COMMON FIXED POINTS OF $w$-COMPATIBLE MAPS IN MODULAR $A$-METRIC SPACES

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Abstract. The aim of this paper is to prove a common fixed point theorem for two $w$-compatible maps in modular $A$-metric spaces. The main result is also illustrated by an example to demonstrate the degree of validity of our hypothesis.

1. Introduction

Modular metric spaces are a natural generalization of metric spaces. The introduction of this new concept was given by V. V. Chistyakov [2], [3]. In the last decade, there has been an enormous progress in modular metric.

In 2017, Aydin and Kutukcu [1] introduced a new structure of generalized metric space and called it modular $A$-metric space.

Now, we give some definitions and results which are used in this paper.

Definition 1.1 ([1]). The modular $A$-metric on $X$ where $X$ is non-empty is defined by a mapping $A_1 : (0, \infty) \times X^n \rightarrow [0, \infty]$ that satisfying following conditions for all $x_i, \ a \in X$ and $\lambda, \lambda_i > 0$ for $i = 1, n$:

(A1) $A_1 (x_1, x_2, x_3, \ldots, x_{n-1}, x_n) \geq 0$,

(A2) $A_1 (x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = \ldots = x_{n-1} = x_n$

(A3) $A_{\lambda_1+\lambda_2+\ldots+\lambda_n} (x_1, x_2, x_3, \ldots, x_{n-1}, x_n) \leq A_{\lambda_1} (x_1, x_1, \ldots, (x_1)_{n-1}, a) + A_{\lambda_2} (x_2, x_2, \ldots, (x_2)_{n-1}, a) + \ldots + A_{\lambda_n} (x_n, x_n, \ldots, (x_n)_{n-1}, a)$

The pair $(X, A)$ is said to be a modular $A$-metric space.
Lemma 1.2 ([1]). Let \((X, A)\) be a modular \(A\)-metric space. If for each \(x_1, \ldots, x_n \in X\), the mapping \(A(\cdot, x_1, x_2, \ldots, x_n) : (0, \infty) \rightarrow [0, \infty]\) is continuous, then the following equality is true

\[ A_{\lambda}(x, x, \ldots, x, y) = A_{\lambda}(y, y, \ldots, y, x) \]

for each \(x, y \in X\) and \(\lambda > 0\).

Theorem 1.3 ([1]). Let \((X, A)\) be a modular \(A\)-metric space and the mapping \(A(\cdot, x_1, x_2, \ldots, x_n) : (0, \infty) \rightarrow [0, \infty]\) is continuous for each \(x_1, x_2, \ldots, x_n \in X\). Then, there are the following inequalities such that

\[ A_{\lambda}(x, x, \ldots, x, z) \leq (n - 1) A_{\frac{\lambda}{n}}(x, x, \ldots, x, y) + A_{\frac{\lambda}{n}}(z, z, \ldots, z, y) \]

and

\[ A_{\lambda}(x, x, \ldots, x, z) \leq (n - 1) A_{\frac{\lambda}{n}}(x, x, \ldots, x, y) + A_{\frac{\lambda}{n}}(y, y, \ldots, y, z) \]

for each \(x, y, z \in X\).

Proposition 1.4 ([1]). Let \((X, A)\) be a modular \(A\)-metric space and the mapping \(A(\cdot, x_1, x_2, \ldots, x_n) : (0, \infty) \rightarrow [0, \infty]\) is continuous for each \(x_1, x_2, \ldots, x_n \in X\). Then, the following inequality

\[ A_{\lambda}(x, x, \ldots, x, y) \leq A_{\frac{\lambda}{n}}(x, x, \ldots, x, y) \leq A_{\frac{\lambda}{n^2}}(x, x, \ldots, x, y) \]

is satisfied for \(\frac{\lambda}{n^2} \leq \frac{\lambda}{n} \leq \lambda\).

Proof. If it is taken \(a = x\) in the condition (A3) and used the inequality in Theorem (1.3), the following inequality is written:

\begin{align*}
A_{\lambda}(x, x, \ldots, x, y) &\leq (n - 1) A_{\frac{\lambda}{n}}(x, x, \ldots, x, x) + A_{\frac{\lambda}{n}}(y, y, \ldots, y, x) \\
&= A_{\frac{\lambda}{n}}(y, y, \ldots, y, x) \\
&\leq (n - 1) A_{\frac{\lambda}{n^2}}(y, y, \ldots, y, y) + A_{\frac{\lambda}{n^2}}(x, x, \ldots, x, y) \\
&= A_{\frac{\lambda}{n^2}}(x, x, \ldots, x, y)
\end{align*}

Thus,

\[ A_{\lambda}(x, x, \ldots, x, y) \leq A_{\frac{\lambda}{n}}(x, x, \ldots, x, y) \leq A_{\frac{\lambda}{n^2}}(x, x, \ldots, x, y) \]

is obtained with Lemma (1.2). \qed
Example 1.5 ([1]). Let $X = \mathbb{R}$. Define a function $A_\lambda : (0, \infty) \times X^n \to [0, \infty]$ by

$$A_\lambda(x_1, x_2, x_3, \ldots, x_n) = \frac{\lambda}{n} |x_1 - x_2| + |x_1 - x_3| + \ldots + |x_1 - x_n| + |x_2 - x_3| + |x_2 - x_4| + \ldots + |x_2 - x_n| + \ldots + |x_{n-2} - x_{n-1}| + |x_{n-2} - x_n| + |x_{n-1} - x_n|$$

$$= \frac{\lambda}{n} \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|$$

for all $\lambda > 0$ and $x_1, x_2, \ldots, x_n \in X$.

Then, $(X, A)$ is a usual modular $A$-metric space on $X$.

Definition 1.6 ([1]). Let $(X, A)$ be a modular $A$-metric space and let $x_0 \in X$. Then, for any $r > 0$, the set

$$B_{A_\lambda}(x_0, r) = \{y \in X : A_\lambda(y, y, \ldots, y, x_0) < r\}$$

is defined as an open ball with center $x_0$ and radius $r$.

Definition 1.7 ([1]). Let $(X, A)$ be a modular $A$-metric space and $Y \subseteq X$.

1. If there exists a $r > 0$ such that $B_{A_\lambda}(x, r) \subseteq Y$ for each $x \in Y$ and $\lambda > 0$, then $Y$ is called be an open set.

2. Let

$$\tau_A := \{Y \subseteq X : x \in Y \text{ iff there exists a } r > 0 \text{ such that } B_{A_\lambda}(x, r) \subseteq Y\}.$$

In this case, $(X, \tau_A)$ is a topological space.

Theorem 1.8 ([1]). Let $A$ be a modular $A$-metric on $X$. In this case, $(X, \tau)$ is a Hausdorff space.

Definition 1.9 ([1]). Let $A$ be a modular $A$-metric on $X$, $\{x_k\}_{k \in \mathbb{N}} \subseteq X$ and $x \in X$.

1. $\{x_k\}$ converges to $x$ if $A_\lambda(x_k, x_k, \ldots, x_k, x) \to 0$ as $k \to \infty$ for all $\lambda > 0$. In other words, for each $\varepsilon > 0$, there exists a natural number $k_0(\varepsilon) \in \mathbb{N}$ such that for all $k \geq k_0$, $A_\lambda(x_k, x_k, \ldots, x_k, x) \leq \varepsilon$. 


(2) \( \{x_k\} \) is said to be a Cauchy sequence if \( A_\lambda(x_k, x_k, \ldots, x_k, x_m) \to 0 \) as \( k, m \to \infty \) for all \( \lambda > 0 \). In other words, for each \( \varepsilon > 0 \), there exists a natural number \( k_0(\varepsilon) \in \mathbb{N} \) such that for all \( k, m \geq k_0 \),
\[
A_\lambda(x_k, x_k, \ldots, x_k, x_m) \leq \varepsilon.
\]

(3) \((X, A)\) is said to be complete modular \( A \)-metric space if every Cauchy sequence in \( X \) is convergent.

**Theorem 1.10** ([1]). Let \( A \) be a modular \( A \)-metric on \( X \). If the sequence \( \{x_k\}_{k \in \mathbb{N}} \subset X \) converges to \( x \) in \( X \), in this case \( x \) is unique.

**Theorem 1.11** ([1]). Let \( A \) be a modular \( A \)-metric on \( X \). If \( \{x_k\}_{k \in \mathbb{N}} \subset X \) is a convergent sequence in \( X \), then \( \{x_k\}_{k \in \mathbb{N}} \) is a Cauchy sequence.

2. \( w \)-Compatible Maps in Modular \( A \)-Metric Spaces

In 1986, Jungck [4] introduced the concept of compatible maps in metric spaces as follows:

**Definition 2.1.** Let \((M, d)\) be a metric space and \( f, g : M \to M \). The mappings \( f \) and \( g \) are said to be compatible if \( \lim_{k \to \infty} d(fgx_k, gfx_k) = 0 \), whenever \( \{x_k\} \) is a sequence in \( M \) such that \( \lim_{k \to \infty} fx_k = \lim_{k \to \infty} gx_k = z \) for some \( z \in M \).

In modular \( A \)-metric spaces, the notion of \( w \)-compatible maps is given as follows:

**Definition 2.2** ([5]). Let \( M \) and \( N \) be two self maps on a modular \( A \)-metric space \((X, A)\). If
\[
\lim_{k \to \infty} A_\lambda(MNx_k, \ldots, MNx_k, NMx_k) = 0
\]
where \( \{x_k\} \) is a sequence in \( X \) which satisfies \( \lim_{k \to \infty} Mx_k = \lim_{k \to \infty} Nx_k = t \) for some point \( t \in X \) and \( \lambda > 0 \), the maps \( M \) and \( N \) are said to be \( w \)-compatible.

**Example 2.3.** Let \( X = \mathbb{R} \) and \( A \) be a function on \( X \) defined by
\[
A_\lambda(x_1, x_2, \ldots, x_n) = \frac{\lambda}{n} \sum_{i=1}^{n} \sum_{j=i+1}^{n} |x_i - x_j|
\]
for all \( \lambda > 0 \) and \( x_1, x_2, \ldots, x_n \in X \). Then, \((X, A)\) is a modular \( A \)-metric space. Let \( M \) and \( N \) be two self maps defined on \( X \) by \( M(x) = x^2 \) and \( N(x) = x^3 \) for each \( x \in \mathbb{R} \). Take \( \{x_k\} \) such that \( \{x_k\} = \frac{k}{k}, k = 1, 2, \ldots \). In this case, the maps \( M \) and \( N \) are \( w \)-compatible maps.
Theorem 2.4. Let \((X, A)\) be a complete modular \(A\)-metric space. Let \(T\) and \(S\) be maps from \(X\) into itself such that

1. \(T(X) \subset S(X)\)
2. \(T\) or \(S\) is continuous
3. \(A_\lambda (T(x_1, T(x_2, \ldots, T(x_k) \leq q A_\lambda (S(x_1, S(x_2, \ldots, S(x_k) \text{ for each } x_1, x_2, \ldots, x_k \in X \text{ and } 0 \leq q < 1}
4. \(T\) and \(S\) are \(w\)-compatible maps.

Then, \(T\) and \(S\) have a unique common fixed point in \(X\).

Proof. Let \(x_0\) be an arbitrary point in \(X\). We can choose a point \(x_1\) in \(X\) such that \(T x_0 = S x_1\) with \(T(X) \subset S(X)\). By generalizing this situation, we can construct a sequence \(\{x_{k+1}\}\) in \(X\) as follows:

\[ y_k = T x_k = S x_{k+1}, \quad k = 0, 1, 2, \ldots \]

From (3), we have

\[
A_\lambda (T x_k, T x_k, \ldots, T x_{k+1}) \\
\leq q A_\lambda (S x_k, S x_k, \ldots, S x_k, S x_{k+1}) \\
= q A_\lambda (T x_{k-1}, T x_{k-1}, \ldots, T x_{k-1}, T x_k) \\
\leq q^2 A_\lambda (S x_{k-1}, S x_{k-1}, \ldots, S x_{k-1}, S x_k) \\
= q^2 A_\lambda (T x_{k-2}, T x_{k-2}, \ldots, T x_{k-2}, T x_{k-1}) \\
\vdots \\
\leq q^k A_\lambda (S x_1, S x_1, \ldots, S x_1, S x_2) \\
= q^k A_\lambda (T x_0, T x_0, \ldots, T x_0, T x_1)
\]

Letting as \(k \to \infty\), we obtain

\[
\lim_{k \to \infty} A_\lambda (T x_k, T x_k, \ldots, T x_{k+1}) \leq \lim_{k \to \infty} q^k A_\lambda (T x_0, T x_0, \ldots, T x_1) = 0.
\]

For all \(k, m \in IN\) and \(k < m\), we have by rectangle inequality that

\[
A_\lambda (T x_k, T x_k, \ldots, T x_m) \\
\leq (n - 1) \sum_{i=k}^{m-2} A_{\frac{\lambda}{n^i}} (T x_i, \ldots, T x_{i+1}) + A_{\frac{\lambda}{n^{m-2}}} (T x_{m-1}, \ldots, T x_m) \\
\leq (n - 1) \sum_{i=k}^{m-2} q^i A_{\frac{\lambda}{n^i}} (T x_0, \ldots, T x_1) + q^{m-1} A_{\frac{\lambda}{n^{m-2}}} (T x_0, \ldots, T x_1)
\]
\[
\begin{align*}
&\leq (n - 1) \left[ q^k + q^{k+1} + \ldots + q^{m-2} + q^{m-1} \right] A_{\frac{\lambda}{n^k}}(T x_0, T x_0, \ldots, T x_1) \\
&\leq (n - 1) q^k \left( \frac{1 - q^{m-k}}{1 - q} \right) A_{\frac{\lambda}{n^k}}(T x_0, T x_0, \ldots, T x_1) \\
&\leq (n - 1) \left( \frac{q^k}{1 - q} \right) A_{\frac{\lambda}{n^k}}(T x_0, T x_0, \ldots, T x_1)
\end{align*}
\]

Letting as \( k \to \infty \), we have

\[
\lim_{k \to \infty} A_{\lambda}(T x_k, T x_k, \ldots, T x_m) \leq \lim_{k \to \infty} (n - 1) \left( \frac{q^k}{1 - q} \right) A_{\frac{\lambda}{n^k}}(T x_0, T x_0, \ldots, T x_1) = 0.
\]

Thus, \( \{T x_k\} \) is a Cauchy sequence in \( X \). Since \( (X, A) \) is complete modular \( A \)-metric space, it has a limit in \( X \) such that

\[
\lim_{k \to \infty} y_k = \lim_{k \to \infty} T x_k = \lim_{k \to \infty} S x_{k+1} = z.
\]

Since the maps \( T \) or \( S \) is continuous (assume that \( S \) is continuous), \( \lim_{k \to \infty} S T x_k = S z \).

Further, the maps \( S \) and \( T \) are \( w \)-compatible, therefore

\[
\lim_{k \to \infty} A_{\lambda}(S T x_k, S T x_k, T S x_k) = 0
\]

implies \( \lim_{k \to \infty} T S x_k = S z \). From (3), we have

\[
A_{\lambda}(S z, \ldots, S z, z) = A_{\lambda}(T S x_k, \ldots, T S x_k, T x_k) \\
\leq q A_{\lambda}(S x_k, \ldots, S x_k, S x_k).
\]

Proceeding limit as \( k \to \infty \), we have \( S z = z \). Again by (3), we obtain

\[
A_{\lambda}(T x_k, \ldots, T x_k, T z) \leq q A_{\lambda}(S x_k, \ldots, S x_k, S z)
\]

and taking limit as \( k \to \infty \), we have \( z = T z \). Thus, we have \( T z = S z = z \) and \( z \) is a common fixed point of \( T \) and \( S \).

Finally, the uniqueness of \( z \) as the common fixed point of \( T \) and \( S \) shows easily as follows:

Suppose that \( z_1 \neq z \) be another common fixed point of \( T \) and \( S \). Then,

\[
A_{\lambda}(z, z, \ldots, z, z_1) = A_{\lambda}(T z, T z, \ldots, T z, T z_1) \leq q A_{\lambda}(S z, S z, \ldots, S z, S z_1) \\
= q A_{\lambda}(z, z, \ldots, z, z_1) \\
< A_{\lambda}(z, z, \ldots, z, z_1)
\]

for \( 0 \leq q < 1 \), so this is a contradiction. Therefore, \( z = z_1 \). Hence, the common fixed point of \( T \) and \( S \) is unique. \( \Box \)
Example 2.5. Let $X = [-1, 1]$ and define $A_{\lambda} : (0, \infty) \times X^n \to [0, \infty]$ as follows:

$$A_{\lambda}(x_1, x_2, \ldots, x_n) = \frac{\lambda}{n} \sum_{i=1}^{n} \sum_{j=i+1}^{n} |x_i - x_j|$$

for all $x_1, x_2, \ldots, x_n \in X$ and all $\lambda > 0$. Clearly, $(X, A)$ is a modular $A$-metric space. Let $T$ and $S$ be maps from $X$ into itself defined as $T(x) = \frac{x}{7}$ and $S(x) = \frac{x}{3}$ for all $x \in X$. Then,

$$T(X) = \left[-\frac{1}{7}, \frac{1}{7}\right] \subseteq \left[-\frac{1}{3}, \frac{1}{3}\right] = S(X)$$

and the maps $T, S$ are continuous. Also,

$$A_{\lambda}(Tx_1, Tx_2, \ldots, Tx_n) \leq qA_{\lambda}(Sx_1, Sx_2, \ldots, Sx_n)$$

holds for all $x_1, x_2, \ldots, x_n \in X$ and $\frac{3}{7} \leq q < 1$. Moreover, the maps $T$ and $S$ are $w$-compatible since

$$\lim_{k \to \infty} A_{\lambda}(TSx_k, \ldots, TSx_k, STx_k) = 0$$

where $\{x_k\} = \frac{1}{k}$ is a sequence for $k = 1, 2, \ldots$ in $X$ such that

$$\lim_{k \to \infty} Sx_k = \lim_{k \to \infty} \frac{1}{3k} = 0$$

$$\lim_{k \to \infty} Tx_k = \lim_{k \to \infty} \frac{1}{7k} = 0$$

for $0 \in X$. Thus, 0 is the unique common fixed point of $T$ and $S$.

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**References**

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