ which is $\mathcal{M}$-regular. The short exact sequence

$$0 \to \mathcal{M} \to \mathcal{M}/x\mathcal{M} \to 0$$

leads the following exact sequence

$$H^{i-1}_\Phi(\mathcal{M}/x\mathcal{M}) \to H^i_\Phi(\mathcal{M}) \to H^i_\Phi(\mathcal{M}) \to \cdots.$$ 

By the assumption, $\text{Supp}_R H^1_\Phi(\mathcal{M}/x\mathcal{M}) \subseteq \{m\}$ for all $i < t - 1$. Combining Proposition 2.6 with [7, Proposition 1.4] we see that $\mathcal{M}/x\mathcal{M}$ is a ZD-module as well as $\Phi$-minimax module. The inductive hypothesis deduces that $H^1_\Phi(\mathcal{M}/x\mathcal{M})$ is artinian for all $i < t - 1$. Hence, one gets that $0 : H^{1-1}_\Phi(\mathcal{M})$ is artinian. Since $\text{Supp}_R H^{i-1}_\Phi(\mathcal{M}) \subseteq \{m\} \subseteq V(xR)$, the artinianness of $H^{i-1}_\Phi(\mathcal{M})$ is followed from [8, Theorem 1.3].

Next, we have a connection on the finiteness and the vanishing of local cohomology modules with respect to a system of ideals. This is also an improvement of [11, Proposition 3.1].

**Theorem 3.3.** Let $(R, m)$ be a local ring, $M$ a $\Phi$-minimax ZD-module and $t$ a positive integer. The following statements are equivalent:

(i) $H^i_\Phi(M) = 0$ for all $i \geq t$;
(ii) $H^i_\Phi(M)$ is finitely generated for all $i \geq t$.

**Proof.** (i) $\Rightarrow$ (ii) Trivial.

(ii) $\Rightarrow$ (i) The proof is by induction on $\dim M$. Let $n = \dim M$. If $n = 0$, then $H^i_\Phi(M) = 0$ for all $i > 0$.

Let $n > 0$, it follows from [10, 1.4] that

$$H^1_\Phi(M) \cong H^1_\Phi(M/\Gamma_\Phi(M))$$
for all $i > 0$. Let $\overline{M} = M/\Gamma_{\Phi}(M)$, it is clear that $\overline{M}$ is $\Phi$-torsion-free. This implies that $\overline{M}$ is $a$-torsion-free for all $a \in \Phi$. In particular, there is an element $x \in m$ which is regular on $\overline{M}$. Now, the short exact sequence

$$0 \to \overline{M} \to M \to M/xM \to 0$$

induces a long exact sequence

$$\cdots \to H^i_{\Phi}(\overline{M}) \to H^i_{\Phi}(M) \to H^i_{\Phi}(M/xM) \to \cdots .$$

By the assumption, $H^i_{\Phi}(M/xM)$ is finitely generated for all $i \geq t$. Since $\dim(M/xM) < \dim(M) \leq n$ and $M/xM$ is a $\mathbb{Z}$D-module, it follows from the inductive hypothesis that $H^i_{\Phi}(M/xM) = 0$ for all $i \geq t$. Now the long exact sequence yields

$$H^i_{\Phi}(M) = xH^i_{\Phi}(M)$$

for all $i \geq t$. By Nakayama's Lemma, we can conclude that $H^i_{\Phi}(M) = 0$ for all $i \geq t$, and the proof is complete.  

It is clear that finitely generated $R$-modules are $\Phi$-minimax $\mathbb{Z}$D-modules. Hence, the following consequence is deduced immediately from Theorem 3.3.

**Corollary 3.4.** Let $(R, m)$ be a local ring, $M$ a finitely generated $R$-module and $t$ a positive integer. The following statements are equivalent:

1. $H^i_{\Phi}(M) = 0$ for all $i \geq t$;
2. $H^i_{\Phi}(M)$ is finitely generated for all $i \geq t$.

**Corollary 3.5.** Let $(R, m)$ be a local ring, $M$ a minimax $R$-module and $t > 1$ a positive integer. The following statements are equivalent:

1. $H^i_{\Phi}(M) = 0$ for all $i \geq t$;
2. $H^i_{\Phi}(M)$ is finitely generated for all $i \geq t$.

**Proof.** (i) $\Rightarrow$ (ii) Trivial. We now prove (ii) $\Rightarrow$ (i). Since $M$ is a minimax $R$-module, there is a short exact sequence

$$0 \to N \to M \to A \to 0,$$

where $N$ is finitely generated and $A$ is artinian. By applying the functor $\Gamma_{\Phi}(\cdot)$ to the above exact sequence, we get a long exact sequence

$$0 \to H^i_{\Phi}(N) \to H^i_{\Phi}(M) \to H^i_{\Phi}(A) \to H^i_{\Phi}(N) \to H^i_{\Phi}(M) \to 0$$

and

$$H^i_{\Phi}(N) \cong H^i_{\Phi}(M)$$

for all $i \geq 2$. By the hypothesis, $H^i_{\Phi}(N)$ is finitely generated for all $i \geq t$. It follows from Corollary 3.4 that $H^i_{\Phi}(N) = 0$ for all $i \geq t$ and which completes the proof.

We recall the cohomological dimension of $M$ with respect to a system of ideals

$$\text{cd}(\Phi, M) = \sup \{ i \mid H^i_{\Phi}(M) \neq 0 \}.$$
Corollary 3.6. Let $(R, m)$ be a local ring and $M$ a finitely generated $R$-module with $\text{cd}(\Phi, M) > 0$. Then $H_{\Phi}^{\text{cd}(\Phi, M)}(M)$ is not finitely generated.

Proof. The assertion follows easily from Corollary 3.4.

Corollary 3.7. Let $(R, m)$ be a local ring, $M$ a finitely generated $R$-module with finite dimension and $t > 1$ a positive integer such that $H_{\Phi}^{t-1}(M)/aH_{\Phi}^{t-1}(M) = 0$ for all $a \in \Phi$.

Proof. Let $a \in \Phi$, in the proof of Theorem 3.3, there is a long exact sequence

$$\cdots \rightarrow H_{\Phi}^{t-1}(M) \rightarrow H_{\Phi}^{t-1}(M) \rightarrow H_{\Phi}^{t-1}(M/xM) \rightarrow 0,$$

where $x \in a$. By the inductive hypothesis of the dimension of $M$, one asserts that $H_{\Phi}^{t-1}(M/xM)/aH_{\Phi}^{t-1}(M/xM) = 0$. Moreover, there is an isomorphism

$$H_{\Phi}^{t-1}(M)/aH_{\Phi}^{t-1}(M) \cong H_{\Phi}^{t-1}(M/xM)/aH_{\Phi}^{t-1}(M/xM),$$

and which complete the proof.

By using Theorem 3.3 and a fact of minimax modules, we have the following result.

Theorem 3.8. Let $M$ be a $\Phi$-minimax $\mathcal{ZD}$-module and $t$ a positive integer. Assume that $H_{\Phi}^{i}(M)$ is a minimax $R$-module for all $i \geq t$. Then $H_{\Phi}^{i}(M)$ is an artinian $R$-module for all $i \geq t$.

Proof. Let $i \geq t$. Since $H_{\Phi}^{i}(M)$ is minimax, we see that $H_{\Phi}^{i}(M)/p \cong H_{\Phi}^{i}(M_p)$ is a finitely generated $R_p$-module for all prime ideals $p$ which is not maximal. By [7, Proposition 3(2)], $M_p$ is a $\mathcal{ZD}$ $R_p$-module. By using Proposition 2.6, we can check that $M_p$ is a $\Phi_p$-minimax $R_p$-module. It follows from Theorem 3.3 that $H_{\Phi}^{i}(M)/p = 0$. This implies that $\text{Supp}_R H_{\Phi}^{i}(M) \subseteq \text{Max}R$ and then we have the artinianness of $H_{\Phi}^{i}(M)$.

The following consequence is an extension of [1, Theorem 2.3].

Corollary 3.9. Let $M$ be an $a$-minimax $\mathcal{ZD}$-module and $t$ a positive integer. Assume that $H_{\Phi}^{i}(M)$ is a minimax $R$-module for all $i \geq t$. Then $H_{\Phi}^{i}(M)$ is an artinian $R$-module for all $i \geq t$.

Acknowledgments. The author is deeply grateful to the referee for careful reading of the manuscript and for the helpful suggestions.

References


Nguyen Minh Tri
Department of Natural Science Education
Dong Nai University
4 Le Quy Don Street, Tan Hiep Ward, Bien Hoa City
Dong Nai Province, Vietnam
Email address: nguyennhintri@dnpu.edu.vn, triminhng@gmail.com