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AN OPTIMAL INEQUALITY FOR WARPED PRODUCT LIGHTLIKE SUBMANIFOLDS

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Abstract. In this paper, we establish several geometric characterizations focusing on the relationship between the squared norm of the second fundamental form and the warping function of SCR-lightlike warped product submanifolds in an indefinite Kaehler manifold. In particular, we find an estimate for the squared norm of the second fundamental form h in terms of the Hessian of the warping function λ for SCR-lightlike warped product submanifolds of an indefinite complex space form. Consequently, we derive an optimal inequality, namely

$$||h||^2 \geq 2q\{\Delta(ln\lambda) + ||\nabla(ln\lambda)||^2 + \frac{c}{2}p\},$$

for SCR-lightlike warped product submanifolds in an indefinite complex space form. We also provide one non-trivial example for this class of warped products in an indefinite Kaehler manifold.

1. Introduction

In 1969, Bishop and O'Neill [4] introduced a natural framework to construct negatively curved manifolds by giving the notion of warped product manifolds. From the application point of view, warped product manifolds have been successfully employed in the study of black holes and space-time near bodies with large gravitational force (cf., [15]). In furtherance of it, Chen [5] studied CR-warped product submanifolds of Kaehler manifolds and proved that warped product CR-submanifolds of the type $N_\perp \times_\lambda N_T$ do not exist in Kaehler manifolds such that N_T and $N_\perp$ respectively denotes a complex submanifold and a totally real submanifold of a Kaehler manifold. Then, he proved the existence of CR-warped product submanifolds of the type $N_T \times_\lambda N_\perp$ and also established a geometric inequality for the second fundamental form in terms of the warping function for this kind of warped product submanifolds. After that, many researchers investigated warped product submanifolds in different ambient space

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settings to find geometrical inequalities arising from warped products (cf., [1], [2], [6], [7], [12], [14]).

On the other hand, one may observe that most of the available works on warped product submanifolds is explored on manifolds with non-degenerate metric. Thus, the available results may not be suitable to study those topics in mathematical physics, where indefinite metrics are employed. For instance, warped products have been significantly utilized in the study of black holes and cosmological models (cf., [11], [15]). In cosmological models, there do exist some points, where the warping function becomes zero. These points are called singular points. Further, at singular points, the metric of the product manifold becomes degenerate. Thus to deal with the degenerate metric, one possible solution is to use the techniques of semi-Riemannian geometry (cf., [17]). In this context, Duggal [9] introduced the concept of warped product lightlike manifolds, where he constructed two types of warped product lightlike manifolds. On a similar note, Sahin [16] studied warped product lightlike submanifolds of semi-Riemannian manifolds. Very recently, Kumar [13] investigated the non-existence of warped product SCR-lightlike submanifolds of the type $N_{\perp} \times_{\lambda} N_T$ of indefinite Kaehler manifolds and proved the existence of warped product SCR-lightlike submanifolds of the type $N_T \times_{\lambda} N_{\perp}$ in indefinite Kaehler manifolds. But till date, no attempts have been made to find geometrical inequalities associated with warped product lightlike submanifolds in semi-Riemannian manifolds.

In this paper, we emphasize on the relationship between the second fundamental form and the warping function for SCR-lightlike warped product submanifolds in indefinite Kaehler manifolds. Particularly, we derive an estimate for the squared norm of the second fundamental form in terms of the Hessian of the warping function λ for SCR-lightlike warped product submanifolds of an indefinite complex space form. Consequently, we establish a geometric inequality giving a lower bound for the squared norm of the second fundamental form for SCR-lightlike warped product submanifolds in indefinite complex space forms. Finally, we provide an example for this class of warped products in an indefinite Kaehler manifold.

2. Preliminaries

2.1. Geometry of lightlike submanifolds

In this section, we recall some basic formulae and notations on lightlike submanifolds for later use following [10].

Assume a submanifold (N_n, g) of a semi-Riemannian manifold $(\tilde{N}_{m+n}, \tilde{g})$, where the metric $\tilde{g}$ is of constant index q satisfying $m, n \geq 1$, $1 \leq q \leq m+n-1$. If $\tilde{g}$ is degenerate on TN, then T_pN and $T_pN^{\perp}$ both are degenerate and not complementary to one another. Thus, there exists a subspace $Rad(T_pN) = T_pN \cap T_pN^{\perp}$, known as a radical subspace. If $Rad(TN) : p \in N \to Rad(T_pN)$

is a smooth distribution on N of rank $r(>0)$, then N is known as an r-lightlike submanifold of $\tilde{N}$ and $Rad(TN)$ is called the radical distribution on N. While the radical distribution $Rad(TN)$ of TN is defined as

$$Rad(TN) = \cup_{p\in N}\{\xi \in T_pN | g(u,\xi) = 0,\ \forall\ u \in T_pN, \xi \neq 0\}.$$

Thus, the tangent bundle TN and the normal bundle $TN^{\perp}$ are decomposed as

$$TN = Rad(TN) \perp S(TN) \text{ and } TN^{\perp} = Rad(TN) \perp S(TN^{\perp}).$$

Theorem 2.1. [10] *For an r-lightlike submanifold $(N, g, S(TN), S(TN^{\perp}))$ of a semi-Riemannian manifold $(\tilde{N}, \tilde{g})$, there exists a complementary vector bundle $ltr(TN)$ of $Rad(TN)$ in $S(TN^{\perp})^{\perp}$ and basis of $\Gamma(ltr(TN)\mid_{\mathrm{u}})$ consisting of smooth section $\{N_i\}$ of $S(TN^{\perp})^{\perp}\mid_{\mathrm{u}}$, where u is a coordinate neighborhood of N satisfying*

$$\tilde{g}(N_i, N_j) = 0, \quad \tilde{g}(N_i, \xi_j) = \delta_{ij},\ \text{for}\quad i,j \in \{1,2,..,r\},$$

where $\{\xi_1, ..., \xi_r\}$ is the lightlike basis of $\Gamma(Rad(TN))$.

As a result, the decomposition of the tangent bundle $T\tilde{N}$ is given by

$$T\tilde{N}|_N = TN \oplus tr(TN) = S(TN) \perp (Rad(TN) \oplus ltr(TN)) \perp S(TN^{\perp}). \tag{1}$$

Further, the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_{Y_1}Y_2 = \nabla_{Y_1}Y_2 + h(Y_1,Y_2),\ \tilde{\nabla}_{Y_1}V = -A_VY_1 + \nabla^{\perp}_{Y_1}V,$$

for $X, Y \in \Gamma(TN)$ and $V \in \Gamma(tr(TN))$, where $\tilde{\nabla}$ represents the Levi-Civita connection on $\tilde{N}$. In view of decomposition (1), the Gauss and Weingartan formulae become

$$\tilde{\nabla}_{Y_1}Y_2 = \nabla_{Y_1}Y_2 + h^l(Y_1,Y_2) + h^s(Y_1,Y_2), \tag{2}$$

$$\tilde{\nabla}_{Y_1}N' = -A_{N'}Y_1 + \nabla^l_{Y_1}N' + D^s(Y_1,N'), \tag{3}$$

$$\tilde{\nabla}_{Y_1}W = -A_WY_1 + \nabla^s_{Y_1}W + D^l(Y_1,W), \tag{4}$$

for $Y_1, Y_2 \in \Gamma(TN)$, $W \in \Gamma(S(TN^{\perp}))$ and $N' \in \Gamma(ltr(TN))$.
Furthermore, employing Eqs. (2)-(4), we derive

$$g(A_WY_1,Y_2) = \tilde{g}(h^s(Y_1,Y_2),W) + \tilde{g}(Y_2, D^l(Y_1,W)), \tag{5}$$

$$\tilde{g}(D^s(Y_1,N'),W) = \tilde{g}(A_WY_1,N'). \tag{6}$$

Let us denote by $\tilde{R}$ and R the curvature tensors of $\tilde{\nabla}$ and ∇ respectively then we have

$$\begin{aligned}\tilde{R}(Y_1,Y_2)Y_3 =& R(Y_1,Y_2)Y_3 + A_{h^l(Y_1,Y_3)}Y_2 - A_{h^l(Y_2,Y_3)}Y_1 + A_{h^s(Y_1,Y_3)}Y_2\\ &- A_{h^s(Y_2,Y_3)}Y_1 + (\nabla_{Y_1}h^l)(Y_2,Y_3) - (\nabla_{Y_2}h^l)(Y_1,Y_3)\\ &+ D^l(Y_1,h^s(Y_2,Y_3)) - D^l(Y_2,h^s(Y_1,Y_3)) + (\nabla_{Y_1}h^s)(Y_2,Y_3)\\ &- (\nabla_{Y_2}h^s)(Y_1,Y_3) + D^s(Y_1,h^l(Y_2,Y_3)) - D^s(Y_2,h^l(Y_1,Y_3)).\end{aligned}$$

Further, the equation of Codazzi is given by

$$(\tilde{R}(Y_1, Y_2)Y_3)^{\perp} = (\nabla_{Y_1} h^l)(Y_2, Y_3) - (\nabla_{Y_2} h^l)(Y_1, Y_3) + D^l(Y_1, h^s(Y_2, Y_3)) - D^l(Y_2, h^s(Y_1, Y_3)) + (\nabla_{Y_1} h^s)(Y_2, Y_3) - (\nabla_{Y_2} h^s)(Y_1, Y_3) + D^s(Y_1, h^l(Y_2, Y_3)) - D^s(Y_2, h^l(Y_1, Y_3)), \tag{7}$$

where

$$(\nabla_{Y_1} h^s)(Y_2, Y_3) = \nabla^s_{Y_1} h^s(Y_2, Y_3) - h^s(\nabla_{Y_1} Y_2, Y_3) - h^s(Y_2, \nabla_{Y_1} Y_3), \tag{8}$$

$$(\nabla_{Y_1} h^l)(Y_2, Y_3) = \nabla^l_{Y_1} h^l(Y_2, Y_3) - h^l(\nabla_{Y_1} Y_2, Y_3) - h^l(Y_2, \nabla_{Y_1} Y_3), \tag{9}$$

for $Y_1, Y_2, Y_3 \in \Gamma(TN)$.

2.2. Indefinite Kaehler manifolds

Let $(\tilde{N}, \tilde{g})$ be an indefinite almost Hermitian manifold with an almost complex structure $\tilde{J}$ of the type (1, 1) [3]. Then

$$\tilde{J}^2 = -I, \; \tilde{g}(\tilde{J}Y_1, \tilde{J}Y_2) = \tilde{g}(Y_1, Y_2), \; \forall \; Y_1, Y_2 \in \Gamma(T\tilde{N}). \tag{10}$$

An indefinite almost Hermitian manifold is called an indefinite Kaehler manifold if

$$(\tilde{\nabla}_{Y_1} \tilde{J})Y_2 = 0, \tag{11}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{N}$.
An indefinite complex space form $\tilde{N}(c)$ is an indefinite Kaehler manifold $\tilde{N}$ with constant holomorphic curvature c and its curvature $\tilde{R}$ is given by

$$\tilde{R}(Y_1, Y_2)Y_3 = \frac{c}{4}\{\tilde{g}(Y_2, Y_3)Y_1 - \tilde{g}(Y_1, Y_3)Y_2 + \tilde{g}(\tilde{J}Y_2, Y_3)\tilde{J}Y_1 - \tilde{g}(\tilde{J}Y_1, Y_3)\tilde{J}Y_2 + 2\tilde{g}(Y_1, \tilde{J}Y_2)\tilde{J}Y_3\} \tag{12}$$

for $Y_1, Y_2, Y_3 \in \Gamma(T\tilde{N})$.

2.3. Screen Cauchy-Riemann SCR-lightlike submanifolds

Definition 2.2. [8] *Let $(N, g, S(TN))$ be a real lightlike submanifold of an indefinite Kaehler manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Then N is called a Screen Cauchy-Riemann (SCR)-lightlike submanifold, if the following conditions are satisfied:*

(A) *There exists a real non-null distribution $D \subset S(TN)$ such that*

$$S(TN) = D \oplus D^{\perp}, \tilde{J}D^{\perp} \subset S(TN^{\perp}), D \cap D^{\perp} = 0,$$

where $D^{\perp}$ is an orthogonal complementary distribution to D in $S(TN)$.

(B) *$Rad(TN)$ is invariant with respect to $\tilde{J}$.*

Further, it follows that $TN = D' \oplus D^{\perp}$ where $D' = D \perp Rad(TN)$.

3. SCR-lightlike warped product submanifolds of indefinite Kaehler manifolds

In [13], Kumar investigated SCR-lightlike warped product submanifolds of indefinite Kaehler manifolds. He proved that there does not exist SCR-lightlike warped product submanifolds of the type $N_\perp \times_\lambda N_T$ in indefinite Kaehler manifolds. Further, the existence of SCR-lightlike warped product submanifolds of the type $N_T \times_\lambda N_\perp$ was obtained by developing several results in terms of the canonical structures. Next, it is obvious to seek geometric estimates arising for this class of warped products. Therefore, in this section, we find some characterization theorems giving geometric estimates for SCR-lightlike warped product submanifolds of the type $N_T \times_\lambda N_\perp$ in indefinite Kaehler manifolds. Before proceeding, we recall an important result on warped product manifolds given by Bishop and O'Neill [4] as follows.

Theorem 3.1. [4] *For a warped product manifold $N = N_1 \times_\lambda N_2$, one has*

$$\nabla_{Y_1} Y_2 \in \Gamma(TN_1),$$

$$\nabla_{Y_1} Z_1 = \nabla_{Z_1} Y_1 = (Y_1 ln\lambda) Z_1, \tag{13}$$

$$\nabla_{Z_1} Z_2 = -\frac{g(Z_1, Z_2)}{\lambda} \nabla\lambda.$$

for $Y_1, Y_2 \in \Gamma(TN_1)$ and $Z_1, Z_2 \in \Gamma(TN_2)$.

Note: Throughout this paper, we shall denote an indefinite Kaehler manifold by $\tilde{N}$, an indefinite complex space form by $\tilde{N}(c)$ and warped product by **w.p.**, unless otherwise mentioned.
Firstly, we give a basic lemma for later use.

Lemma 3.2. *Consider $N = N_T \times_\lambda N_\perp$ be a SCR-lightlike* **w.p.** *submanifold of $\tilde{N}$. Then*

$$\tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) = (Y_1 ln\lambda)||Z_1||^2 \tag{14}$$

and

$$\tilde{g}(h^s(Y_1, Z_1), \tilde{J}Z_1) = -(\tilde{J}Y_1 ln\lambda)||Z_1||^2, \tag{15}$$

where $Y_1 \in \Gamma(D')$ and $Z_1 \in \Gamma(D^\perp)$.

Proof. For $Y_1 \in \Gamma(D')$ and $Z_1 \in \Gamma(D^\perp)$, employing Eqs. (2), (10), (11) and (13), we attain

$$\begin{aligned}\tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) &= \tilde{g}(\tilde{\nabla}_{Z_1}\tilde{J}Y_1 - \nabla_{Z_1}\tilde{J}Y_1, \tilde{J}Z_1)\\ &= \tilde{g}(\tilde{\nabla}_{Z_1}\tilde{J}Y_1, \tilde{J}Z_1) - (\tilde{J}Y_1 ln\lambda)\tilde{g}(Z_1, \tilde{J}Z_1)\\ &= \tilde{g}(\tilde{J}\tilde{\nabla}_{Z_1}Y_1, \tilde{J}Z_1) = \tilde{g}(\tilde{\nabla}_{Z_1}Y_1, Z_1) = g(\nabla_{Z_1}Y_1, Z_1)\\ &= (Y_1 ln\lambda)||Z_1||^2.\end{aligned}$$

Similarly, using Eqs. (2), (10), (11) and (13), for $Y_1 \in \Gamma(D')$ and $Z_1 \in \Gamma(D^{\perp})$, we derive

$$\tilde{g}(h^s(Y_1, Z_1), \tilde{J}Z_1) = -(\tilde{J}Y_1 ln\lambda)||Z_1||^2.$$

This completes the proof. □

Corollary 3.3. *Let $N = N_T \times_\lambda N_\perp$ be a SCR-lightlike* ***w.p.*** *submanifold of $\tilde{N}$. Then*

(i) $\tilde{g}(h^s(\tilde{J}Y_1, \nabla_{Y_1} Z_1), \tilde{J}Z_1) = (Y_1 ln\lambda)^2||Z_1||^2$,
(ii) $\tilde{g}(h^s(Y_1, \nabla_{\tilde{J}Y_1} Z_1), \tilde{J}Z_1) = -(\tilde{J}Y_1 ln\lambda)^2||Z_1||^2$,

for $Y_1 \in \Gamma(D')$ and $Z_1 \in \Gamma(D^{\perp})$.

Proof. The result follows from Eq. (13) and Lemma 3.2. □

Theorem 3.4. *Consider $N = N_T \times_\lambda N_\perp$ be a SCR-lightlike* ***w.p.*** *submanifold of $\tilde{N}$. Then we have*

$$\begin{aligned}||h^s(\tilde{J}Y_1, Z_1)||^2 + ||h^s(Y_1, Z_1)||^2 =&(Y_1 ln\lambda)^2||Z_1||^2 + (\tilde{J}Y_1 ln\lambda)^2||Z_1||^2\\ &+ 2\tilde{g}(\tilde{J}h^s(Y_1, Z_1), h^s(\tilde{J}Y_1, Z_1)),\end{aligned}$$

for $Y_1 \in \Gamma(D')$ and $Z_1 \in \Gamma(D^{\perp})$.

Proof. For $Y_1 \in \Gamma(D')$ and $Z_1 \in \Gamma(D^{\perp})$, using Eqs. (2), (11), (13) and (14), we get

$$\begin{aligned}||h^s(\tilde{J}Y_1, Z_1)||^2 =&\tilde{g}(h^s(\tilde{J}Y_1, Z_1), h^s(\tilde{J}Y_1, Z_1))\\ =&\tilde{g}(\tilde{\nabla}_{Z_1}\tilde{J}Y_1 - \nabla_{Z_1}\tilde{J}Y_1, h^s(\tilde{J}Y_1, Z_1))\\ =&\tilde{g}(\tilde{\nabla}_{Z_1}\tilde{J}Y_1, h^s(\tilde{J}Y_1, Z_1)) - (\tilde{J}Y_1 ln\lambda)\tilde{g}(Z_1, h^s(\tilde{J}Y_1, Z_1))\\ =&\tilde{g}(\tilde{J}\tilde{\nabla}_{Z_1}Y_1, h^s(\tilde{J}Y_1, Z_1))\\ =&\tilde{g}(\tilde{J}\nabla_{Z_1}Y_1, h^s(\tilde{J}Y_1, Z_1)) + \tilde{g}(\tilde{J}h^s(Y_1, Z_1), h^s(\tilde{J}Y_1, Z_1))\\ =&(Y_1 ln\lambda)\tilde{g}(\tilde{J}Z_1, h^s(\tilde{J}Y_1, Z_1)) + \tilde{g}(\tilde{J}h^s(Y_1, Z_1), h^s(\tilde{J}Y_1, Z_1))\\ =&(Y_1 ln\lambda)^2||Z_1||^2 + \tilde{g}(\tilde{J}h^s(Y_1, Z_1), h^s(\tilde{J}Y_1, Z_1)).\end{aligned} \tag{16}$$

Similarly, from Eqs. (2), (10), (11), (13) and (15), for $Y_1 \in \Gamma(D')$ and $Z_1 \in \Gamma(D^{\perp})$, we derive

$$||h^s(Y_1, Z_1)||^2 =(\tilde{J}Y_1 ln\lambda)^2||Z_1||^2 + \tilde{g}(\tilde{J}h^s(Y_1, Z_1), h^s(\tilde{J}Y_1, Z_1)). \tag{17}$$

Then on adding Eqs. (16) and (17), the proof follows. □

4. SCR-lightlike warped product submanifolds in an indefinite complex space form

Consider (N, g) be an n-dimensional Riemannian manifold and λ be a smooth function defined on N. Then the Hessian of λ is given by

$$H^{\lambda}(Y_1, Y_2) = Y_1 Y_2 \lambda - (\nabla_{Y_1} Y_2)\lambda, \tag{18}$$

for $Y_1, Y_2 \in \Gamma(TN)$.
Assume that $\{Y_1, ..., Y_n\}$ be an orthogonal basis of TN, then the Laplacian of λ is defined by

$$\Delta\lambda = \sum_{i=1}^{n} \{(\nabla_{Y_i} Y_i)\lambda - Y_i Y_i \lambda\}. \tag{19}$$

Next, we present an important result on SCR-lightlike **w.p.** submanifolds in $\tilde{N}(c)$ involving the squared norm of the second fundamental form and the Hessian of the warping function λ.

Theorem 4.1. *Assume that $N = N_T \times_{\lambda} N_{\perp}$ be a SCR-lightlike* **w.p.** *submanifold of $\tilde{N}(c)$. Then*

$$\begin{aligned} ||h^s(\tilde{J}Y_1, Z_1)||^2 + ||h^s(Y_1, Z_1)||^2 &= \{H^{ln\lambda}(Y_1, Y_1) + H^{ln\lambda}(\tilde{J}Y_1, \tilde{J}Y_1)\}||Z_1||^2 \\ &+ \{\frac{c}{2}||Y_1||^2 + (Y_1 ln\lambda)^2 + (\tilde{J}Y_1 ln\lambda)^2\}||Z_1||^2, \end{aligned} \tag{20}$$

for $Y_1 \in \Gamma(D')$ and $Z_1 \in \Gamma(D^{\perp})$.

Proof. For $Y_1 \in \Gamma(D')$ and $Z_1 \in \Gamma(D^{\perp})$, taking in account Eq. (12), we get

$$\tilde{R}(Y_1, \tilde{J}Y_1, Z_1, \tilde{J}Z_1) = -\frac{c}{2}||Y_1||^2||Z_1||^2. \tag{21}$$

On the other hand, from Eq. (7), for $Y_1 \in \Gamma(D')$ and $Z_1 \in \Gamma(D^{\perp})$, we attain

$$\begin{aligned} (\tilde{R}(Y_1, \tilde{J}Y_1)Z_1)^{\perp} =& (\nabla_{Y_1} h^l)(\tilde{J}Y_1, Z_1) - (\nabla_{\tilde{J}Y_1} h^l)(Y_1, Z_1) \\ &+ D^l(Y_1, h^s(\tilde{J}Y_1, Z_1)) - D^l(\tilde{J}Y_1, h^s(Y_1, Z_1)) \\ &+ (\nabla_{Y_1} h^s)(\tilde{J}Y_1, Z_1)) - (\nabla_{\tilde{J}Y_1} h^s)(Y_1, Z_1) \\ &+ D^s(Y_1, h^l(\tilde{J}Y_1, Z_1)) - D^s(\tilde{J}Y_1, h^l(Y_1, Z_1)). \end{aligned} \tag{22}$$

Taking the inner product of Eq. (22) w.r.t. $\tilde{J}Z_1$ and using Eqs. (8) and (9), we derive

$$\begin{aligned} \tilde{R}(Y_1, \tilde{J}Y_1, Z_1, \tilde{J}Z_1) =& \tilde{g}(\nabla^s_{Y_1} h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) - \tilde{g}(h^s(\nabla_{Y_1}\tilde{J}Y_1, Z_1), \tilde{J}Z_1) \\ &- \tilde{g}(h^s(\tilde{J}Y_1, \nabla_{Y_1} Z_1), \tilde{J}Z_1) - \tilde{g}(\nabla^s_{\tilde{J}Y_1} h^s(Y_1, Z_1), \tilde{J}Z_1) \\ &+ \tilde{g}(h^s(\nabla_{\tilde{J}Y_1} Y_1, Z_1), \tilde{J}Z_1) + \tilde{g}(h^s(Y_1, \nabla_{\tilde{J}Y_1} Z_1), \tilde{J}Z_1) \\ &+ \tilde{g}(D^s(Y_1, h^l(\tilde{J}Y_1, Z_1)), \tilde{J}Z_1) \\ &- \tilde{g}(D^s(\tilde{J}Y_1, h^l(Y_1, Z_1)), \tilde{J}Z_1). \end{aligned} \tag{23}$$

From Eq. (4), we acquire

$$\tilde{g}(\tilde{\nabla}_{Y_1} h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) = -\tilde{g}(A_{h^s(\tilde{J}Y_1,Z_1)}Y_1, \tilde{J}Z_1) + \tilde{g}(\nabla^s_{Y_1} h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) + \tilde{g}(D^l(h^s(\tilde{J}Y_1, Z_1), Y_1), \tilde{J}Z_1),$$

which further yields

$$\tilde{g}(\tilde{\nabla}_{Y_1} h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) = \tilde{g}(\nabla^s_{Y_1} h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1). \tag{24}$$

Since $\tilde{\nabla}$ is a metric connection on $\tilde{N}$, therefore

$$\tilde{g}(\tilde{\nabla}_{Y_1} h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) = Y_1\tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) - \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{\nabla}_{Y_1}\tilde{J}Z_1), \tag{25}$$

where $Y_1 \in \Gamma(D')$ and $Z_1 \in \Gamma(D^\perp)$. Then from Eqs. (24) and (25), we obtain

$$\tilde{g}(\nabla^s_{Y_1} h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) = Y_1\tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) - \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{\nabla}_{Y_1}\tilde{J}Z_1). \tag{26}$$

Next using Eqs. (2), (11), (13), (14) and (16) in Eq. (26), we derive

$$\begin{aligned}\tilde{g}(\nabla^s_{Y_1} h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) =& Y_1\{(Y_1 ln\lambda)||Z_1||^2\} - \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}\tilde{\nabla}_{Y_1}Z_1)\\ =& Y_1(Y_1 ln\lambda)||Z_1||^2 + 2(Y_1 ln\lambda)^2||Z_1||^2\\ & - \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}\nabla_{Y_1}Z_1)\\ & - \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}h^s(Y_1, Z_1))\\ =& Y_1(Y_1 ln\lambda)||Z_1||^2 + 2(Y_1 ln\lambda)^2||Z_1||^2\\ & - (Y_1 ln\lambda)\tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1)\\ & - \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}h^s(Y_1, Z_1))\\ =& Y_1(Y_1 ln\lambda)||Z_1||^2 + 2(Y_1 ln\lambda)^2||Z_1||^2\\ & - ||h^s(\tilde{J}Y_1, Z_1)||^2.\end{aligned} \tag{27}$$

Similarly, one has

$$\begin{aligned}\tilde{g}(\nabla^s_{\tilde{J}Y_1} h^s(Y_1, Z_1), \tilde{J}Z_1) =& -\tilde{J}Y_1(\tilde{J}Y_1 ln\lambda)||Z_1||^2 - 2(\tilde{J}Y_1 ln\lambda)^2||Z_1||^2\\ & + ||h^s(Y_1, Z_1)||^2.\end{aligned} \tag{28}$$

Further from Eqs. (5) and (14), we derive

$$\begin{aligned}g(A_{\tilde{J}Z_1}Z_1, \tilde{J}Y_1) &= \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) + \tilde{g}(D^l(Z_1, \tilde{J}Z_1), \tilde{J}Y_1)\\ &= (Y_1 ln\lambda)||Z_1||^2 + \tilde{g}(D^l(Z_1, \tilde{J}Z_1), \tilde{J}Y_1).\end{aligned}$$

As N_T is totally geodesic, therefore for $Y_1 \in \Gamma(TN_T)$, we have $\nabla_{Y_1}Y_1 \in \Gamma(TN_T)$. Thus replacing Y_1 by $\nabla_{Y_1}Y_1$, the above equation becomes

$$g(A_{\tilde{J}Z_1}Z_1, \tilde{J}\nabla_{Y_1}Y_1) = (\nabla_{Y_1}Y_1 ln\lambda)||Z_1||^2 + \tilde{g}(D^l(Z_1, \tilde{J}Z_1), \tilde{J}\nabla_{Y_1}Y_1). \tag{29}$$

Then for $Y_1 \in \Gamma(D')$ and $Z_1 \in \Gamma(D^{\perp})$, using Eqs. (2), (5) and (11), we derive

$$\begin{aligned} g(A_{\tilde{J}Z_1}Z_1, \tilde{J}\nabla_{Y_1}Y_1) =& \tilde{g}(A_{\tilde{J}Z_1}Z_1, \tilde{J}\tilde{\nabla}_{Y_1}Y_1) = \tilde{g}(A_{\tilde{J}Z_1}Z_1, \tilde{\nabla}_{Y_1}\tilde{J}Y_1) \\ =& g(A_{\tilde{J}Z_1}Z_1, \nabla_{Y_1}\tilde{J}Y_1) \\ =& \tilde{g}(h^s(Z_1, \nabla_{Y_1}\tilde{J}Y_1), \tilde{J}Z_1) + \tilde{g}(D^l(Z_1, \tilde{J}Z_1), \nabla_{Y_1}\tilde{J}Y_1), \end{aligned}$$

which further gives

(30)

$$\tilde{g}(h^s(Z_1, \nabla_{Y_1}\tilde{J}Y_1), \tilde{J}Z_1) = g(A_{\tilde{J}Z_1}Z_1, \tilde{J}\nabla_{Y_1}Y_1) - \tilde{g}(D^l(Z_1, \tilde{J}Z_1), \nabla_{Y_1}\tilde{J}Y_1).$$

Then from Eqs. (29) and (30), we have

$$\tilde{g}(h^s(Z_1, \nabla_{Y_1}\tilde{J}Y_1), \tilde{J}Z_1) = (\nabla_{Y_1}Y_1 ln\lambda)||Z_1||^2. \tag{31}$$

By replacing Y_1 by $\tilde{J}Y_1$ in Eq. (31), we obtain

$$\tilde{g}(h^s(Z_1, \nabla_{\tilde{J}Y_1}Y_1), \tilde{J}Z_1) = -(\nabla_{\tilde{J}Y_1}\tilde{J}Y_1 ln\lambda)||Z_1||^2. \tag{32}$$

On the other hand, from Eqs. (2), (4), (6), (11) and (13), we derive

$$\begin{aligned} \tilde{g}(D^s(Y_1, h^l(\tilde{J}Y_1, Z_1)), \tilde{J}Z_1) &= \tilde{g}(A_{\tilde{J}Z_1}Y_1, h^l(\tilde{J}Y_1, Z_1)) \\ &= -\tilde{g}(\tilde{\nabla}_{Y_1}\tilde{J}Z_1, h^l(\tilde{J}Y_1, Z_1)) \\ &= -\tilde{g}(\tilde{J}\tilde{\nabla}_{Y_1}Z_1, h^l(\tilde{J}Y_1, Z_1)) \\ &= -\tilde{g}(\tilde{J}\nabla_{Y_1}Z_1, h^l(\tilde{J}Y_1, Z_1)) \\ &= 0. \end{aligned} \tag{33}$$

Similarly, we have

$$\tilde{g}(D^s(\tilde{J}Y_1, h^l(Y_1, Z_1)), \tilde{J}Z_1) = 0. \tag{34}$$

Further using Eqs. (27)-(28), (31)-(34) and Corollary 3.3 in Eq. (23), we get

$$\begin{aligned} \tilde{R}(Y_1, \tilde{J}Y_1, Z_1, \tilde{J}Z_1) = & \{Y_1(Y_1 ln\lambda) - \nabla_{Y_1}Y_1 ln\lambda + \tilde{J}Y_1(\tilde{J}Y_1 ln\lambda)\}||Z_1||^2 \\ & - \nabla_{\tilde{J}Y_1}\tilde{J}Y_1 ln\lambda||Z_1||^2 + (Y_1 ln\lambda)^2||Z_1||^2 + (\tilde{J}Y_1 ln\lambda)^2||Z_1||^2 \\ & - ||h^s(\tilde{J}Y_1, Z_1)||^2 - ||h^s(Y_1, Z_1)||^2. \end{aligned} \tag{35}$$

Then using Eq. (18) in Eq. (35), we obtain

$$\begin{aligned} \tilde{R}(Y_1, \tilde{J}Y_1, Z_1, \tilde{J}Z_1) =& \{H^{ln\lambda}(Y_1, Y_1) + H^{ln\lambda}(\tilde{J}Y_1, \tilde{J}Y_1)\}||Z_1||^2 + (Y_1 ln\lambda)^2||Z_1||^2 \\ & + (\tilde{J}Y_1 ln\lambda)^2||Z_1||^2 - ||h^s(\tilde{J}Y_1, Z_1)||^2 - ||h^s(Y_1, Z_1)||^2. \end{aligned} \tag{36}$$

From Eqs. (21) and (36), we attain

$$\begin{aligned} -\frac{c}{2}||Y_1||^2||Z_1||^2 =& \{H^{ln\lambda}(Y_1, Y_1) + H^{ln\lambda}(\tilde{J}Y_1, \tilde{J}Y_1)\}||Z_1||^2 + (Y_1 ln\lambda)^2||Z_1||^2 \\ & + (\tilde{J}Y_1 ln\lambda)^2||Z_1||^2 - ||h^s(\tilde{J}Y_1, Z_1)||^2 - ||h^s(Y_1, Z_1)||^2. \end{aligned}$$

Hence, the proof follows. □

Corollary 4.2. *For a SCR-lightlike* **w.p.** *submanifold* $N = N_T \times_\lambda N_\perp$ *of* $\tilde{N}(c)$*, one has*

$$||h^s(\tilde{J}Y_1, Z_1)||^2 + ||h^s(Y_1, Z_1)||^2 = H^{ln\lambda}(Y_1, Y_1) + H^{ln\lambda}(\tilde{J}Y_1, \tilde{J}Y_1) + \frac{c}{2} + (Y_1 ln\lambda)^2 + (\tilde{J}Y_1 ln\lambda)^2,$$

for $Y_1 \in \Gamma(TN_T)$ *and* $Z_1 \in \Gamma(TN_\perp)$.

Proof. Particularly, for unit vectors $Y_1 \in \Gamma(TN_T)$ and $Z_1 \in \Gamma(TN_\perp)$, the proof follows directly from Eq. (20). □

Theorem 4.3. *Consider* $N = N_T \times_\lambda N_\perp$ *be a SCR-lightlike* **w.p.** *submanifold of* $\tilde{N}(c)$*. Then*

$$\begin{aligned} ||h^s(\tilde{J}Y_1, Z_1)||^2 + ||h^s(Y_1, Z_1)||^2 =& \frac{c}{2}||Y_1||^2||Z_1||^2 - (Y_1 ln\lambda)^2||Z_1||^2 \\ &- (\tilde{J}Y_1 ln\lambda)^2||Z_1||^2 - \tilde{g}(A_{\tilde{J}Z_1} Z_1, [Y_1, \tilde{J}Y_1]) \\ &+ \tilde{g}(D^l(Z_1, \tilde{J}Z_1), [Y_1, \tilde{J}Y_1]), \end{aligned}$$

for $Y_1 \in \Gamma(D')$ *and* $Z_1 \in \Gamma(D^\perp)$.

Proof. For $Y_1 \in \Gamma(D')$ and $Z_1 \in \Gamma(D^\perp)$, using Eq. (12), we have

$$\tilde{R}(Y_1, \tilde{J}Y_1, Z_1, \tilde{J}Z_1) = -\frac{c}{2}||Y_1||^2||Z_1||^2. \tag{37}$$

On the other hand, taking into account the Codazzi equation (7) with Eqs. (8) and (9), for $Y_1 \in \Gamma(D')$ and $Z_1 \in \Gamma(D^\perp)$, we acquire

$$\begin{aligned} \tilde{R}(Y_1, \tilde{J}Y_1, Z_1, \tilde{J}Z_1) =& \tilde{g}(\nabla^s_{Y_1} h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) - \tilde{g}(h^s(\nabla_{Y_1}\tilde{J}Y_1, Z_1), \tilde{J}Z_1) \\ &- \tilde{g}(h^s(\tilde{J}Y_1, \nabla_{Y_1} Z_1), \tilde{J}Z_1) - \tilde{g}(\nabla^s_{\tilde{J}Y_1} h^s(Y_1, Z_1), \tilde{J}Z_1) \\ &+ \tilde{g}(h^s(\nabla_{\tilde{J}Y_1} Y_1, Z_1), \tilde{J}Z_1) + \tilde{g}(h^s(Y_1, \nabla_{\tilde{J}Y_1} Z_1), \tilde{J}Z_1) \\ &+ \tilde{g}(D^s(Y_1, h^l(\tilde{J}Y_1, Z_1)), \tilde{J}Z_1) \\ &- \tilde{g}(D^s(\tilde{J}Y_1, h^l(Y_1, Z_1)), \tilde{J}Z_1). \end{aligned} \tag{38}$$

Then from Eqs. (2), (4), (11), (13) and (16) and Lemma 3.2, we derive

$$\begin{aligned} \tilde{g}(\nabla^s_{Y_1} h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) =& \tilde{g}(\tilde{\nabla}_{Y_1} h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_1) \\ =& -\tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{\nabla}_{Y_1}\tilde{J}Z_1) \\ =& -\tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}\tilde{\nabla}_{Y_1} Z_1) \\ =& -(Y_1 ln\lambda)^2||Z_1||^2 - \tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}h^s(Y_1, Z_1)) \\ =& -||h^s(\tilde{J}Y_1, Z_1)||^2. \end{aligned} \tag{39}$$

Similarly, we have

$$\tilde{g}(\nabla^s_{\tilde{J}Y_1} h^s(Y_1, Z_1), \tilde{J}Z_1) = ||h^s(Y_1, Z_1)||^2. \tag{40}$$

Further from Eq. (5), we get

(41)
$$\tilde{g}(h^s(\nabla_{Y_1}\tilde{J}Y_1, Z_1), \tilde{J}Z_1) = g(A_{\tilde{J}Z_1}Z_1, \nabla_{Y_1}\tilde{J}Y_1) - \tilde{g}(D^l(Z_1, \tilde{J}Z_1), \nabla_{Y_1}\tilde{J}Y_1).$$

Similarly, one has

$$\tilde{g}(h^s(\nabla_{\tilde{J}Y_1}Y_1, Z_1), \tilde{J}Z_1) = g(A_{\tilde{J}Z_1}Z_1, \nabla_{\tilde{J}Y_1}Y_1) - \tilde{g}(D^l(Z_1, \tilde{J}Z_1), \nabla_{\tilde{J}Y_1}Y_1). \quad (42)$$

Further following Eqs. (2), (4), (6), (11) and (13), we attain

$$\tilde{g}(D^s(Y_1, h^l(\tilde{J}Y_1, Z_1)), \tilde{J}Z_1) = 0 \quad (43)$$

and

$$\tilde{g}(D^s(\tilde{J}Y_1, h^l(Y_1, Z_1)), \tilde{J}Z_1) = 0. \quad (44)$$

Now employing Eqs. (39)-(44) and Corollary 3.3 in Eq. (38), we obtain

$$\begin{aligned}\tilde{R}(Y_1, \tilde{J}Y_1, Z_1, \tilde{J}Z_1) = &- ||h^s(\tilde{J}Y_1, Z_1)||^2 - ||h^s(Y_1, Z_1)||^2 - (Y_1 ln\lambda)^2||Z_1||^2 \\ &+ (\tilde{J}Y_1 ln\lambda)^2||Z_1||^2 - \tilde{g}(A_{\tilde{J}Z_1}Z_1, [Y_1, \tilde{J}Y_1]) \\ &+ \tilde{g}(D^l(Z_1, \tilde{J}Z_1), [Y_1, \tilde{J}Y_1]).\end{aligned} \quad (45)$$

Further using Eqs. (37) and (45), we derive

$$\begin{aligned}-\frac{c}{2}||Y_1||^2||Z_1||^2 = &- ||h^s(\tilde{J}Y_1, Z_1)||^2 - ||h^s(Y_1, Z_1)||^2 - (Y_1 ln\lambda)^2||Z_1||^2 \\ &- (\tilde{J}Y_1 ln\lambda)^2||Z_1||^2 - \tilde{g}(A_{\tilde{J}Z_1}Z_1, [Y_1, \tilde{J}Y_1]) \\ &+ \tilde{g}(D^l(Z_1, \tilde{J}Z_1), [Y_1, \tilde{J}Y_1]).\end{aligned}$$

Hence, the proof follows. □

5. An optimal inequality for SCR-lightlike warped product submanifolds

In this section, we establish a geometric inequality giving a lower bound for the squared norm of the second fundamental form for a SCR-lightlike **w.p.** submanifold in $\tilde{N}(c)$ as follows:

Theorem 5.1. *Consider $N = N_T \times_\lambda N_\perp$ be a SCR-lightlike **w.p.** submanifold of $\tilde{N}(c)$. Then the second fundamental form satisfies*

$$||h||^2 \geq 2q\{\Delta(ln\lambda) + ||\nabla(ln\lambda)||^2 + \frac{c}{2}p\},$$

where q is the dimension of $N_\perp$, $\Delta(ln\lambda)$ is the Laplacian of $ln\lambda$ and $\nabla(ln\lambda)$ is the gradient of $ln\lambda$.

Proof. For $Y_1 \in \Gamma(D')$ and $Z_1 \in \Gamma(D^\perp)$, from Eq. (12), we get

$$\tilde{R}(Y_1, \tilde{J}Y_1, Z_1, \tilde{J}Z_1) = -\frac{c}{2}||Y_1||^2||Z_1||^2. \quad (46)$$

On the other hand, using the Codazzi equation (7) with Eqs. (8) and (9), we obtain

$$\begin{aligned}\tilde{R}(Y_1,\tilde{J}Y_1,Z_1,\tilde{J}Z_1)=&\tilde{g}(\nabla^s_{Y_1}h^s(\tilde{J}Y_1,Z_1),\tilde{J}Z_1)-\tilde{g}(h^s(\nabla_{Y_1}\tilde{J}Y_1,Z_1),\tilde{J}Z_1)\\&-\tilde{g}(h^s(\tilde{J}Y_1,\nabla_{Y_1}Z_1),\tilde{J}Z_1)-\tilde{g}(\nabla^s_{\tilde{J}Y_1}h^s(Y_1,Z_1),\tilde{J}Z_1)\\&+\tilde{g}(h^s(\nabla_{\tilde{J}Y_1}Y_1,Z_1),\tilde{J}Z_1)+\tilde{g}(h^s(Y_1,\nabla_{\tilde{J}Y_1}Z_1),\tilde{J}Z_1)\\&+\tilde{g}(D^s(Y_1,h^l(\tilde{J}Y_1,Z_1)),\tilde{J}Z_1)\\&-\tilde{g}(D^s(\tilde{J}Y_1,h^l(Y_1,Z_1)),\tilde{J}Z_1).\end{aligned}\tag{47}$$

Since $\tilde{\nabla}$ is a metric connection and using Eq. (4), we acquire

$$\tilde{g}(\nabla^s_{Y_1}h^s(\tilde{J}Y_1,Z_1),\tilde{J}Z_1)=Y_1\tilde{g}(h^s(\tilde{J}Y_1,Z_1),\tilde{J}Z_1)-\tilde{g}(h^s(\tilde{J}Y_1,Z_1),\tilde{\nabla}_{Y_1}\tilde{J}Z_1).\tag{48}$$

Then employing Eqs. (13) and (14), the first term on R. H. S. of Eq. (48) takes the form

$$Y_1\tilde{g}(h^s(\tilde{J}Y_1,Z_1),\tilde{J}Z_1)=Y_1(Y_1ln\lambda)||Z_1||^2+2(Y_1ln\lambda)^2||Z_1||^2.\tag{49}$$

Further using Eqs. (2), (11), (13) and (14), the last term on R. H. S. of Eq. (48) becomes

$$\begin{aligned}\tilde{g}(h^s(\tilde{J}Y_1,Z_1),\tilde{\nabla}_{Y_1}\tilde{J}Z_1)=&\tilde{g}(h^s(\tilde{J}Y_1,Z_1),\tilde{J}\tilde{\nabla}_{Y_1}Z_1)\\=&(Y_1ln\lambda)^2||Z_1||^2+\tilde{g}(h^s(\tilde{J}Y_1,Z_1),\tilde{J}h^s(Y_1,Z_1))\\=&||h^s(\tilde{J}Y_1,Z_1)||^2\end{aligned}\tag{50}$$

Using Eqs. (49) and (50) in Eq. (48), we derive

$$\begin{aligned}\tilde{g}(\nabla^s_{Y_1}h^s(\tilde{J}Y_1,Z_1),\tilde{J}Z_1)=&\{Y_1(Y_1ln\lambda)+2(Y_1ln\lambda)^2\}||Z_1||^2\\&-||h^s(\tilde{J}Y_1,Z_1)||^2.\end{aligned}\tag{51}$$

Similarly, we have

$$\begin{aligned}\tilde{g}(\nabla^s_{\tilde{J}Y_1}h^s(Y_1,Z_1),\tilde{J}Z_1)=&-\{\tilde{J}Y_1(\tilde{J}Y_1ln\lambda)+2(\tilde{J}Y_1ln\lambda)^2\}||Z_1||^2\\&+||h^s(Y_1,Z_1)||^2.\end{aligned}\tag{52}$$

As N_T is totally geodesic in N, this gives $\nabla_{\tilde{J}Y_1}Y_1\in\Gamma(TN_T)$ for $Y_1\in\Gamma(TN_T)$ and hence from Eq. (15), we attain

$$\tilde{g}(h^s(\nabla_{\tilde{J}Y_1}Y_1,Z_1),\tilde{J}Z_1)=-(\tilde{J}\nabla_{\tilde{J}Y_1}Y_1ln\lambda)||Z_1||^2.\tag{53}$$

Then using Eqs. (2), (13) and (15), we have

$$\begin{aligned}\tilde{g}(h^s(\nabla_{Y_1}\tilde{J}Y_1,Z_1),\tilde{J}Z_1)=&-(\tilde{J}\nabla_{Y_1}\tilde{J}Y_1ln\lambda)||Z_1||^2\\=&-g((\tilde{J}\nabla_{Y_1}\tilde{J}Y_1ln\lambda)Z_1,Z_1)\\=&-g(\nabla_{Z_1}\tilde{J}\nabla_{Y_1}\tilde{J}Y_1,Z_1)=-\tilde{g}(\tilde{\nabla}_{Z_1}\tilde{J}\nabla_{Y_1}\tilde{J}Y_1,Z_1)\\=&\tilde{g}(\tilde{J}\nabla_{Y_1}\tilde{J}Y_1,\tilde{\nabla}_{Z_1}Z_1).\end{aligned}$$

Further using Eqs. (2), (10) and (11), the above equation yields

$$\begin{aligned}\tilde{g}(h^s(\nabla_{Y_1}\tilde{J}Y_1, Z_1), \tilde{J}Z_1) =& g(\tilde{J}\nabla_{Y_1}\tilde{J}Y_1, \nabla_{Z_1}Z_1) + \tilde{g}(\tilde{J}\nabla_{Y_1}\tilde{J}Y_1, h^l(Z_1, Z_1))\\ =&\tilde{g}(\tilde{J}\tilde{\nabla}_{Y_1}\tilde{J}Y_1, \nabla_{Z_1}Z_1) - \tilde{g}(\tilde{J}h(Y_1, \tilde{J}Y_1), \nabla_{Z_1}Z_1)\\ &+ \tilde{g}(\tilde{J}\nabla_{Y_1}\tilde{J}Y_1, h^l(Z_1, Z_1))\\ =& - \tilde{g}(\tilde{\nabla}_{Y_1}Y_1, \nabla_{Z_1}Z_1) - \tilde{g}(\tilde{J}h(Y_1, \tilde{J}Y_1), \nabla_{Z_1}Z_1)\\ &+ \tilde{g}(\tilde{J}\nabla_{Y_1}\tilde{J}Y_1, h^l(Z_1, Z_1))\\ =& - g(\nabla_{Y_1}Y_1, \nabla_{Z_1}Z_1) - \tilde{g}(\tilde{J}h(Y_1, \tilde{J}Y_1), \nabla_{Z_1}Z_1)\\ &+ \tilde{g}(\tilde{J}\nabla_{Y_1}\tilde{J}Y_1, h^l(Z_1, Z_1))\\ =& - \tilde{g}(\nabla_{Y_1}Y_1, \tilde{\nabla}_{Z_1}Z_1) + \tilde{g}(\nabla_{Y_1}Y_1, h(Z_1, Z_1))\\ &+ \tilde{g}(\tilde{J}\nabla_{Y_1}\tilde{J}Y_1, h^l(Z_1, Z_1))\\ &- \tilde{g}(\tilde{J}\tilde{\nabla}_{\tilde{J}Y_1}Y_1 - \tilde{J}\nabla_{\tilde{J}Y_1}Y_1, \tilde{\nabla}_{Z_1}Z_1 - h(Z_1, Z_1)),\end{aligned}$$

which further gives

$$\begin{aligned}\tilde{g}(h^s(\nabla_{Y_1}\tilde{J}Y_1, Z_1), \tilde{J}Z_1) =&\tilde{g}(\tilde{\nabla}_{Z_1}\nabla_{Y_1}Y_1, Z_1) + \tilde{g}(\nabla_{Y_1}Y_1, h(Z_1, Z_1))\\ &- \tilde{g}(\tilde{J}\tilde{\nabla}_{\tilde{J}Y_1}Y_1, \tilde{\nabla}_{Z_1}Z_1) + \tilde{g}(\tilde{J}\nabla_{\tilde{J}Y_1}Y_1, \tilde{\nabla}_{Z_1}Z_1)\\ &+ \tilde{g}(\tilde{J}\tilde{\nabla}_{\tilde{J}Y_1}Y_1, h(Z_1, Z_1)) - \tilde{g}(\tilde{J}\nabla_{\tilde{J}Y_1}Y_1, h(Z_1, Z_1))\\ &+ \tilde{g}(\tilde{J}\nabla_{Y_1}\tilde{J}Y_1, h^l(Z_1, Z_1))\\ =&g(\nabla_{Z_1}\nabla_{Y_1}Y_1, Z_1) - \tilde{g}(\tilde{\nabla}_{\tilde{J}Y_1}\tilde{J}Y_1, \tilde{\nabla}_{Z_1}Z_1)\\ &- \tilde{g}(\tilde{\nabla}_{Z_1}\tilde{J}\nabla_{\tilde{J}Y_1}Y_1, Z_1) + \tilde{g}(\nabla_{Y_1}Y_1, h(Z_1, Z_1))\\ &+ \tilde{g}(\tilde{J}\tilde{\nabla}_{Y_1}\tilde{J}Y_1, h^l(Z_1, Z_1))\\ =&(\nabla_{Y_1}Y_1 ln\lambda)||Z_1||^2 - \tilde{g}(\nabla_{\tilde{J}Y_1}\tilde{J}Y_1, \tilde{\nabla}_{Z_1}Z_1)\\ &- \tilde{g}(\nabla_{Z_1}\tilde{J}\nabla_{\tilde{J}Y_1}Y_1, Z_1) + \tilde{g}(\nabla_{Y_1}Y_1, h(Z_1, Z_1))\\ &- \tilde{g}(\tilde{\nabla}_{Y_1}Y_1, h^l(Z_1, Z_1))\\ =&(\nabla_{Y_1}Y_1 ln\lambda)||Z_1||^2 + \tilde{g}(\tilde{\nabla}_{Z_1}\nabla_{\tilde{J}Y_1}\tilde{J}Y_1, Z_1)\\ &- (\tilde{J}\nabla_{\tilde{J}Y_1}Y_1 ln\lambda)||Z_1||^2\\ =&(\nabla_{Y_1}Y_1 ln\lambda)||Z_1||^2 + (\nabla_{\tilde{J}Y_1}\tilde{J}Y_1 ln\lambda)||Z_1||^2\\ &- (\tilde{J}\nabla_{\tilde{J}Y_1}Y_1 ln\lambda)||Z_1||^2.\end{aligned} \tag{54}$$

Next using Eqs. (2), (4), (6), (11) and (13), we obtain

$$\tilde{g}(D^s(Y_1, h^l(\tilde{J}Y_1, Z_1)), \tilde{J}Z_1) = 0 \tag{55}$$

and

$$\tilde{g}(D^s(\tilde{J}Y_1, h^l(Y_1, Z_1)), \tilde{J}Z_1) = 0. \tag{56}$$

Then employing Eqs. (51)-(56) with Corollary 3.3 in Eq. (47), we derive

$$\begin{aligned}\tilde{R}(Y_1, \tilde{J}Y_1, Z_1, \tilde{J}Z_1) = &\{Y_1(Y_1 ln\lambda) - (\nabla_{Y_1} Y_1 ln\lambda) + \tilde{J}Y_1(\tilde{J}Y_1 ln\lambda)\}||Z_1||^2\\ &- (\nabla_{\tilde{J}Y_1}\tilde{J}Y_1 ln\lambda)||Z_1||^2 - ||h^s(Y_1, Z_1)||^2 - ||h^s(\tilde{J}Y_1, Z_1)||^2\\ &+ \{(Y_1 ln\lambda)^2 + (\tilde{J}Y_1 ln\lambda)^2\}||Z_1||^2. \end{aligned} \tag{57}$$

From Eqs. (46) and (57), we get

$$\begin{aligned}-\frac{c}{2}||Y_1||^2||Z_1||^2 = &\{Y_1(Y_1 ln\lambda) - (\nabla_{Y_1} Y_1 ln\lambda) + \tilde{J}Y_1(\tilde{J}Y_1 ln\lambda)\}||Z_1||^2\\ &- (\nabla_{\tilde{J}Y_1}\tilde{J}Y_1 ln\lambda)||Z_1||^2 - ||h^s(Y_1, Z_1)||^2 - ||h^s(\tilde{J}Y_1, Z_1)||^2\\ &+ \{(Y_1 ln\lambda)^2 + (\tilde{J}Y_1 ln\lambda)^2\}||Z_1||^2,\end{aligned}$$

which further gives

$$\begin{aligned}||h^s(Y_1, Z_1)||^2 + ||h^s(\tilde{J}Y_1, Z_1)||^2 = &\{Y_1(Y_1 ln\lambda) - (\nabla_{Y_1} Y_1 ln\lambda)\}||Z_1||^2\\ &+ \{\tilde{J}Y_1(\tilde{J}Y_1 ln\lambda) - (\nabla_{\tilde{J}Y_1}\tilde{J}Y_1 ln\lambda)\}||Z_1||^2\\ &+ \{(Y_1 ln\lambda)^2 + (\tilde{J}Y_1 ln\lambda)^2 + \frac{c}{2}||Y_1||^2\}||Z_1||^2.\end{aligned} \tag{58}$$

Consider the local orthonormal frames of vector fields $\{Y_1, Y_2, Y_3, ..., Y_p, Y_{p+1} = \tilde{J}Y_1, Y_{p+2} = \tilde{J}Y_2, ..., Y_{2p} = \tilde{J}Y_p, Y_{2p+1} = \xi_1, Y_{2p+2} = \xi_2, ..., Y_{2p+r} = \xi_r, Y_{2p+r+1} = \tilde{J}\xi_1, Y_{2p+r+2} = \tilde{J}\xi_2, ..., Y_{2p+2r} = \tilde{J}\xi_r\}$ and $\{Z_1, Z_2, Z_3, ..., Z_q\}$ on N_T and $N_\perp$ respectively. Next, choosing Y and Z as basic vector fields in Eq. (58) and then summing both sides over $i = 1, 2, ..., 2p + 2r$ and $j = 1, 2, ..., q$ and using Eq. (19), we derive

$$\begin{aligned}\sum_{i=1}^{2p+2r}\sum_{j=1}^{q}\{||h^s(Y_i, Z_j)||^2 + ||h^s(\tilde{J}Y_i, Z_j)||^2\} = \sum_{i=1}^{2p+2r}\sum_{j=1}^{q}\{&\frac{c}{2}g(Y_i, Y_i)g(Z_j, Z_j)\\ &+ \{Y_i(Y_i ln\lambda) - \nabla_{Y_i} Y_i ln\lambda\}g(Z_j, Z_j)\\ &+ \tilde{J}Y_i(\tilde{J}Y_i ln\lambda)g(Z_j, Z_j)\\ &- (\nabla_{\tilde{J}Y_i}\tilde{J}Y_i ln\lambda)g(Z_j, Z_j)\\ &+ \{(Y_i ln\lambda)^2 + (\tilde{J}Y_i ln\lambda)^2\}g(Z_j, Z_j)\},\end{aligned}$$

which further yields

$$||h^s(D', D^\perp)||^2 = \frac{c}{2}pq + q\Delta(ln\lambda) + q||\nabla ln\lambda||^2. \tag{59}$$

On the other hand, the second fundamental form can be written as

$$||h||^2 = ||h(D', D')||^2 + ||h(D^\perp, D^\perp)||^2 + 2||h(D', D^\perp)||^2. \tag{60}$$

Then in view of degeneracy of $ltr(TN)$, Eq. (60) yields

$$||h||^2 = ||h^s(D', D')||^2 + ||h^s(D^\perp, D^\perp)||^2 + 2||h^s(D', D^\perp)||^2,$$

which further gives

$$||h||^2 \geq 2||h^s(D', D^\perp)||^2. \tag{61}$$

Now using Eq. (59) in Eq. (61), we obtain

$$||h||^2 \geq 2q\{\Delta(ln\lambda) + ||\nabla(ln\lambda)||^2 + \frac{c}{2}p\},$$

which completes the proof. □

6. Example

Finally, we present one non-trivial example of SCR-lightlike **w.p.** submanifolds in $\tilde{N}$.

Example 6.1. *Consider N be a 5-dimensional submanifold of $(R_1^8, \tilde{g})$ with*

$$x^1 = -u^1 - u^2, \quad x^2 = u^1 - u^2, \quad x^3 = u^3, \quad x^4 = u^4, \quad x^5 = u^3 cosu^5,$$

$$x^6 = u^4 cosu^5, \quad x^7 = u^3 sinu^5, \quad x^8 = u^4 sinu^5,$$

where $u^5 \in R - \{\frac{n\pi}{2},\ n \in Z\}$. Then TN is spanned by Z_1, Z_2, Z_3, Z_4, Z_5, where

$$Z_1 = -\partial x_1 + \partial x_2, \quad Z_2 = -\partial x_1 - \partial x_2,$$

$$Z_3 = \partial x_3 + cosu^5 \partial x_5 + sinu^5 \partial x_7, \quad Z_4 = \partial x_4 + cosu^5 \partial x_6 + sinu^5 \partial x_8,$$

$$Z_5 = -u^3 sinu^5 \partial x_5 - u^4 sinu^5 \partial x_6 + u^3 cosu^5 \partial x_7 + u^4 cosu^5 \partial x_8.$$

Thus, N is a 2-lightlike submanifold with $Rad(TN) = Span\{Z_1, Z_2\}$. As $\tilde{J}Z_3 = Z_4$ gives that $D = Span\{Z_3, Z_4\}$. Further by direct calculations, $S(TN^{\perp}) = Span\{W = u^4 sinu^5 \partial x_5 - u^3 sinu^5 \partial x_6 - u^4 cosu^5 \partial x_7 + u^3 cosu^5 \partial x_8\}$ and $\tilde{J}Z_5 = W$. On the other hand, $ltr(TN)$ is spanned by

$$N_1 = \frac{1}{2}(\partial x_1 + \partial x_2), \quad N_2 = \frac{1}{2}(\partial x_1 - \partial x_2).$$

Hence, $D' = Span\{Z_1, Z_2, Z_3, Z_4\}$. Thus, N is a proper SCR-lightlike submanifold of R_1^8. Clearly D' is integrable. We denote the leaves of D' and $D^{\perp}$ by N_T and $N_{\perp}$ respectively. Then, the induced metric of $N = N_T \times_{\lambda} N_{\perp}$ is given by

$$ds^2 = 2(du_3^2 + du_4^2) + ((u^3)^2 + (u^4)^2)du_5^2.$$

Thus, N is a proper SCR-lightlike **w.p.** *submanifold of the type $N_T \times_{\lambda} N_{\perp}$ in R_1^8, with $\lambda = \sqrt{(u^3)^2 + (u^4)^2}$.*

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