ON $n$-HYPONORMALITY FOR BACKWARD EXTENSIONS OF BERGMAN WEIGHTED SHIFTS†

YANWU DONG, GUIJUN ZHENG AND CHUNJI LI∗

Abstract. In this paper, we discuss the backward extensions of Bergman shifts $W_{\alpha(m)}$, where

$\alpha(m) : \sqrt{\frac{m}{m+1}}, \sqrt{\frac{m+1}{m+2}}, \ldots, (m \in \mathbb{N}).$

We obtained a complete description of the $n$-hyponormality for backward one, two and three step extensions.

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1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $L(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T$ in $L(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \succeq TT^*$, and subnormal if $T = N|_{\mathcal{K}}$, where $N$ is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. For $A, B \in L(\mathcal{H})$, let $[A, B] = AB - BA$. We say that an $n$-tuple $T = (T_1, \cdots, T_n)$ of operators in $L(\mathcal{H})$ is hyponormal if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of $n$ copies of $\mathcal{H}$. For a positive integer $k$, $T \in L(\mathcal{H})$ is $k$-hyponormal if $(I, T, \cdots, T^k)$ is hyponormal. It is well known from Bram-Halmos criterion that $T$ is subnormal if and only if $T$ is $k$-hyponormal for all $k \in \mathbb{N}$([3], [4]). Thus the implications ‘subnormal $\Rightarrow$ $\cdots \Rightarrow$ 2-hyponormal $\Rightarrow$ hyponormal’ hold, but each converse is not true in general. Since Curto in 1990 introduced a bridge between the hyponormality and subnormality in the concept of $k$-hyponormality ([2]), many operator theorists have studied these classes of operators until now.
In the study of these classes, the weighted shifts have played an important roles ([1], [2], [3], [4], [6], [7], [8], [14], etc.).

Let \( \{e_n\}_{n=0}^{\infty} \) be the canonical orthonormal basis for Hilbert space \( l^2(\mathbb{Z}_+) \) and let \( \alpha := \{\alpha_n\}_{n=0}^{\infty} \) be a bounded sequence of positive numbers. Let \( W_{\alpha} \) be a unilateral weighted shift defined by \( W_{\alpha} e_n := \alpha_n e_{n+1} \) \((n \geq 0)\). The moments of \( W_{\alpha} \) are usually defined by \( \gamma_0 := 1, \gamma_i := \alpha_0^2 \cdots \alpha_{i-1}^2 \) \((i \geq 1)\). We consider \( k \) variables \( x_i (i = 1, \cdots, k) \) satisfying \( 0 < x_k \leq \cdots \leq x_2 \leq x_1 \) and denote an augmented weighted sequence by

\[
\alpha(x_1, \cdots, x_k): x_k, \cdots, x_2, x_1, \alpha_0, \alpha_1, \cdots, (k \geq 1).
\]

Let \( W_{\alpha} \) be a \( p \)-hyponormal weighted shift and let \( k, p, q \in \mathbb{N} \) with \( q \leq p \). Then we may consider \( W_{\alpha(x_1, \cdots, x_k)} \) as a backward \( k \)-step extension of \( W_{\alpha} \) and describe the set

\[
HE_k(\alpha, q) := \{ (x_1, \cdots, x_k) : W_{\alpha(x_1, \cdots, x_k)} \text{ is } q \text{-hyponormal} \}.
\]

Many works have been done in this problem ([7], [8], [10], [11], [13], etc.). In this paper, we discuss the backward extensions of Bergman shifts \( W_{\alpha(m)} \), where

\[
\alpha(m) := \sqrt{\frac{m}{m+1}}, \sqrt{\frac{m+1}{m+2}}, \cdots, (m \in \mathbb{N}).
\]

We obtained a complete description of the \( n \)-hyponormality for one, two and three backward extensions.

The calculations in this paper were obtained through computer experiments using the software tool Scientific WorkPlace ([15]). Some lemmas to be used in this paper are as follows.

**Lemma 1.1 ([9, Lemma 3.1])**: Let \( W_{\alpha} \) be a \( p \)-hyponormal weighted shift, \( q \leq p \). Then \( W_{\alpha(x_1, \cdots, x_k)} \) is \( q \)-hyponormal if and only if the Hankel matrices

\[
M_{q+1}(k, i) := \begin{bmatrix}
\frac{1}{x_1}, \frac{1}{x_2}, \cdots, & \frac{1}{x_1}, & \gamma_0, & \cdots, & \gamma_{q-k+i} \\
\frac{1}{x_1}, \frac{1}{x_2}, \cdots, & \gamma_0, & \gamma_1, & \cdots, & \gamma_{q-k+1+i} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{x_1}, & \gamma_{k-i-2}, & \gamma_{k-i-1}, & \cdots, & \gamma_{q-1} \\
\gamma_0, & \gamma_{k-i-1}, & \gamma_{k-i}, & \cdots, & \gamma_{q} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_{q-k+i}, & \cdots, & \gamma_{q-1}, & \gamma_{q}, & \cdots, & \gamma_{2q-k+i}
\end{bmatrix}
\]

are positive for all \( i \) with \( 0 \leq i \leq k-1 \). Therefore we have

\[
HE_k(\alpha, q) := \{(x_1, \cdots, x_k) : M_{q+1}(k, i) \geq 0, \ 0 \leq i \leq k-1\}.
\]

**Lemma 1.2 ([5, Lemma 2.1])**. For \( \omega \geq 0 \), the determinant \( A_n(\omega) \) of the matrix with \((i, j)\) entry \( \frac{1}{\omega_{i+j+1}} \) \((1 \leq i, j \leq n)\) is

\[
1^{\Gamma(\cdot)} \text{ is the gamma function.}
\]
\[ W_n(\omega) = \left(1!2! \cdots (n-1)\right)^2 \frac{\Gamma (\omega + 1) \Gamma (\omega + 2) \cdots \Gamma (\omega + n)}{\Gamma (\omega + n + 1) \Gamma (\omega + n + 2) \cdots \Gamma (\omega + 2n)} \]  

(1.2)

For our convenience, we record the following five determinants which will be useful in the sequel:

\[ \Delta_{m,n}^{(1)} = \begin{vmatrix} \frac{1}{m} & \frac{1}{m+1} & \cdots & \frac{1}{m+n-1} \\ \frac{1}{m+1} & \frac{1}{m+2} & \cdots & \frac{1}{m+n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m+n-1} & \frac{1}{m+n} & \cdots & \frac{1}{m+2n-2} \end{vmatrix}, \]

\[ \Delta_{m,n}^{(2)} = \begin{vmatrix} \frac{1}{m} & \frac{1}{m+1} & \cdots & \frac{1}{m+n-1} \\ \frac{1}{m+1} & \frac{1}{m+2} & \cdots & \frac{1}{m+n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m+n-1} & \frac{1}{m+n} & \cdots & \frac{1}{m+2n-1} \end{vmatrix}, \]

\[ \Delta_{m,n}^{(3)} = \begin{vmatrix} \frac{1}{m} & \frac{1}{m+1} & \cdots & \frac{1}{m+n} \\ \frac{1}{m+1} & \frac{1}{m+2} & \cdots & \frac{1}{m+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m+n} & \frac{1}{m+n+1} & \cdots & \frac{1}{m+2n} \end{vmatrix}, \]

\[ \Delta_{m,n}^{(4)} = \begin{vmatrix} \frac{1}{m} & \frac{1}{m+1} & \cdots & \frac{1}{m+n} \\ \frac{1}{m+1} & \frac{1}{m+2} & \cdots & \frac{1}{m+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m+n} & \frac{1}{m+n+1} & \cdots & \frac{1}{m+2n} \end{vmatrix}, \]

\[ \Delta_{m,n}^{(5)} = \begin{vmatrix} \frac{1}{m} & \frac{1}{m+1} & \frac{1}{m+2} & \cdots & \frac{1}{m+n} \\ \frac{1}{m+1} & \frac{1}{m+2} & \frac{1}{m+3} & \cdots & \frac{1}{m+n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m+n} & \frac{1}{m+n+1} & \frac{1}{m+n+2} & \cdots & \frac{1}{m+2n-1} \end{vmatrix}. \]

By using Lemma 1.2, we can obtain the following formulas, we omit the tedious calculations here.

**Lemma 1.3.** When \( n \geq 3 \), for the above notation, we have the followings: \(^2\)

\(^2\) \( () \) is the binomial function.
\[ \Delta_{m,n}^{(1)} = \left( \frac{(n-1)!}{(n!)^m} \right)^2 \left( \prod_{l=n}^{m-1} \binom{m+l-1}{n} \right)^{-1}, \]
\[ \Delta_{m,n}^{(2)} = \left( \frac{(n-1)!}{(n!)^m} \right)^2 \binom{m+n}{n} \left( \prod_{l=n}^{m-1} \binom{m+l}{n} \right)^{-1}, \]
\[ \Delta_{m,n}^{(3)} = \left( \frac{(n-1)!}{(n!)^m} \right)^2 \left( \prod_{l=n}^{m-1} \binom{m+l+1}{n} \right)^{-1}, \]
\[ \Delta_{m,n}^{(4)} = \left( \frac{(n-1)!}{(n!)^m} \right)^2 \left( \prod_{l=n}^{m-1} \binom{m+l+2}{n} \right)^{-1}, \]
\[ \Delta_{m,n}^{(5)} = \left( \frac{(n-1)!}{(n!)^m} \right)^2 \left( \prod_{l=n}^{m-2} \binom{m+2}{n} \right)^{-1}. \]

For the binomial function, we can easily obtain the following formulas through simple calculation.

**Lemma 1.4** If we let \( \Omega = \binom{m+n-3}{n-2}^{-1} \), then we have the followings:

1. \( \binom{m+n-1}{n}^{-1} = \frac{(n-1)!}{(n-2)!(m+n-2)!} \)
2. \( \binom{m+n-1}{n-1}^{-1} = \frac{(n-1)!}{(n-2)(m+n-2)!} \)
3. \( \binom{m+n-1}{n-2}^{-1} = \frac{(n-1)!}{(n-2)(m+n-2)!} \)
4. \( \binom{m+n-2}{n-1}^{-1} = \frac{(n-1)!}{m+n-2} \)
5. \( \binom{m+n-2}{n-2}^{-1} = \frac{m!}{m+n-2} \)

2. Main results

2.1. One-step backward extensions. For \( m \) be a positive and we consider a weight sequence as follows:

\[ \alpha(x; m) : \sqrt{x}, \sqrt[4]{\frac{m}{m+1}}, \sqrt[4]{\frac{m+1}{m+2}}, \ldots, (m \geq 2). \]  

(2.1)

We can rewrite the result as following.

**Theorem 2.1** ([5, Theorem 3.2]). Let \( W_{\alpha(x; m)} \) be a weighted shift with weight \( \alpha(x; m) \) in (2.1). Then \( W_{\alpha(x; m)} \) is \( n \)-hyponormal if and only if

\[ 0 < x \leq H_{1}(m, n) := \frac{m-1}{m} \left( 1 - \left( \frac{m+n-1}{n} \right)^{-2} \right)^{-1}. \]
Remark. $H_1(m,n) \leq \frac{m}{m+1}$, for any $n \in \mathbb{N}$. In fact, let $X := \binom{m+n-1}{n}$, then
\[
\frac{m}{m+1} - H_1(m,n) = \frac{m}{m+1} - \frac{m-1}{m} \left( \frac{X^2}{X^2 - 1} \right) = \frac{(X-m)(X+m)}{m(X-1)(X+1)(m+1)},
\]
and since $m \geq 2$, we can show that:

1. $X = \frac{m+n-1}{m(m-1)} \geq m$, by mathematical induction on $n$,
2. $X \geq n + 1$.

Thus we know $\frac{m}{m+1} \geq H_1(m,n)$, and $\lim_{n \to \infty} X = +\infty$. Therefore, we obtain $\lim_{n \to \infty} H_1(m,n) = \frac{m}{m+1}$.

Proposition 2.2. Let $W_{\alpha(x,m)}$ be a weighted shift with weight $\alpha(x;m)$ in (2.1). Then $W_{\alpha(x,m)}$ is subnormal if and only if $0 < x \leq \frac{m-1}{m}$.

We give the following computational values.

<table>
<thead>
<tr>
<th>$n = 2$</th>
<th>$n = 3$</th>
<th>$n = 4$</th>
<th>$n = 5$</th>
<th>$n = 6$</th>
<th>$n = 7$</th>
<th>$\ldots$</th>
<th>$n = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 2$</td>
<td>$\frac{9}{10}$</td>
<td>$\frac{8}{11}$</td>
<td>$\frac{25}{33}$</td>
<td>$\frac{18}{23}$</td>
<td>$\frac{49}{65}$</td>
<td>$\frac{12}{17}$</td>
<td>$\approx 0.508$</td>
</tr>
<tr>
<td>$m = 3$</td>
<td>$\frac{24}{25}$</td>
<td>$\frac{20}{21}$</td>
<td>$\frac{75}{85}$</td>
<td>$\frac{147}{155}$</td>
<td>$\frac{1568}{1632}$</td>
<td>$\frac{864}{936}$</td>
<td>$\approx 0.667$</td>
</tr>
<tr>
<td>$m = 4$</td>
<td>$\frac{25}{27}$</td>
<td>$\frac{100}{108}$</td>
<td>$\frac{123}{135}$</td>
<td>$\frac{784}{855}$</td>
<td>$\frac{5292}{5955}$</td>
<td>$\frac{1035}{11272}$</td>
<td>$\approx 0.750$</td>
</tr>
<tr>
<td>$m = 5$</td>
<td>$\frac{45}{47}$</td>
<td>$\frac{245}{257}$</td>
<td>$\frac{3920}{4009}$</td>
<td>$\frac{63594}{64225}$</td>
<td>$\frac{35280}{35820}$</td>
<td>$\frac{87120}{88899}$</td>
<td>$\approx 0.800$</td>
</tr>
<tr>
<td>$m = 6$</td>
<td>$\frac{147}{155}$</td>
<td>$\frac{1568}{1632}$</td>
<td>$\frac{2646}{2780}$</td>
<td>$\frac{52920}{54800}$</td>
<td>$\frac{177870}{180800}$</td>
<td>$\frac{52720}{54360}$</td>
<td>$\approx 0.833$</td>
</tr>
<tr>
<td>$m = 7$</td>
<td>$\frac{176}{181}$</td>
<td>$\frac{1861}{1912}$</td>
<td>$\frac{3175}{3238}$</td>
<td>$\frac{63593}{65075}$</td>
<td>$\frac{21441}{21948}$</td>
<td>$\frac{67263}{69000}$</td>
<td>$\approx 0.857$</td>
</tr>
<tr>
<td>$m = 8$</td>
<td>$\frac{244}{249}$</td>
<td>$\frac{6048}{6205}$</td>
<td>$\frac{37900}{38825}$</td>
<td>$\frac{182952}{186275}$</td>
<td>$\frac{731808}{753750}$</td>
<td>$\frac{17667936}{18092485}$</td>
<td>$\approx 0.875$</td>
</tr>
<tr>
<td>$m = 9$</td>
<td>$\frac{162}{165}$</td>
<td>$\frac{1800}{1825}$</td>
<td>$\frac{27225}{27562}$</td>
<td>$\frac{78408}{80600}$</td>
<td>$\frac{369082}{379475}$</td>
<td>$\frac{1030629}{1063550}$</td>
<td>$\approx 0.899$</td>
</tr>
<tr>
<td>$m = 10$</td>
<td>$\frac{225}{232}$</td>
<td>$\frac{3090}{3114}$</td>
<td>$\frac{27225}{27562}$</td>
<td>$\frac{184041}{188000}$</td>
<td>$\frac{1092001}{1122757}$</td>
<td>$\frac{4601025}{4741588}$</td>
<td>$\approx 0.900$</td>
</tr>
</tbody>
</table>

From the table, we can obtain the $k$-hyponormalities easily, for example, $W_{\alpha(x;5)}$ is 4-hyponormal if and only if $0 < x \leq \frac{3920}{4899}$, where $\alpha(x;5) : \sqrt{x}, \sqrt[5]{\frac{5}{6}}, \sqrt[5]{\frac{6}{7}}, \ldots$.

2.2. Two-step backward extensions. Now we discuss the two-step backward extensions of weighted shift. For $m$ be a positive and we consider a weight sequence as follows:

\[
\alpha(x,y;m) : \sqrt[y]{x}, \sqrt[m+1]{\frac{m}{m+1}}, \sqrt[m+2]{\frac{m+1}{m+2}}, \ldots, (m \geq 3).
\]

Theorem 2.3. Let $W_{\alpha(x,y;m)}$ be a weighted shift with weight $\alpha(x,y;m)$ in (2.2). Then $W_{\alpha(x,y;m)}$ is $n$-hyponormal if and only if
(i) \(0 < x \leq \frac{m-1}{m} \left(1 - \left(\frac{m+n-1}{n}\right)^{-2}\right)^{-1}\),

(ii) \(0 < y \leq \min\left\{\frac{x}{A_2 x^2 + A_1 x + A_0}, x\right\}\), where

\[
A_2 = \frac{m}{m-2} - \frac{2mn(m+n-1)}{(m-1)^2} + \frac{m^2}{(m-1)^2} \left(m + n - 1\right)^2 - \frac{m}{m-2} \left(m + n - 2\right)^{-2},
\]

\[
A_1 = \frac{2}{1-m} \left(m \left(m + n - 1\right)^2 - n \left(m + n - 1\right)\right),
\]

\[
A_0 = \left(m + n - 1\right)^2.
\]

Proof. Let \(\alpha(x, y; m)\) be given in (2.2). Then the moments of \(W_\alpha\) are as follows:

\[
\gamma_0 = 1, \quad \gamma_k = \frac{m}{m+k} (k \geq 1).
\]

From Lemma 1.1, we know that \(W_\alpha(x, y; m)\) is \(n\)-hyponormal if and only if two Hankel matrices \(M_{n+1}(2, 0)\) and \(M_{n+1}(2, 1)\) are positive. First we consider the positivity of matrix \(M_{n+1}(2, 1)\), where

\[
M_{n+1}(2, 1) = \left[
\begin{array}{cccc}
\frac{1}{x} & \frac{1}{m} & \frac{1}{m+1} & \cdots & \frac{1}{m+n-1} \\
1 & \frac{m}{m+1} & \frac{m}{m+2} & \cdots & \frac{m}{m+n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{m}{m+n-1} & \frac{m}{m+n} & \frac{m}{m+n+1} & \cdots & \frac{m}{m+2n-1}
\end{array}
\right].
\]

Since

\[
\det M_{n+1}(2, 1) = m^{n+1} \left(\left(\frac{1}{mx} - \frac{1}{m-1}\right) \Delta^{(1)}_{m+1, n} + \Delta^{(1)}_{m-1, n+1}\right),
\]

and by Lemma 1.3, we have \(\det M_{n+1}(2, 1) \geq 0\) if and only if

\[
0 < x \leq \frac{(m-1) \Delta^{(1)}_{m+1, n}}{m \left(\Delta^{(1)}_{m+1, n} - (m-1) \Delta^{(1)}_{m-1, n+1}\right)} = \frac{m-1}{m} \left(1 - \left(\frac{m+n-1}{n}\right)^{-2}\right)^{-1}.
\]

Next we consider the positivity of matrix \(M_{n+1}(2, 0)\), where

\[
M_{n+1}(2, 0) = \left[
\begin{array}{cccc}
\frac{1}{x^2} & \frac{1}{x} & \frac{1}{m} & \cdots & \frac{1}{m+n-1} \\
\frac{1}{x} & \frac{1}{m} & \frac{1}{m+1} & \cdots & \frac{1}{m+n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{m}{m+n-2} & \frac{m}{m+n-1} & \frac{m}{m+n} & \cdots & \frac{m}{m+2n-2}
\end{array}
\right].
\]

Since

\[
\frac{\det M_{n+1}(2, 0)}{m^{n+1}} = \left(\frac{1}{mxy} - \frac{1}{m-2}\right) \Delta^{(1)}_{m, n} - \left(\frac{1}{mxy} - \frac{1}{m-1}\right)^2 \Delta^{(1)}_{m+2, n-1}
\]
and by Lemma 1.3, we have \( \det M_{n+1}(2,0) \geq 0 \) if and only if \( 0 < y \leq \frac{x}{A_2 x^2 + A_1 x + A_0} \), where \( A_0, A_1 \) and \( A_2 \) are as in (2.3). The proof is complete.

By Theorem 2.3, we can obtain the following results.

**Corollary 2.4 ([5, Theorem 3.6]).** Let \( \alpha (x, y; 3) : \sqrt{y}, \sqrt{\frac{x}{2}}, \sqrt{\frac{2}{5}}, \ldots \) be a weighted shift. Then \( W_{\alpha (x, y; 3)} \) is \( n \)-hyponormal if and only if

1. \( 0 < x \leq \frac{2}{3} (n+1)^2 (n+2)^2 \),
2. \( 0 < y \leq \frac{\alpha^2}{A_2 x^2 + A_1 x + A_0}, x \),

where \[
A_0 = \frac{1}{36} n^2 (n+1)^2 (n+2)^2,
A_1 = -\frac{1}{12} n (n-1) (n+2) (n+3) (n^2 + 2n + 4),
A_2 = \frac{n (n+2) (n-1) (n+3) (n^4 + 4n^3 + 9n^2 + 10n - 8)}{16 (n+1)^2}.
\]

**Corollary 2.5.** Let \( \alpha (x, y; 4) : \sqrt{y}, \sqrt{\frac{x}{2}}, \sqrt{\frac{2}{5}}, \sqrt{\frac{3}{6}}, \ldots \) be a weighted shift. Then \( W_{\alpha (x, y; 4)} \) is \( n \)-hyponormal if and only if

1. \( 0 < x \leq \frac{3}{4} (n+1)^2 (n+2)^2 (n+3)^2 \),
2. \( 0 < y \leq \frac{\alpha^2}{A_2 x^2 + A_1 x + A_0}, x \),

where \[
A_0 = \frac{1}{576} n^2 (n+1)^2 (n+2)^2 (n+3)^2,
A_1 = -\frac{1}{216} n (n-1) (n+1) (n+4) (n^4 + 6n^3 + 17n^2 + 24n + 36),
A_2 = \frac{n (n-1) (n+1) (n+4) (n+3)}{324 (n+1)^2 (n+2)^2 (n+3)^2}
\times (n^8 + 12n^7 + 66n^6 + 216n^5 + 477n^4 + 756n^3 + 680n^2 + 96n - 360).
\]

### 2.3. Three-step backward extensions

Next we discuss the three-step backward extensions of weighted shift. For \( m \) be a positive and we consider a weight sequence as follows:

\[
\alpha (x, y, z; m) : \sqrt{z}, \sqrt{y}, \sqrt{\frac{x}{m+1}}, \sqrt{\frac{m+1}{m+2}}, \sqrt{\frac{m+2}{m+3}}, \ldots, (m \geq 4).
\]

**Theorem 2.6.** Let \( 0 < z \leq y \leq x \) and \( W_{\alpha (x, y, z; m)} \) be a weighted shift with weight \( \alpha (x, y, z; m) \) in (2.4). Then \( W_{\alpha (x, y, z; m)} \) is \( n \)-hyponormal if and only if
(i) $0 < x \leq \frac{m}{m-1} \left( 1 - \left( \frac{m + n - 1}{n} \right)^{-2} \right)^{-1}$,
(ii) $0 < y \leq \min \left\{ \frac{A_0 x^2 + A_1 x + A_0}{x}, x \right\}$, where $A_0, A_1$ and $A_2$ are as in (2.3),
(iii) $0 < z \leq \min \left\{ \frac{9m^3(m-1)^2(m+1)(m-2)^2(m-3)(n+1)^2x y B_1 x - B_0}{(c_5 x^2 + c_5 x + c_5)^2 x y + c_2 x^2 y + c_1 x y + c_0 x}, y \right\}$, where

$$
B_0 = m - 1, \quad B_1 = m \left( 1 - \frac{(n-1)^2 \Omega^2}{(m+n-2)^2} \right),
$$

$$
C_0 = -9m^3(m-1)^3(m-2)^2(m-3)(m+1)(n+1)^2,
C_1 = 18m^2(m-1)^3(m-2)^2(m-3)(m+1)(n-1)(m+n-1)(n+1)^2,
C_2 = -18m^4(m-1)^2(m-2)(m-3)(m+1)(n+1)^2
\times \left( \frac{(m-2) n^2 + (m-2)^2 n - 2 (m-1)^2}{m} + \frac{n(n-1)^2 \Omega^2}{(m+n-2)} \right),
$$

$$
C_3 = -9(m-1)^3(m-2)^2(m-3)(n+1)^2(m+n-1)^2(m+n-2)^2 \Omega^{-2},
C_4 = -9m(m-1)^2(m-2)^2(m-3)(n+1)^2(m+n-1)^2(m+n-2)^2
\times \left( \frac{(m-1) (2m+1) n^2 + (2m+1)(m-2) n - m(m-1)}{(m+n-1)(m+n-2)^2} - 3 \Omega^{-2} \right),
$$

and

$$
C_5 = 9m^2(m-1)(m-2)(n+1)^2(m+n-1)^2(m+n-2)^2
\times (c_{51} \Omega^2 - 3(m-2)(m-3) \Omega^{-2} + c_{52}),
$$

$$
C_6 = 9m^2(n+1)^2(m+n-1)^2(m+n-2)^2
\times \left( \frac{(c_{61} - c_{62} \Omega^2 + c_{63} \Omega^4)}{(m+n-1)^2(m+n-2)^4} + m(m-2)^2(m-3) \Omega^{-2} \right),
$$

with

$$
c_{51} = \frac{m^2 n (m+1)(n-1)^2 ((2m-5) n^2 + (m-2) (2m-5) n - (m-1) (m-3))}{(m+n-1)^2(m+n-2)^4},
$$

$$
c_{52} = \frac{m (q_1(m) n^4 + q_2(m) n^3 + q_3(m) n^2 - q_4(m) n + q_5(m))}{(m+n-2)^2(m+n-1)^2},
$$

$$
q_1(m) = 2(m-3)(2m+1)(m-2),
q_2(m) = 4(2m+1)(m-3)(m-2)^2,
q_3(m) = 2(m-3)(2m^4 - 15m^3 + 27m^2 - 6m - 11),
q_4(m) = 2(m-1)(m-2)(m-3)(4m^2 - 5m - 3),
$$
Proof. Let follows:

\[ q_5(m) = 3(m^3 - 5m^2 + 4m + 2)(m - 1)^2, \]

\[ c_{61} = -m^2(m + n - 2)^2(p_1(m) n^4 + p_2(m) n^3 + p_3(m) n^2 - p_4(m) n + p_5(m)), \]

\[ c_{62} = m^3(m + 1)(n - 1)^2(p_6(m) n^4 + p_7(m) n^3 + p_8(m) n^2 - p_9(m) n + p_{10}(m)), \]

\[ c_{63} = m^3 n^2(m + 1)(n - 1)^4, \]

\[ p_1(m) = (2m + 1)(m - 3)(m - 2)^2, \]

\[ p_2(m) = 2(2m + 1)(m - 3)(m - 2)^3, \]

\[ p_3(m) = (m - 2)(m - 3)(2m^2 - 16m^3 + 28m^2 - 5m - 12), \]

\[ p_4(m) = (m - 1)(m - 3)(5m^2 - 5m - 4)(m - 2)^2, \]

\[ p_5(m) = (3m^4 - 17m^3 + 25m^2 - 3m - 12)(m - 1)^2, \]

\[ p_6(m) = (m - 2)(2m - 5), \]

\[ p_7(m) = 2(2m - 5)(m - 2)^2, \]

\[ p_8(m) = (m - 4)(2m^2 - 12m^2 + 22m - 13), \]

\[ p_9(m) = (m - 1)(m - 2)(m - 3)(3m - 4), \]

\[ p_{10}(m) = (m - 1)^2(m - 2)^2. \]

Also from Lemma 1.1 we know that \( W_{\alpha(x, y, z; m)} \) is \( n \)-hyponormal if and only if three Hankel matrices \( M_{n+1}(3, 0) \), \( M_{n+1}(3, 1) \) and \( M_{n+1}(3, 2) \) are positive. We have discussed the positivity of matrices \( M_{n+1}(3, 1) (= M_{n+1}(2, 0)) \) and \( M_{n+1}(3, 2) (= M_{n+1}(2, 1)) \) in Theorem 2.3. Therefore we just need to consider the positivity of matrix \( M_{n+1}(3, 0) \).

Since \( \det M_{n+1}(3, 0) = m^{n+1} \Lambda_{n+1}(x, y, z; m) \), with

\[
\Lambda_{n+1}(x, y, z; \Gamma) = \begin{vmatrix}
\frac{1}{m} & \frac{1}{1} & \cdots & \frac{1}{m^{n+1}} \\
\frac{1}{m^y} & \frac{1}{m} & \cdots & \frac{1}{m^{n+2}} \\
\frac{1}{m^x} & \frac{1}{1} & \cdots & \frac{1}{m^{n+1}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{m^{n+2}} & \frac{1}{m^{n+1}} & \cdots & \frac{1}{1}
\end{vmatrix}
\]

\[
= \left( \frac{1}{m} \right)^{n+1} \Delta_1 \Delta_2 - \frac{1}{m^2} \left( \frac{1}{m^{n+1}} \right)^2 \Delta_3 \Delta_4 + 2 \left( \frac{1}{m^2} \right)^2 \left( \frac{1}{m^{n+1}} \right)^2 \Delta_5 \Delta_6
\]

\[
= \left( \frac{1}{m^2} - \frac{1}{m^{n+2}} \right) \left( \frac{1}{m^2} - \frac{1}{m^{n+1}} \right) \Delta_1 \Delta_2
\]

\[
+ 2 \left( \frac{1}{m^2} - \frac{1}{m^{n+1}} \right) \left( \frac{1}{m^2} - \frac{1}{m^{n+1}} \right) \Delta_3 \Delta_4
\]

Also from Lemma 1.1 we know that \( W_{\alpha(x, y, z; m)} \) is \( n \)-hyponormal if and only if three Hankel matrices \( M_{n+1}(3, 0) \), \( M_{n+1}(3, 1) \) and \( M_{n+1}(3, 2) \) are positive. We have discussed the positivity of matrices \( M_{n+1}(3, 1) (= M_{n+1}(2, 0)) \) and \( M_{n+1}(3, 2) (= M_{n+1}(2, 1)) \) in Theorem 2.3. Therefore we just need to consider the positivity of matrix \( M_{n+1}(3, 0) \).

Since \( \det M_{n+1}(3, 0) = m^{n+1} \Lambda_{n+1}(x, y, z; m) \), with
Corollary 2.7 ([12, Theorem 3.6])

in Theorem 2.6.

if and only if

and by Lemma 1.2, Lemma 1.3 and Lemma 1.4, we obtain

where

\[ \Delta_1 = \Delta_{m+1,n-1}^{(1)}, \quad \Delta_2 = \Delta_{m-1,n}^{(1)}, \quad \Delta_3 = \Delta_{m,n-1}^{(2)}, \]

\[ \Delta_4 = \Delta_{m-2,n}^{(2)}, \quad \Delta_5 = \Delta_{m+3,n-2}^{(1)}, \quad \Delta_6 = \Delta_{m-1,n-1}^{(3)}, \]

\[ \Delta_7 = \Delta_{m-1,n-1}^{(4)}, \quad \Delta_8 = \Delta_{m-2,n}^{(5)}, \quad \Delta_9 = \Delta_{m-3,n}^{(3)}, \quad \Delta_{10} = \Delta_{m-3,n+1}^{(1)}, \]

and by Lemma 1.2, Lemma 1.3 and Lemma 1.4, we obtain \( \det M_{n+1}(3,0) \geq 0 \)

if and only if

\[ 0 < z \leq \frac{9m^3 (m-1)^2 (m+1) (m-2)^2 (m-3) (n+1)^2 xy (B_1 x - B_0)}{(C_6 x^3 + C_5 x^2 + C_4 x + C_3) y^2 + C_2 x^2 y + C_1 xy + C_0 x}, \]

where \( B_0, B_1 \) and \( C_i \) \((i = 0, 1, 2, \ldots, 6)\) are described in (iii). The proof is complete. \( \square \)

The authors in [12] obtained the following results, which is the case of \( m = 4 \)

in Theorem 2.6.

Corollary 2.7 ([12, Theorem 3.6]). Let \( 0 < x \leq y \leq z \) and

\[ \alpha(x,y,z) : \sqrt{z}, \sqrt{y}, \sqrt{x}, \sqrt[4]{\frac{4}{5}}, \sqrt[5]{\frac{5}{6}}, \ldots. \]

Then \( W_{\alpha(x,y,z)} \) is \( n \)-hyponormal if and only if

(i) \( 0 < x \leq \frac{3}{4} \frac{(n+1)^2 (n+2)^2 (n+3)^2}{n(n+4)(n^2+2n+3)(n^2+6n+11)}, \)

(ii) \( 0 < y \leq \min \left\{ \frac{3}{324} x^2 + A_1 x + A_0, x \right\} \), where

\[ A_0 = \frac{1}{576} n^2 (n+1)^2 (n+2)^2 (n+3)^2, \]

\[ A_1 = -\frac{1}{216} n (n-1) (n+3) (n+4) r_1(n), \]

\[ A_2 = \frac{n (n-1) (n+4) (n+3)}{324} (n+2)^2 (n+1)^2 r_2(n), \]

with

\[ r_1(n) = n^4 + 6n^3 + 17n^2 + 24n + 36, \]

\[ r_2(n) = n^8 + 12n^7 + 66n^6 + 216n^5 + 477n^4 + 756n^3 + 680n^2 + 96n - 360, \]

(iii) \( 0 < z \leq \min \left\{ \frac{103680(n+1)^2 y (B_1 x - B_0)}{(C_6 x^3 + C_5 x^2 + C_4 x + C_3) y^2 + C_2 x^2 y + C_1 xy + C_0 x}, y \right\} \), where
\[ B_0 = 3, \quad B_1 = \frac{4(n-1)(n+3)(n^2+2)(n^2+4n+6)}{n^2(n+2)^2(n+1)^2}, \]
\[ C_0 = -311,040 \frac{(n+1)^2}{n^2}, \quad C_1 = 155,520 \frac{(n-1)(n+1)^2(n+3)}{n^2(n+2)}, \]
\[ C_2 = -\frac{207,360(n-1)(n-2)(n+3)(n+4)}{n(n+2)}, \quad C_3 = -27n^2 \frac{(n-1)^2(n+2)^2(n+3)^2}{n^2}, \]
\[ C_4 = 108(n-1)(n-2)(n+1)^2(n+3)(n+4) r_3(n), \quad C_5 = -\frac{144(n-1)(n-2)(n+3)(n+4)}{n(n+2)} r_4(n), \]
\[ C_6 = \frac{64(n-2)(n-1)^2(n+3)^2(n+4)}{n(n+1)^2(n+2)} r_5(n), \]

with
\[
\begin{align*}
r_3(n) & = n^6 + 6n^5 + 18n^4 + 32n^3 + 69n^2 + 90n - 72, \\
r_4(n) & = n^{10} + 10n^9 + 47n^8 + 136n^7 + 299n^6 + 562n^5 \\
& \quad + 265n^4 - 1428n^3 - 3060n^2 - 2448n + 2160, \\
r_5(n) & = n^{10} + 10n^9 + 51n^8 + 168n^7 + 435n^6 + 930n^5 \\
& \quad + 701n^4 - 1540n^3 - 5076n^2 - 5616n + 4320.
\end{align*}
\]

**Remark.** (1) In Theorem 2.1, we assume \( m \geq 2 \), the authors in [5, Theorem 3.2] obtained a result for \( m = 1 \). That is, let \( \alpha(x;1) : \sqrt{x}, \sqrt[3]{x}, \sqrt[4]{x}, \ldots \), then \( W_{\alpha(x;1)} \) is \( n \)-hyponormal if and only if \( 0 < x \leq \frac{1}{2(1+\frac{1}{2}+\cdots+\frac{1}{n})} \).

(2) In Theorem 2.3, we assume \( m \geq 3 \), and in Theorem 2.6, we assume \( m \geq 4 \). We can obtain similar results \( m = 1, 2 \) for two-step backward extensions, and \( m = 1, 2, 3 \) for three-step backward extensions. We leave them to interested readers.

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