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SLANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE NEARLY KAEHLER MANIFOLDS

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Abstract. In the present paper, we introduce the study of slant lightlike submanifolds of indefinite nearly Kaehler manifolds. After proving some geometric results for the existence of slant lightlike submanifolds of indefinite nearly Kaehler manifolds, we give a non-trivial example of this class of lightlike submanifolds. Then, we derive some conditions for the integrability of the distributions associated with slant lightlike submanifolds of indefinite nearly Kaehler manifolds. Consequently, we study totally umbilical slant lightlike submanifolds of indefinite nearly Kaehler manifolds. Subsequently, we investigate minimal slant lightlike submanifolds of indefinite nearly Kaehler manifolds.

1. Introduction

The concept of slant immersions (or slant submanifolds) is among the most significant contributions in differential geometry. Firstly, the notion of slant immersions was brought up by Chen [9], as a generalization of invariant and anti-invariant immersions. Then, Chen [10] generalized the concept of slant immersions to define slant submanifolds in complex geometry. Moreover, the theory of slant submanifolds in contact geometry was investigated and developed by Lotta [3, 4]. Further, the concept of slant submanifolds in Sasakian manifolds was explored by Cabrerizo et al. [14]. Afterwards, several new generalizations of slant submanifolds namely, bi-slant submanifolds, hemi-slant submanifolds, semi-slant submanifolds came into the existence and the said classes were dealt in detail by Carriazo [1], Sahin [7] and Papaghiuc [17].

On the other hand, due to interesting geometric features and broad application area, the focus of geometers shifted towards the study of lightlike submanifolds. For instance, the theory of lightlike submanifolds has been successfully employed to study the concept of black holes, asymptotically flat spacetimes, Killing horizon, electromagnatic and radiation fields (see [15] and [19]). Therefore, Sahin [5, 8] initiated the study of slant lightlike submanifolds of indefinite

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almost Hermitian manifolds and indefinite Sasakian manifolds. In this continuation, Gupta et al. [18] studied the theory of slant lightlike submanifolds of indefinite cosymplectic manifolds. Later, different generalized classes of slant lightlike submanifolds viz. screen slant lightlike submanifolds, pointwise slant lightlike submanifolds, semi-slant lightlike submanifolds were considered in indefinite Kaehler manifolds (for detail, see [6], [13], [20]).

One may note that most of the available works on slant lightlike submanifolds and allied classes have been considered in indefinite Kaehler manifolds and this concept is yet to be explored in indefinite nearly Kaehler manifolds. Thus, in view of interesting geometric features of slant lightlike submanifolds and indefinite nearly Kaehler manifolds, it is worth studying slant lightlike submanifolds in indefinite nearly Kaehler manifolds.

To this end, we study slant lightlike submanifolds of indefinite nearly Kaehler manifolds. After proving some geometric results for the existence of slant lightlike submanifolds of indefinite nearly Kaehler manifolds, we present a non-trivial example of this class of lightlike submanifolds. Then, we give a characterization theorem for induced connection to be a metric connection on a slant lightlike submanifold of an indefinite nearly Kaehler manifold. Further, we derive some conditions for the integrability of the distributions associated with such submanifolds. Consequently, we study totally umbilical slant lightlike submanifolds of indefinite nearly Kaehler manifolds. Subsequently, we investigate minimal slant lightlike submanifolds of indefinite nearly Kaehler manifolds.

2. Preliminaries

2.1. Lightlike Submanifolds

In this section, we will review essential formulae and notation for lightlike submanifolds, following Duggal and Bejancu [15].

Assume an n-dimensional submanifold (K, g) of an $(m+n)$ real dimensional semi-Riemannian manifold $(\tilde{K}, \tilde{g})$ such that $\tilde{g}$ is a metric with constant index q satisfying $m, n \geq 1$, $1 \leq q \leq m+n-1$. If the metric $\tilde{g}$ is degenerate on TK, then T_pK and $T_pK^{\perp}$ both are degenerate and there exists a radical subspace $Rad(T_pK)$ such that $Rad(T_pK) = T_pK \cap T_pK^{\perp}$. If $Rad(TK) : p \in K \rightarrow Rad(T_pK)$ is a smooth distribution on K of rank $r(>0)$, then K is known as an r-lightlike submanifold of $\tilde{K}$. Moreover, let $S(TK)$ be the screen distribution in TK such that $TK = Rad(TK) \perp S(TK)$. Similarly, let $S(TK^{\perp})$ be the screen transversal vector bundle in $TK^{\perp}$ such that $TK^{\perp} = Rad(TK) \perp S(TK^{\perp})$. On the other hand, let $tr(TK)$ and $ltr(TK)$ be vector bundles in $T\tilde{K}|_K$ and $S(TK^{\perp})^{\perp}$, respectively with the property that $tr(TK) = ltr(TK) \perp S(TK^{\perp})$. It follows that

$$T\tilde{K}|_K = TK \oplus tr(TK) = S(TK) \perp (Rad(TK) \oplus ltr(TK)) \perp S(TK^{\perp}).$$

Consider $\tilde{\nabla}$ and ∇ respectively denote the Levi-Civita connection on $\tilde{K}$ and the torsion-free linear connection on K. Then, the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_Y Z = \nabla_Y Z + h^l(Y, Z) + h^s(Y, Z), \tag{1}$$

$$\tilde{\nabla}_Y N = -A_N Y + \nabla^l_Y N + D^s(Y, N), \tag{2}$$

$$\tilde{\nabla}_Y W = -A_W Y + D^l(Y, W) + \nabla^s_Y W, \tag{3}$$

where for any $Y, Z \in \Gamma(TK), N \in \Gamma(ltr(TK))$ and for $W \in \Gamma(S(TK^{\perp}))$. Further employing Eqs. (1) and (3), we derive

$$g(A_W Y, Z) = \tilde{g}(h^s(Y, Z), W) + \tilde{g}(Z, D^l(Y, W)). \tag{4}$$

Let us denote the projection morphism of TK on screen distribution $S(TK)$ by S, then we have

$$\nabla_Y SZ = \nabla^*_Y SZ + h^*(Y, SZ),\ \nabla_Y \xi = -A^*_\xi Y + \nabla^{*t}_Y \xi. \tag{5}$$

where $\{h^*(Y, SZ), \nabla^{*t}_Y \xi\} \in \Gamma(Rad(TK))$ and $\{\nabla^*_Y SZ, A^*_\xi Y\} \in \Gamma(S(TK))$. Further, employing Eqs. (2), (3) and (5), we attain

$$\tilde{g}(h^l(Y, SZ), \xi) = g(A^*_\xi Y, SZ). \tag{6}$$

As $\tilde{\nabla}$ is a metric connection on $\tilde{K}$, therefore for any $Y, Z, W \in \Gamma(TK)$, one has

$$(\nabla_Y g)(Z, W) = \tilde{g}(h^l(Y, Z), W) + \tilde{g}(h^l(Y, W), Z), \tag{7}$$

this gives that the induced connection ∇ on K is not necessarily a metric connection (in general).
Consider $\tilde{R}$ denotes the curvature tensor of $\tilde{\nabla}$. Then by direct calculations, the equation of Codazzi is given by

$$\begin{aligned}(\tilde{R}(Y, Z)W)^{\perp} =& (\nabla_Y h^l)(Z, W) - (\nabla_Z h^l)(Y, W) + D^l(Y, h^s(Z, W)) \\ & - D^l(Z, h^s(Y, W)) + (\nabla_Y h^s)(Z, W) - (\nabla_Z h^s)(Y, W) \\ & + D^s(Y, h^l(Z, W)) - D^s(Z, h^l(Y, W)),\end{aligned} \tag{8}$$

where

$$(\nabla_Y h^s)(Z, W) = \nabla^s_Y h^s(Z, W) - h^s(\nabla_Y Z, W) - h^s(Z, \nabla_Y W),$$

$$(\nabla_Y h^l)(Z, W) = \nabla^l_Y h^l(Z, W) - h^l(\nabla_Y Z, W) - h^l(Z, \nabla_Y W),$$

for $Y, Z, W \in \Gamma(TK)$.

2.2. Indefinite Nearly Kaehler Manifolds

An indefinite almost Hermitian manifold $(\tilde{K}, \tilde{J}, \tilde{g})$ is a $2n$-dimensional semi-Riemannian manifold $\tilde{K}$ with semi-Riemannian metric $\tilde{g}$ of constant index q, $0 < q < 2n$, and a $(1,1)$ tensor field $\tilde{J}$ on $\tilde{K}$ such that

$$\tilde{J}^2 Y = -Y, \ \tilde{g}(\tilde{J}Y, \tilde{J}Z) = \tilde{g}(Y, Z),$$

for any $Y, Z \in \Gamma(T\tilde{K})$.
In [2], Gray defined nearly Kaehler manifolds as follows:

Definition 2.1. *Let $(\tilde{K}, \tilde{J}, \tilde{g})$ be an indefinite almost Hermitian manifold and $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{K}$ with respect to $\tilde{g}$. Then $\tilde{K}$ is called an indefinite nearly Kaehler manifold if*

$$(\tilde{\nabla}_Y \tilde{J})Z + (\tilde{\nabla}_Z \tilde{J})Y = 0, \quad \forall \quad Y, Z \in \Gamma(T\tilde{K}). \tag{9}$$

On the other hand, the indefinite RK-manifolds are considered to be a more general class of indefinite almost Hermitian manifolds than indefinite nearly Kaehler manifolds. An indefinite almost Hermitian manifold $(\tilde{K}, \tilde{g}, \tilde{\nabla}, \tilde{J})$ is said to be an indefinite RK-manifold, if the curvature tensor $\tilde{R}$ is invariant under almost complex structure $\tilde{J}$, that is,

$$\tilde{R}(\tilde{J}Y_1, \tilde{J}Y_2, \tilde{J}Z_1, \tilde{J}Z_2) = \tilde{R}(Y_1, Y_2, Z_1, Z_2),$$

for $Y_1, Y_2, Z_1, Z_2 \in \Gamma(T\tilde{K})$. Further, an indefinite RK-manifold of constant holomorphic sectional curvature c and of constant type α is known as a generalized complex space form and is denoted by $\tilde{K}(c, \alpha)$. Its curvature tensor $\tilde{R}$ [12] is given as follows :

$$\begin{aligned} \tilde{R}(Y, Z)W = & \frac{c + 3\alpha}{4}\{\tilde{g}(Z, W)Y - \tilde{g}(Y, W)Z\} \\ & + \frac{c - \alpha}{4}\{\tilde{g}(Y, \tilde{J}W)\tilde{J}Z - \tilde{g}(Z, \tilde{J}W)\tilde{J}Y + 2\tilde{g}(Y, \tilde{J}Z)\tilde{J}W\}, \end{aligned} \tag{10}$$

where $Y, Z, W \in \Gamma(T\tilde{K})$.

3. Slant Lightlike Submanifolds

It is well known that radical distribution in $TK = Rad(TK) \perp S(TK)$ is degenerate. Therefore one cannot use vectors employed in $Rad(TK)$ to study the angle between vectors. In this context, Lemma 3.2 of [5] plays a crucial role and allows that the concept of angle between vectors can be established for the screen distribution $S(TK)$. In [5], Sahin defined the concept of slant lightlike submanifolds in indefinite almost Hermitian manifolds. On a similar note, we define a slant lightlike submanifold in indefinite nearly Kaehler manifolds as follows:

Definition 3.1. *A q-lightlike submanifold K of an indefinite nearly Kaehler manifold $\tilde{K}$ with index $2q$ is called a slant lightlike submanifold of $\tilde{K}$, if following conditions hold:*

(i) *$Rad(TK)$ is a distribution on K such that*

$$\tilde{J}Rad(TK) \cap Rad(TK) = \{0\}.$$

(ii) *For each non-zero vector field tangent to D at $z \in U \subset K$, the angle $\theta(Z)$ between $\tilde{J}Z$ and the vector space D_z is constant, that is, it is independent of the choice of $z \in U \subset K$ and $z \in D_z$, where D is a complementary distribution to $\tilde{J}Rad(TK) \oplus \tilde{J}ltr(TK)$ in the screen distribution $S(TK)$.*

Here $\theta(Z)$ is called slant angle of distribution D. If the slant distribution $D \neq \{0\}$ and $\theta \neq 0, \pi/2$, then the slant lightlike submanifold is said to be proper.

Remark 3.2. *In view of Definition 3.1, the tangent bundle TK of K has the following decomposition*

$$TK = Rad(TK) \perp (\tilde{J}Rad(TK) \oplus \tilde{J}ltr(TK)) \perp D.$$

Note: In the forthcoming part, an indefinite nearly Kaehler manifold and a slant lightlike submanifold shall be denoted by $\tilde{K}$ and **s.l.s.**, respectively, unless otherwise stated.

For any $Y \in \Gamma(TK)$, we write

$$\tilde{J}Y = fY + \omega Y, \tag{11}$$

where fY and ωY respectively denote the tangential component and the transversal component of $\tilde{J}Y$.

Similarly, for any $W \in \Gamma(tr(TK))$,

$$\tilde{J}W = tW + nW, \tag{12}$$

where tW and nW respectively denote the tangential and transversal components of $\tilde{J}W$.

Consider the projection morphisms ϕ_1, ϕ_2, η_1 and η_2 of TK on distributions $Rad(TK)$, $\tilde{J}(Rad(TK))$, $\tilde{J}(ltr(TK))$ and D respectively. Then for any $Y \in \Gamma(TK)$, we have

$$Y = \phi_1 Y + \phi_2 Y + \eta_1 Y + \eta_2 Y, \tag{13}$$

then applying $\tilde{J}$ to Eq. (13), we obtain

$$\tilde{J}Y = \tilde{J}\phi_1 Y + \tilde{J}\phi_2 Y + \tilde{J}\eta_1 Y + \tilde{J}\eta_2 Y,$$

which further yields

$$\tilde{J}Y = \tilde{J}\phi_1 Y + \tilde{J}\phi_2 Y + f\eta_2 Y + \omega\eta_1 Y + \omega\eta_2 Y. \tag{14}$$

Moreover, Eq. (14) can be written as

$$\tilde{J}Y = fY + \omega\eta_1 Y + \omega\eta_2 Y, \tag{15}$$

where $fY = \tilde{J}\phi_1 Y + \tilde{J}\phi_2 Y + f\eta_2 Y$.
Differentiating Eq. (14) and using Eqs. (1)-(3) with Eq. (12) and then equating the components of $Rad(TK), \tilde{J}Rad(TK), \tilde{J}ltr(TK), D, ltr(TK)$ and $S(TK^{\perp})$ on both sides, we obtain

$$\begin{aligned}\phi_1(\nabla_Y \tilde{J}\phi_1 Z) + \phi_1(\nabla_Y \tilde{J}\phi_2 Z) + \phi_1(\nabla_Y f\eta_2 Z) + \phi_1(\nabla_Z \tilde{J}\phi_1 Y) + \phi_1(\nabla_Z \tilde{J}\phi_2 Y)\\ +\phi_1(\nabla_Z f\eta_2 Y) = \phi_1(A_{\omega\eta_1 Z} Y) + \phi_1(A_{\omega\eta_2 Z} Y) + \phi_1(A_{\omega\eta_1 Y} Z) + \phi_1(A_{\omega\eta_2 Y} Z)\\ +\tilde{J}\phi_2\nabla_Y Z + \tilde{J}\phi_2\nabla_Z Y. \qquad (16)\end{aligned}$$

$$\begin{aligned}\phi_2(\nabla_Y \tilde{J}\phi_1 Z) + \phi_2(\nabla_Y \tilde{J}\phi_2 Z) + \phi_2(\nabla_Y f\eta_2 Z) + \phi_2(\nabla_Z \tilde{J}\phi_1 Y) + \phi_2(\nabla_Z \tilde{J}\phi_2 Y)\\ +\phi_2(\nabla_Z f\eta_2 Y) = \phi_2(A_{\omega\eta_1 Z} Y) + \phi_2(A_{\omega\eta_2 Z} Y) + \phi_2(A_{\omega\eta_1 Y} Z) + \phi_2(A_{\omega\eta_2 Y} Z)\\ +\tilde{J}\phi_1\nabla_Y Z + \tilde{J}\phi_1\nabla_Z Y. \qquad (17)\end{aligned}$$

$$\begin{aligned}\eta_1(\nabla_Y \tilde{J}\phi_1 Z) + \eta_1(\nabla_Y \tilde{J}\phi_2 Z) + \eta_1(\nabla_Y f\eta_2 Z) + \eta_1(\nabla_Z \tilde{J}\phi_1 Y) + \eta_1(\nabla_Z \tilde{J}\phi_2 Y)\\ +\eta_1(\nabla_Z f\eta_2 Y) = \eta_1(A_{\omega\eta_1 Z} Y) + \eta_1(A_{\omega\eta_2 Z} Y) + \eta_1(A_{\omega\eta_1 Y} Z)\\ +\eta_1(A_{\omega\eta_2 Y} Z) + 2th^l(Y, Z).\end{aligned}$$

$$\begin{aligned}\eta_2(\nabla_Y \tilde{J}\phi_1 Z) + \eta_2(\nabla_Y \tilde{J}\phi_2 Z) + \eta_2(\nabla_Y f\eta_2 Z) + \eta_2(\nabla_Z \tilde{J}\phi_1 Y) + \eta_2(\nabla_Z \tilde{J}\phi_2 Y)\\ +\eta_2(\nabla_Z f\eta_2 Y) = \eta_2(A_{\omega\eta_1 Z} Y) + \eta_2(A_{\omega\eta_2 Z} Y) + \eta_2(A_{\omega\eta_1 Y} Z) + \eta_2(A_{\omega\eta_2 Y} Z)\\ +f\eta_2\nabla_Y Z + f\eta_2\nabla_Z Y + 2th^s(Y, Z). \qquad (18)\end{aligned}$$

$$\begin{aligned}h^l(Y, \tilde{J}\phi_1 Z) + h^l(Y, \tilde{J}\phi_2 Z) + h^l(Y, f\eta_2 Z) + h^l(Z, \tilde{J}\phi_1 Y) + h^l(Z, \tilde{J}\phi_2 Y)\\ +h^l(Z, f\eta_2 Y) = \omega\eta_1\nabla_Y Z + \omega\eta_1\nabla_Z Y - \nabla^l_Y\omega\eta_1 Z - \nabla^l_Z\omega\eta_1 Y\\ -D^l(Y, \omega\eta_2 Z) - D^l(Z, \omega\eta_2 Y). \qquad (19)\end{aligned}$$

$$\begin{aligned}h^s(Y, \tilde{J}\phi_1 Z) + h^s(Y, \tilde{J}\phi_2 Z) + h^s(Y, f\eta_2 Z) + h^s(Z, \tilde{J}\phi_1 Y) + h^s(Z, \tilde{J}\phi_2 Y)\\ +h^s(Z, f\eta_2 Y) = \omega\eta_2\nabla_Y Z + \omega\eta_2\nabla_Z Y - \nabla^s_Y\omega\eta_2 Z - \nabla^s_Z\omega\eta_2 Y\\ -D^s(Y, \omega\eta_1 Z) - D^s(Z, \omega\eta_1 Y) + 2nh^s(Y, Z). \qquad (20)\end{aligned}$$

Lemma 3.3. *For a* **s.l.s.** *K of $\tilde{K}$, one has $\omega\eta_2 Y \in \Gamma(S(TK^{\perp}))$, for any $Y \in \Gamma(TK)$.*

Proof. We have $\omega\eta_2 Y \in \Gamma(S(TK^{\perp}))$, if and only if, $\tilde{g}(\omega\eta_2 Y, \xi) = 0$ for any $Y \in \Gamma(TK)$ and $\xi \in \Gamma(Rad(TK))$. Now consider $\tilde{g}(\omega\eta_2 Y, \xi) = \tilde{g}(\tilde{J}\eta_2 Y - f\eta_2 Y, \xi) = \tilde{g}(\tilde{J}\eta_2 Y, \xi) = -g(\eta_2 Y, \tilde{J}\xi) = 0$, which implies that $\omega\eta_2 Y$ has no component in $ltr(TK)$. Thus the proof is completed. □

Note: From Lemma 3.3, we have $\omega D \subset S(TK^{\perp})$, which implies that there exist $\mu \subset S(TK^{\perp})$ such that

$$S(TK^{\perp}) = \tilde{J}D \perp \mu$$

and

$$T\tilde{K} = S(T\tilde{K}) \perp \{Rad(TK) \oplus ltr(TK)\} \perp \{\omega(D) \perp \mu\}.$$

Next, we prove the following results.

Theorem 3.4. *(Existence Theorem) A q – lightlike submanifold K of $\tilde{K}$ is a **s.l.s.**, if and only if,*

(i) $\tilde{J}ltr(TK)$ *is a distribution on* K.

(ii) $f^2\eta_2 Y = -\cos^2\theta(\eta_2 Y)$, *for* $Y \in \Gamma(TK)$.

Proof. Firstly, we consider K be a **s.l.s.** of $\tilde{K}$, then $\tilde{J}Rad(TK)$ is a distribution on $S(TK)$ and using Lemma 3.1 of [5], it follows that $\tilde{J}ltr(TK)$ is also a distribution on K such that $\tilde{J}ltr(TK)$ is contained in $S(TK)$, which proves condition (i). In addition, the angle between $\tilde{J}\eta_2 Y$ and D_y is constant. Thus we have

$$\cos\theta(\eta_2 Y) = \frac{\tilde{g}(\tilde{J}\eta_2 Y, f\eta_2 Y)}{|\tilde{J}\eta_2 Y||f\eta_2 Y|} = \frac{-\tilde{g}(\eta_2 Y, \tilde{J}f\eta_2 Y)}{|\eta_2 Y||f\eta_2 Y|} = \frac{-g(\eta_2 Y, f^2\eta_2 Y)}{|\eta_2 Y||f\eta_2 Y|}. \tag{21}$$

On the other hand, we also have

$$\cos\theta(\eta_2 Y) = \frac{|f\eta_2 Y|}{|\tilde{J}\eta_2 Y|}. \tag{22}$$

Thus from Eqs. (21) and (22), we get

$$\cos^2\theta(\eta_2 Y) = \frac{-g(\eta_2 Y, f^2\eta_2 Y)}{|\eta_2 Y|^2}.$$

As we know that $\theta(\eta_2 Y)$ is constant on slant distribution D, thus we conclude that

$$f^2\eta_2 Y = -\cos^2\theta(\eta_2 Y),$$

which proves (ii).

Conversely, let K be a q – lightlike submanifold of $\tilde{K}$ such that conditions (i) and (ii) are satisfied. Then condition (i) implies that $\tilde{J}Rad(TK)$ is a distribution on K. Further, Lemma 3.2 of [5] implies any distribution which is complementary to $\tilde{J}Rad(TK) \oplus \tilde{J}ltr(TK)$ within $S(TK)$, is a Riemannian distribution . Thus one has

$$g(f\eta_2 Y, f\eta_2 Y) = -g(f^2\eta_2 Y, \eta_2 Y) = \cos^2\theta(\eta_2 Y)g(\eta_2 Y, \eta_2 Y), \tag{23}$$

for any $\eta_2 Y \in D_Y$, which further gives

$$\cos^2\theta(\eta_2 Y) = \frac{g(f\eta_2 Y, f\eta_2 Y)}{g(\eta_2 Y, \eta_2 Y)}. \tag{24}$$

Hence, the proof is completed. □

Theorem 3.5. *(Existence Theorem) A q – lightlike submanifold K of $\tilde{K}$ is a **s.l.s**, if and only if,*

(i) $\tilde{J}ltr(TK)$ *is a distribution on* K.
(ii) $t\omega\eta_2 Y = -\sin^2\theta(\eta_2 Y)$, *for* $Y \in \Gamma(TK)$.

Proof. Firstly, consider K be a **s.l.s.** of $\tilde{K}$, then $\tilde{J}Rad(TK)$ is a distribution on $S(TK)$ and from Lemma 3.1 of [5], it follows that $\tilde{J}ltr(TK)$ is also a distribution on K such that $\tilde{J}ltr(TK)$ is contained in $S(TK)$, this proves (i). Next, applying $\tilde{J}$ to Eq. (14) and using Eqs. (11) and (12), we get

$$-Y = -\phi_1 Y - \phi_2 Y + f^2\eta_2 Y + \omega f\eta_2 Y + \tilde{J}\omega\eta_1 Y + t\omega\eta_2 Y + n\omega\eta_2 Y.$$

As $\tilde{J}\omega\eta_1 Y = -\eta_1 Y \in \Gamma(S(TK))$, then equating the tangential components on both sides, we derive

$$-Y = -\phi_1 Y - \phi_2 Y + f^2\eta_2 Y - \eta_1 Y + t\omega\eta_2 Y, \tag{25}$$

which further gives

$$-\eta_2 Y = f^2\eta_2 Y + t\omega\eta_2 Y. \tag{26}$$

As K is a **s.l.s.**, thus using Theorem 3.4, we have $f^2\eta_2 Y = -\cos^2\theta(\eta_2 Y)$. Further from Eq. (26), we acquire

$$t\omega\eta_2 Y = -\sin^2\theta(\eta_2 Y),$$

which proves (ii).
Conversely, consider a $q-$ lightlike submanifold K of $\tilde{K}$ such that conditions (i) and (ii) are satisfied. From condition (ii), one has $t\omega\eta_2 Y = -\sin^2\theta(\eta_2 Y)$. Further employing Eq. (26), we obtain

$$f^2\eta_2 Y = -(1 - \sin^2\theta(\eta_2 Y)) = -\cos^2\theta(\eta_2 Y).$$

Then following similar steps as in proof of Theorem 3.4, the result follows. □

Corollary 3.6. *For a* **s.l.s.** K *of* $\tilde{K}$, *one has*

$$g(f\eta_2 Y_1, f\eta_2 Y_2) = \cos^2\theta g(\eta_2 Y_1, \eta_2 Y_2) \tag{27}$$

and

$$\tilde{g}(\omega\eta_2 Y_1, \omega\eta_2 Y_2) = \sin^2\theta g(\eta_2 Y_1, \eta_2 Y_2), \tag{28}$$

for $Y_1, Y_2 \in \Gamma(TK)$.

Next, we present a non-trivial example of a **s.l.s.** of $\tilde{K}$.

Example 3.7. *Let* $(R^{12}_2, \tilde{g})$ *be a semi-Euclidean space of signature* $(-, -, +, +, +, +, +, +, +, +)$ *w. r. t. the canonical basis* $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial x_9, \partial x_{10}\}$ *with an almost complex structure* $\tilde{J}$ *defined by*

$$\begin{aligned}\tilde{J}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) = (&-x_2, x_1, -x_4, x_3, \\ &- x_6\sin\alpha - x_9\cos\alpha, x_5\sin\alpha - x_{10}\cos\alpha, -x_8\sin\alpha - x_{11}\cos\alpha, \\ &x_7\sin\alpha - x_{12}\cos\alpha, x_5\cos\alpha + x_{10}\sin\alpha, x_6\cos\alpha - x_9\sin\alpha, \\ &x_7\cos\alpha + x_{12}\sin\alpha, x_8\cos\alpha - x_{11}\sin\alpha),\end{aligned}$$

where $\alpha \in \left(0, \frac{\pi}{2}\right)$.

Let K *be a 5-dimensional submanifold of* $(R_2^{12}, \tilde{g})$ *given by*

$$x^1 = u^1, \quad x^2 = u^2, \quad x^3 = u^1, \quad x^4 = u^5 \quad x^5 = u^4 \sin\theta,$$

$$x^6 = \frac{k}{\sqrt{2}} u^3 \sin\theta, \quad x^7 = \frac{k}{\sqrt{2}} u^3 \sin\theta, \quad x^8 = \frac{k}{\sqrt{2}} u^3 \cos\theta, \quad x^9 = u^4 \cos\theta,$$

$$x^{10} = \frac{k}{\sqrt{2}} u^3 \cos\theta, \quad x^{11} = ku^4, \quad x^{12} = u^3, \ \textit{where } \theta \in R - \left\{ \frac{n\pi}{2}, \ n \in Z \right\}.$$

Then TK *is spanned by* Z_1, Z_2, Z_3, Z_4, Z_5, *where*

$$Z_1 = \partial x_1 + \partial x_3, \quad Z_2 = \partial x_2,$$

$$Z_3 = \frac{k}{\sqrt{2}} \sin\theta \partial x_6 + \frac{k}{\sqrt{2}} \sin\theta \partial x_7 + \frac{k}{\sqrt{2}} \cos\theta \partial x_8 + \frac{k}{\sqrt{2}} \cos\theta \partial x_{10} + \partial x_{12},$$

$$Z_4 = \sin\theta \partial x_5 + \cos\theta \partial x_9 + k \partial x_{11}, \quad Z_5 = \partial x_4.$$

It is clear that K *is a 1-lightlike submanifold with* $Rad(TK) = Span\{Z_1\}$ *and the lightlike transversal bundle* $ltr(TK)$ *is spanned by*

$$N_1 = \frac{1}{2}\{-\partial x_1 + \partial x_3\}.$$

It follows that $\tilde{J} Z_1 = Z_2 + Z_5$ *and* $\tilde{J} N_1 = \frac{1}{2}\{-Z_2 + Z_5\}$, *which implies that* $\tilde{J} Rad(TK)$ *and* $\tilde{J} ltr(TK)$ *are distributions on* K. *Hence,* $D = Span\{Z_3, Z_4\}$ *is a slant distribution w. r. t.* $\tilde{J}$ *with slant angle* α. *Further, by direct calculations,* $S(TK^{\perp})$ *is spanned by*

$$\begin{aligned} W = & - k \sin\theta \partial x_5 + \frac{1}{\sqrt{2}} \sin\theta \partial x_6 + \frac{1}{\sqrt{2}} \sin\theta \partial x_7 + \frac{1}{\sqrt{2}} \cos\theta \partial x_8 \\ & - k \cos\theta \partial x_9 + \frac{1}{\sqrt{2}} \cos\theta \partial x_{10} + \partial x_{11} - \frac{k}{2} \partial x_{12}. \end{aligned}$$

Therefore, K *is a proper* ***s.l.s.*** *of* R_2^{12}.

In next theorem, we give a characterization result enabling ∇ to be a metric connection on **s.l.s.** of $\tilde{K}$. Thus we have

Theorem 3.8. *A necessary and sufficient condition for the induced connection* ∇ *on a* ***s.l.s.*** K *of* $\tilde{K}$ *to be a metric connection is that*

$$\nabla_Y \tilde{J}\xi + \nabla_{\tilde{J}\xi} Y \in \Gamma(\tilde{J} Rad(TK)), \ \nabla_{\tilde{J}\xi} fY - A_{\omega Y} \tilde{J}\xi \in \Gamma(Rad(TK))$$

and

$$th(Y, \tilde{J}\xi) = 0,$$

for $Y \in \Gamma(TK)$ *and* $\xi \in \Gamma(Rad(TK))$.

Proof. For any $Y \in \Gamma(TK)$ and $\xi \in \Gamma(Rad(TK))$, we have $\tilde{\nabla}_Y \xi = -\tilde{\nabla}_Y \tilde{J}^2 \xi$. Further considering Eq. (9), we acquire $\tilde{\nabla}_Y \xi = -\tilde{J}\tilde{\nabla}_Y \tilde{J}\xi + \tilde{\nabla}_{\tilde{J}\xi}\tilde{J}Y - \tilde{J}\tilde{\nabla}_{\tilde{J}\xi}Y$. Then employing Eqs. (1), (11) and (12), we derive

$$\begin{aligned}\nabla_Y \xi + h(Y,\xi) =& -\tilde{J}(\nabla_Y \tilde{J}\xi + \nabla_{\tilde{J}\xi}Y) + \nabla_{\tilde{J}\xi} fY + h(\tilde{J}\xi, fY)\\ &- A_{\omega Y}\tilde{J}\xi + \nabla^t_{\tilde{J}\xi}\omega Y - 2th(Y,\tilde{J}\xi) - 2nh(Y,\tilde{J}\xi).\end{aligned} \tag{29}$$

Further equating the tangential components on both sides of Eq. (29), we get

$$\nabla_Y \xi = -\tilde{J}(\nabla_Y \tilde{J}\xi + \nabla_{\tilde{J}\xi}Y) + (\nabla_{\tilde{J}\xi} fY - A_{\omega Y}\tilde{J}\xi) - 2th(Y,\tilde{J}\xi). \tag{30}$$

Hence, from Eq. (30), $\nabla_Y \xi \in \Gamma(Rad(TK))$, if and only if, $\nabla_Y \tilde{J}\xi + \nabla_{\tilde{J}\xi}Y \in \Gamma(\tilde{J}Rad(TK))$, $\nabla_{\tilde{J}\xi} fY - A_{\omega Y}\tilde{J}\xi \in \Gamma(Rad(TK))$ and $th(Y,\tilde{J}\xi) = 0$, which completes the proof. □

4. Integrability of the Distributions

In this section, we will investigate several conditions for the integrability of the distributions associated with a **s.l.s.** K of $\tilde{K}$. Firstly, we present a basic lemma.

Lemma 4.1. *For a* **s.l.s.** K *of* $\tilde{K}$*, we have*

$$\begin{aligned}(\nabla_Y f)Z + (\nabla_Z f)Y =& A_{\omega\eta_1 Z}Y + A_{\omega\eta_1 Y}Z + A_{\omega\eta_2 Z}Y\\ &+ A_{\omega\eta_2 Y}Z + 2th(Y,Z)\end{aligned} \tag{31}$$

and

$$\begin{aligned}\nabla^l_Y\omega\eta_1 Z + \nabla^l_Z\omega\eta_1 Y + \nabla^s_Y\omega\eta_2 Z + \nabla^s_Z\omega\eta_2 Y =& 2nh^s(Y,Z) + \omega\nabla_Y Z + \omega\nabla_Z Y\\ &- D^l(Y,\omega\eta_2 Z) - D^l(Z,\omega\eta_2 Y)\\ &- D^s(Y,\omega\eta_1 Z) - D^s(Z,\omega\eta_1 Y)\\ &- h(Y,fZ) - h(fY,Z),\end{aligned}$$

where

$$(\nabla_Y f)Z = \nabla_Y fZ - f\nabla_Y Z, \tag{32}$$

for any $Y, Z \in \Gamma(TK)$.

Proof. Employing Eqs. (9), (12) and (15), then equating the tangential and transversal components, the assertion follows. □

Theorem 4.2. *Assume a* **s.l.s.** K *of* $\tilde{K}$*. Then the slant distribution* D *is integrable, if and only if,*

$$\nabla_{Z_1} fZ_2 + \nabla_{Z_2} fZ_1 - A_{\omega\eta_2 Z_2}Z_1 - A_{\omega\eta_2 Z_1}Z_2 - 2th(Z_1,Z_2) - 2f\nabla_{Z_2}Z_1 \in \Gamma(D),$$

for each $Z_1, Z_2 \in \Gamma(D)$.

Proof. For $Z_1, Z_2 \in \Gamma(D)$, using Eqs. (31) and (32), we derive

$$f[Z_1, Z_2] = \nabla_{Z_1} f Z_2 + \nabla_{Z_2} f Z_1 - A_{\omega\eta_2 Z_2} Z_1 - A_{\omega\eta_2 Z_1} Z_2 \\ - 2th(Z_1, Z_2) - 2f\nabla_{Z_2} Z_1,$$

Hence, the result follows. □

Corollary 4.3. *Let K be a* **s.l.s.** *of $\tilde{K}$. Then the slant distribution D is integrable, if the following conditions hold*

$$(i)\ \phi_1(\nabla_{Z_1} f\eta_2 Z_2) + \phi_1(\nabla_{Z_2} f\eta_2 Z_1) = \phi_1(A_{\omega\eta_2 Z_2} Z_1) + \phi_1(A_{\omega\eta_2 Z_1} Z_2) \\ + 2\tilde{J}\phi_2 \nabla_{Z_2} Z_1,$$

$$(ii)\ \phi_2(\nabla_{Z_1} f\eta_2 Z_2) + \phi_2(\nabla_{Z_2} f\eta_2 Z_1) = \phi_2(A_{\omega\eta_2 Z_2} Z_1) + \phi_2(A_{\omega\eta_2 Z_1} Z_2) \\ + 2\tilde{J}\phi_1 \nabla_{Z_2} Z_1,$$

$$(iii)\ h^l(Z_1, f\eta_2 Z_2) + h^l(Z_2, f\eta_2 Z_1) = 2\omega\eta_1 \nabla_{Z_2} Z_1 - D^l(Z_1, \omega\eta_2 Z_2) \\ - D^l(Z_2, \omega\eta_2 Z_1),$$

where $Z_1, Z_2 \in \Gamma(D)$.

Proof. The proof follows directly from Eqs. (16), (17) and (19). □

Theorem 4.4. *The anti-invariant distribution $\tilde{J}ltr(TK)$ of a* **s.l.s.** *K of $\tilde{K}$ is integrable, if and only if,*

$$A_{\omega\eta_1 Y_1} Y_2 + A_{\omega\eta_1 Y_2} Y_1 + 2th(Y_1, Y_2) + 2f\nabla_{Y_2} Y_1 = 0,$$

for any $Y_1, Y_2 \in \Gamma(\tilde{J}ltr(TK))$.

Proof. Employing Eqs. (31) and (32), for any $Y_1, Y_2 \in \Gamma(\tilde{J}ltr(TK))$, we have

$$f[Y_1, Y_2] = -A_{\omega\eta_1 Y_1} Y_2 - A_{\omega\eta_1 Y_2} Y_1 - 2th(Y_1, Y_2) - 2f\nabla_{Y_2} Y_1.$$

Hence, the result follows from above equation. □

Theorem 4.5. *The distribution $Rad(TK)$ of a* **s.l.s.** *K of $\tilde{K}$ is integrable, if and only if,*

(i) $\phi_1(\nabla_{\xi_1} \tilde{J}\phi_1 \xi_2) + \phi_1(\nabla_{\xi_2} \tilde{J}\phi_1 \xi_1) = 2\tilde{J}\phi_2 \nabla_{\xi_2} \xi_1,$
(ii) $\eta_2(\nabla_{\xi_1} f\eta_2 \xi_2) + \eta_2(\nabla_{\xi_2} f\eta_2 \xi_1) = 2f\eta_2 \nabla_{\xi_2} \xi_1 + 2th^s(\xi_1, \xi_2),$
(iii) $h^l(\xi_1, \tilde{J}\phi_1 \xi_2) + h^l(\xi_2, \tilde{J}\phi_1 \xi_1) = 2\omega\eta_1 \nabla_{\xi_2} \xi_1,$
(iv) $h^s(\xi_1, \tilde{J}\phi_1 \xi_2) + h^s(\xi_2, \tilde{J}\phi_1 \xi_1) = 2\omega\eta_2 \nabla_{\xi_2} \xi_1 + 2nh^s(\xi_1, \xi_2),$

for $\xi_1, \xi_2 \in \Gamma(Rad(TK))$.

Proof. Using Eqs. (16), (18), (19) and (20), for $\xi_1, \xi_2 \in \Gamma(Rad(TK))$, we have

$$\phi_1(\nabla_{\xi_1}\tilde{J}\phi_1\xi_2) + \phi_1(\nabla_{\xi_2}\tilde{J}\phi_1\xi_1) = \tilde{J}\phi_2[\xi_1, \xi_2] + 2\tilde{J}\phi_2\nabla_{\xi_2}\xi_1, \tag{33}$$

$$\begin{aligned} \eta_2(\nabla_{\xi_1}f\eta_2\xi_2) + \eta_2(\nabla_{\xi_2}f\eta_2\xi_1) =& f\eta_2[\xi_1, \xi_2] + 2f\eta_2\nabla_{\xi_2}\xi_1 \\ &+ 2th^s(\xi_1, \xi_2), \end{aligned} \tag{34}$$

$$h^l(\xi_1, \tilde{J}\phi_1\xi_2) + h^l(\xi_2, \tilde{J}\phi_1\xi_1) = \omega\eta_1[\xi_1, \xi_2] + 2\omega\eta_1\nabla_{\xi_2}\xi_1, \tag{35}$$

and

$$\begin{aligned} h^s(\xi_1, \tilde{J}\phi_1\xi_2) + h^s(\xi_2, \tilde{J}\phi_1\xi_1) =& \omega\eta_2[\xi_1, \xi_2] + 2\omega\eta_2\nabla_{\xi_2}\xi_1 \\ &+ 2nh^s(\xi_1, \xi_2) \end{aligned} \tag{36}$$

respectively. Thus proof follows from Eqs. (33)–(36). □

Theorem 4.6. *For a* **s.l.s.** *K of $\tilde{K}$, the distribution $\tilde{J}Rad(TK)$ is integrable, if and only if,*

(i) $\phi_2(\nabla_{Y_1}\tilde{J}\phi_2Y_2) + \phi_2(\nabla_{Y_2}\tilde{J}\phi_2Y_1) = 2\tilde{J}\phi_1\nabla_{Y_2}Y_1$,
(ii) $\eta_2(\nabla_{Y_1}f\eta_2Y_2) + \eta_2(\nabla_{Y_2}f\eta_2Y_1) = 2f\eta_2\nabla_{Y_2}Y_1 + 2th^s(Y_1, Y_2)$,
(iii) $h^l(Y_1, \tilde{J}\phi_2Y_2) + h^l(Y_2, \tilde{J}\phi_2Y_1) = 2\omega\eta_1\nabla_{Y_2}Y_1$,
(iv) $h^s(Y_1, \tilde{J}\phi_2Y_2) + h^s(Y_2, \tilde{J}\phi_2Y_1) = 2\omega\eta_2\nabla_{Y_2}Y_1 + 2nh^s(Y_1, Y_2)$,

for any $Y_1, Y_2 \in \Gamma(\tilde{J}(Rad(TK)))$.

Proof. Employing Eqs. (17), (18), (19) and (20), for any $Y_1, Y_2 \in \Gamma(\tilde{J}Rad(TK))$, we acquire

$$\phi_2(\nabla_{Y_1}\tilde{J}\phi_2Y_2) + \phi_2(\nabla_{Y_2}\tilde{J}\phi_2Y_1) = \tilde{J}\phi_1[Y_1, Y_2] + 2\tilde{J}\phi_1\nabla_{Y_2}Y_1, \tag{37}$$

$$\begin{aligned} \eta_2(\nabla_{Y_1}f\eta_2Y_2) + \eta_2(\nabla_{Y_2}f\eta_2Y_1) =& f\eta_2[Y_1, Y_2] + 2f\eta_2\nabla_{Y_2}Y_1 \\ &+ 2th^s(Y_1, Y_2), \end{aligned} \tag{38}$$

$$h^l(Y_1, \tilde{J}\phi_2Y_2) + h^l(Y_2, \tilde{J}\phi_2Y_1) = \omega\eta_1[Y_1, Y_2] + 2\omega\eta_1\nabla_{Y_2}Y_1, \tag{39}$$

and

$$\begin{aligned} h^s(Y_1, \tilde{J}\phi_2Y_2) + h^s(Y_2, \tilde{J}\phi_2Y_1) =& \omega\eta_2[Y_1, Y_2] + 2\omega\eta_2\nabla_{Y_2}Y_1 \\ &+ 2nh^s(Y_1, Y_2) \end{aligned} \tag{40}$$

respectively. Then proof follows from Eqs. (37)–(40). □

5. Totally Umbilical Slant Lightlike Submanifolds

In this section, we investigate totally umbilical **s.l.s.** of $\tilde{K}$.

Definition 5.1. [16] *A lightlike submanifold (K, g) of a semi-Riemann ian manifold $(\tilde{K}, \tilde{g})$ is called totally umbilical if there exist transversal curvature vector field $H \in \Gamma(tr(TK))$ on K such that for $Y_1, Y_2 \in \Gamma(TK)$,*

$$h(Y_1, Y_2) = H\tilde{g}(Y_1, Y_2). \tag{41}$$

Using the Eqs. (1)–(3), K is a totally umbilical, if and only if, there exist smooth vector fields $H^l \in \Gamma(ltr(TK))$ and $H^s \in \Gamma(S(TK^{\perp}))$ such that

$$h^l(Y_1, Y_2) = H^l g(Y_1, Y_2); \; h^s(Y_1, Y_2) = H^s g(Y_1, Y_2); \; D^l(Y_1, W) = 0, \tag{42}$$

for $Y_1, Y_2 \in \Gamma(TK)$ and $W \in \Gamma(S(TK^{\perp}))$.
On the other hand, a lightlike submanifold is totally geodesic if $h(Y_1, Y_2) = 0$, for $Y_1, Y_2 \in \Gamma(TK)$. Thus, we say that a lightlike submanifold is totally geodesic, if $H^l = 0$ and $H^s = 0$.

Theorem 5.2. *Consider a totally umbilical **s.l.s.** K of $\tilde{K}$. Then at least one of the following statements hold:*

(a) *K is an anti-invariant submanifold.*
(b) *$D = \{0\}$.*
(c) *If K is a proper **s.l.s.**, then $H^s \in \Gamma(\mu)$.*

Proof. As K is totally umbilical **s.l.s.**, therefore for any $Z = \eta_2 Z \in \Gamma(D)$, from Eqs. (27) and (41), we have

$$h(f\eta_2 Z, f\eta_2 Z) = \cos^2\theta g(\eta_2 Z, \eta_2 Z)H.$$

Employing Eq. (1), we obtain

$$\tilde{\nabla}_{f\eta_2 Z} f\eta_2 Z - \nabla_{f\eta_2 Z} f\eta_2 Z = \cos^2\theta g(\eta_2 Z, \eta_2 Z)H. \tag{43}$$

Then applying $\tilde{J}$ on both sides of Eq. (43) and using Eq. (9), we get

$$\tilde{\nabla}_{f\eta_2 Z} \tilde{J} f\eta_2 Z - \tilde{J}\nabla_{f\eta_2 Z} f\eta_2 Z = \cos^2\theta g(\eta_2 Z, \eta_2 Z)\tilde{J}H,$$

which on using Eq. (11), yields

$$\begin{aligned} \cos^2\theta g(\eta_2 Z, \eta_2 Z)\tilde{J}H = &\tilde{\nabla}_{f\eta_2 Z} f^2\eta_2 Z - \tilde{\nabla}_{f\eta_2 Z}\omega f\eta_2 Z - f\nabla_{f\eta_2 Z} f\eta_2 Z \\ &- \omega\nabla_{f\eta_2 Z} f\eta_2 Z. \end{aligned} \tag{44}$$

Taking into account Theorem 3.4, we have $f^2\eta_2 Z = -\cos^2\theta(\eta_2 Z)$ and hence Eq. (44) reduces to

$$\begin{aligned} \cos^2\theta g(\eta_2 Z, \eta_2 Z)\tilde{J}H = &-\cos^2\theta\tilde{\nabla}_{f\eta_2 Z}\eta_2 Z - A_{\omega f\eta_2 Z} f\eta_2 Z + D^l(f\eta_2 Z, \omega f\eta_2 Z) \\ &+ \nabla^s_{f\eta_2 Z}\omega f\eta_2 Z - f\nabla_{f\eta_2 Z} f\eta_2 Z - \omega\nabla_{f\eta_2 Z} f\eta_2 Z. \end{aligned} \tag{45}$$

Further employing Eq. (3) in Eq. (45) and then equating the transversal components on both sides, we derive

$$\cos^2\theta g(\eta_2 Z,\eta_2 Z)nH^s = -\cos^2\theta g(f\eta_2 Z,\eta_2 Z)H^l - \cos^2\theta g(f\eta_2 Z,\eta_2 Z))H^s + D^l(f\eta_2 Z,\omega f\eta_2 Z) + \nabla^s_{f\eta_2 Z}\omega f\eta_2 Z - \omega\nabla_{f\eta_2 Z} f\eta_2 Z.$$

Then, taking the inner product of above equation w.r.t. $\omega\eta_2 Z$, we obtain

$$0 = -\cos^2\theta\tilde{g}(H^s,\omega\eta_2 Z)g(f\eta_2 Z,\eta_2 Z) + \tilde{g}(\nabla^s_{f\eta_2 Z}\omega f\eta_2 Z,\omega\eta_2 Z) - \tilde{g}(\omega\nabla_{f\eta_2 Z} f\eta_2 Z,\omega\eta_2 Z). \tag{46}$$

Considering Eq. (28) for $Y_1 = Y_2 \in \Gamma(D)$ and then taking the covariant derivative w.r.t. $f\eta_2 Z$, we have

$$\tilde{g}(\nabla^s_{f\eta_2 Z}\omega\eta_2 Z,\omega\eta_2 Z) = \sin^2\theta g(\nabla_{f\eta_2 Z}\eta_2 Z,\eta_2 Z),$$

which further gives

$$\tilde{g}(\nabla^s_{f\eta_2 Z}\omega f\eta_2 Z,\omega\eta_2 Z) = \sin^2\theta g(\nabla_{f\eta_2 Z} f\eta_2 Z,\eta_2 Z). \tag{47}$$

Next using Eqs. (28) and (47) in Eq. (46), we obtain

$$\cos^2\theta g(f\eta_2 Z, f\eta_2 Z)\tilde{g}(H^s,\omega\eta_2 Z) = 0. \tag{48}$$

Thus Eq. (48) yields that either $\eta_2 Z = 0$ or $\theta = \pi/2$ or $H^s \in \Gamma(\mu)$, which proves the result. □

Lemma 5.3. *For a totally umbilical* **s.l.s.** K *of* $\tilde{K}$ *with* $\nabla^s_Z W \in \Gamma(\mu)$, *for any* $Z \in \Gamma(D)$ *and* $W \in \Gamma(S(TK^\perp))$. *Then, the slant distribution* D *always defines a totally geodesic foliation in* K.

Proof. As K is a totally umbilical **s.l.s.**, therefore for any $Z \in \Gamma(D)$ and $\xi \in \Gamma(Rad(TK))$, using Eqs. (1)-(4), we acquire

$$\begin{aligned} g(\nabla_Z Z,\tilde{J}\xi) &= \tilde{g}(\tilde{\nabla}_Z Z,\tilde{J}\xi) = -\tilde{g}(\tilde{J}\tilde{\nabla}_Z Z,\xi) = -\tilde{g}(\tilde{\nabla}_Z\tilde{J}Z,\xi) \\ &= -\tilde{g}(\tilde{\nabla}_Z f\eta_2 Z,\xi) + \tilde{g}(\tilde{\nabla}_Z\omega\eta_2 Z,\xi) = -\tilde{g}(D^l(Z,\omega\eta_2 Z),\xi) \\ &= -\tilde{g}(h^s(Z,\xi),\omega\eta_2 Z) = -g(Z,\xi)\tilde{g}(H^s,\omega\eta_2 Z) \\ &= 0. \end{aligned} \tag{49}$$

On the other hand, for $Z \in \Gamma(D)$ and $W \in \Gamma(\omega D) \subset \Gamma(S(TK^\perp))$, we have

$$\begin{aligned} \tilde{g}(\nabla_Z Z,\tilde{J}W) &= \tilde{g}(\tilde{\nabla}_Z Z,\tilde{J}W) = -\tilde{g}(\tilde{J}\tilde{\nabla}_Z Z,W) = -\tilde{g}(\tilde{\nabla}_Z\tilde{J}Z,W) \\ &= -\tilde{g}(\tilde{\nabla}_Z f\eta_2 Z,W) + \tilde{g}(\tilde{\nabla}_Z\omega\eta_2 Z,W) \\ &= -\tilde{g}(h^s(Z,f\eta_2 Z),W) - \tilde{g}(\nabla^s_Z\omega\eta_2 Z,W) \\ &= -\tilde{g}(H^s,W)g(Z,f\eta_2 Z) \\ &= 0. \end{aligned} \tag{50}$$

Thus the result follows from Eqs. (49)-(50). □

Theorem 5.4. *Consider a proper totally umbilical* **s.l.s.** K *of* $\tilde{K}$, *then we must have* $H^l = 0$.

Proof. Since $\tilde{K}$ is an indefinite nearly Kaehler manifold therefore for any $Z \in \Gamma(D)$, we have $\tilde{\nabla}_Z \tilde{J} Z = \tilde{J} \tilde{\nabla}_Z Z$. Then employing Eqs. (1), (2), (11) and (12), we obtain

$$\begin{aligned} (51) \qquad & \nabla_Z f\eta_2 Z + h^l(Z, f\eta_2 Z) + h^s(Z, f\eta_2 Z) - A_{\omega\eta_2 Z} Z + \nabla^s_Z \omega\eta_2 Z \\ & + D^l(Z, \omega\eta_2 Z) = f\nabla_Z Z + \omega\nabla_Z Z + \tilde{J} h^l(Z, Z) + th^s(Z, Z) + nh^s(Z, Z). \end{aligned}$$

Considering tangential components on both sides of Eq. (51), we get

$$\nabla_Z f\eta_2 Z - A_{\omega\eta_2 Z} Z = f\nabla_Z Z + \tilde{J} h^l(Z, Z) + th^s(Z, Z).$$

Taking inner product on both sides of above equation w.r.t. $\tilde{J}\xi \in \Gamma(\tilde{J} Rad(TK))$, we acquire

$$(52) \qquad g(A_{\omega\eta_2 Z} Z, \tilde{J}\xi) + \tilde{g}(h^l(Z, Z), \xi) = 0.$$

Then using Eq. (4) in Eq. (52), we get

$$\tilde{g}(h^s(Z, \tilde{J}\xi), \omega\eta_2 Z) + \tilde{g}(\tilde{J}\xi, D^l(Z, \omega\eta_2 Z)) + \tilde{g}(h^l(Z, Z), \xi) = 0.$$

In view of Eq. (42), the above equation reduces to

$$(53) \qquad \tilde{g}(H^s, \omega\eta_2 Z) g(Z, \tilde{J}\xi) + \tilde{g}(H^l, \xi) g(Z, Z) = 0.$$

For a proper totally umbilical **s.l.s.** of $\tilde{K}$, following Theorem 5.2, we have $H^s \in \Gamma(\mu)$. Thus Eq. (53) yields

$$\tilde{g}(H^l, \xi) g(Z, Z) = 0.$$

Then the non-degeneracy of the slant distribution D implies that $\tilde{g}(H^l, \xi) = 0$, which further gives

$$(54) \qquad H^l = 0.$$

Hence, the proof is completed. □

Theorem 5.5. *The induced connection* ∇ *on a proper totally umbilical* **s.l.s.** *of* $\tilde{K}$ *is always a metric connection.*

Proof. The proof follows directly from Theorem 5.4 and Eq. (7). □

Theorem 5.6. *Consider* K *be a proper totally umbilical* **s.l.s.** *of* $\tilde{K}$ *such that* $\nabla^s_Z W \in \Gamma(\mu)$, *for* $Z \in \Gamma(D)$ *and* $W \in \Gamma(S(TK^\perp))$, *then we have* $H^s = 0$.

Proof. One may note that for a proper totally umbilical **s.l.s.** of $\tilde{K}$, we have $H^s \in \Gamma(\mu)$. Comparing the transversal components on both sides of Eq. (51), we obtain

$$h^l(Z, f\eta_2 Z) + h^s(Z, f\eta_2 Z) + \nabla^s_Z \omega\eta_2 Z + D^l(Z, \omega\eta_2 Z) = \omega\nabla_Z Z + nh^s(Z, Z).$$

Then using Eq. (42), we acquire

$$g(Z, f\eta_2 Z) H^l + g(Z, f\eta_2 Z) H^s + \nabla^s_Z \omega\eta_2 Z = \omega\nabla_Z Z + g(Z, Z) nH^s.$$

Taking the inner product on both sides of above equation w.r.t $\tilde{J}H^s$, we get

$$\tilde{g}(\nabla^s_Z \omega\eta_2 Z, \tilde{J}H^s) = g(Z,Z)\tilde{g}(H^s, H^s). \tag{55}$$

As $\tilde{\nabla}$ is a metric connection on $\tilde{K}$, therefore $(\tilde{\nabla}_Z \tilde{g})(\omega\eta_2 Z, \tilde{J}H^s) = 0$, which further gives

$$\tilde{g}(\nabla^s_Z \omega\eta_2 Z, \tilde{J}H^s) = -\tilde{g}(\nabla^s_Z \tilde{J}H^s, \omega\eta_2 Z) = 0. \tag{56}$$

Thus from Eqs. (55) and (56), we derive

$$g(Z,Z)\tilde{g}(H^s, H^s) = 0.$$

As slant distribution D is non-degenerate, thus we have

$$H^s = 0.$$

Hence, the proof follows. □

Then, using Theorem 5.4 and Theorem 5.6, we have the following result.

Theorem 5.7. *Every totally umbilical proper* **s.l.s.** *K of $\tilde{K}$ such that $\nabla^s_Z W \in \Gamma(\mu)$, for $Z \in \Gamma(D)$ and $W \in \Gamma(S(TK^\perp))$ is always totally geodesic.*

Next, we prove an important characterization theorem on the non-existence of a totally umbilical **s.l.s.** K in a generalized complex space form $\tilde{K}(c, \alpha)$.

Theorem 5.8. *There does not exist any proper totally umbilical* **s.l.s.** *K in a generalized complex space form $\tilde{K}(c, \alpha)$ provided $c \neq \alpha$.*

Proof. Employing Eq. (10), for $Z \in \Gamma(D), Y \in \Gamma(\tilde{J}ltr(TK))$ and $\xi \in \Gamma(Rad(TK))$, we get

$$\tilde{g}(\tilde{R}(Z, \tilde{J}Z)Y, \xi) = -\frac{c-\alpha}{2} g(Z,Z)\tilde{g}(\tilde{J}Y, \xi). \tag{57}$$

On the other hand, equation of Codazzi (8) yields

$$\tilde{g}(\tilde{R}(Z, \tilde{J}Z)Y, \xi) = \tilde{g}((\nabla_Z h^l)(\tilde{J}Z, Y), \xi) - \tilde{g}((\nabla_{\tilde{J}Z} h^l)(Z, Y), \xi). \tag{58}$$

Then from Eqs. (57) and (58), we acquire

$$-\frac{c-\alpha}{2} g(Z,Z)\tilde{g}(\tilde{J}Y, \xi) = \tilde{g}((\nabla_Z h^l)(\tilde{J}Z, Y), \xi) - \tilde{g}((\nabla_{\tilde{J}Z} h^l)(Z, Y), \xi). \tag{59}$$

As K is a totally umbilical **s.l.s.**, thus from Eq. (42), we have

$$(\nabla_Z h^l)(\tilde{J}Z, Y) = -g(\nabla_Z \tilde{J}Z, Y)H^l - \tilde{g}(\tilde{J}Z, \nabla_Z Y)H^l. \tag{60}$$

As $\tilde{g}(\tilde{J}Z, Y) = 0$, for any $Z \in \Gamma(D)$ and $Y \in \Gamma(\tilde{J}ltr(TK))$, then taking covariant derivative w.r.t. Z, we get $g(\nabla_Z \tilde{J}Z, Y) = -\tilde{g}(\tilde{J}Z, \nabla_Z Y)$. Thus Eq. (60) reduces to

$$(\nabla_Z h^l)(\tilde{J}Z, Y) = 0. \tag{61}$$

Similarly, it follows that

$$(\nabla_{\tilde{J}Z} h^l)(Z, Y) = 0. \tag{62}$$

Then using Eqs. (61) and (62) in Eq. (59), we acquire

$$-\frac{c-\alpha}{2}g(Z,Z)\tilde{g}(\tilde{J}Y,\xi)=0.$$

Hence, in view of non-degeneracy of the slant distribution D, we obtain $c=\alpha$, which proves the result. □

6. Minimal Slant Lightlike Submanifolds

In [15], Duggal and Bejancu defined a minimal lightlike submanifold K by considering K to be a hypersurface of a 4-dimensional Minkowski space. Later, the general definition of a minimal lightlike submanifold of a semi-Riemannian manifold was given by Bejan and Duggal [11] as follows:

Definition 6.1. *A lightlike submanifold $(K,g,S(TK))$ isometrically immersed in a semi-Riemannian manifold $(\tilde{K},\tilde{g})$ is said to be a minimal lightlike submanifold if the following conditions hold:*

(i) $h^s(\xi_1,\xi_2)=0,\ for\ all\ \xi_1,\xi_2\in\Gamma(Rad(TK))$.
(ii) $trace\ h|_{S(TK)}=0$.

Note: In view of Definition 6.1, a **s.l.s.** K of $\tilde{K}$ is minimal if it is totally geodesic.

Example 6.2. *Let $(\tilde{K},\tilde{g})=(R^{10}_2,\tilde{g})$ be a semi-Euclidean space of signature $(-,-,+,+,+,+,+,+,+,+)$ w. r. t. the canonical basis $\{\partial x_1,\partial x_2,\partial x_3,\partial x_4,\partial x_5,\partial x_6,\partial x_7,\partial x_8,\partial x_9,\partial x_{10}\}$. Consider an almost complex structure $\tilde{J}$ defined by*

$$\begin{aligned}\tilde{J}(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10})&=(-x_2,x_1,-x_4,x_3,-x_6,x_5,\\&x_9\cos\alpha-x_8\sin\alpha,-x_{10}\cos\alpha+x_7\sin\alpha,x_7\cos\alpha+x_{10}\sin\alpha,\\&x_8\cos\alpha-x_9\sin\alpha),\end{aligned}$$

where $\alpha\in\left(0,\frac{\pi}{2}\right)$.
Let K be a 5-dimensional submanifold of $(R^{10}_2,\tilde{g})$ given by

$$x^1=u^1,\quad x^2=u^2,\quad x^3=u^1\cos\theta,\quad x^4=u^3\cos\theta,\quad x^5=u^1\sin\theta,$$

$$x^6=u^3\sin\theta,\quad x^7=\cos u^4\cosh u^5,\quad x^8=\cos u^4\sinh u^5,$$

$$x^9=\sin u^4\sinh u^5,\quad x^{10}=\sin u^4\cosh u^5,\ where\ \theta,u^4\in R-\left\{\frac{n\pi}{2},n\in Z\right\}.$$

Then TK is spanned by Z_1,Z_2,Z_3,Z_4,Z_5, where

$$Z_1=\partial x_1+\cos\theta\partial x_3+\sin\theta\partial x_5,\quad Z_2=\partial x_2,\quad Z_3=\cos\theta\partial x_4+\sin\theta\partial x_6,$$

$$\begin{aligned}Z_4=&-\sin u^4\cosh u^5\partial x_7-\sin u^4\sinh u^5\partial x_8\\&+\cos u^4\sinh u^5\partial x_9+\cos u^4\cosh u^5\partial x_{10},\end{aligned}$$

$$Z_5 = \cos u^4 \sinh u^5 \partial x_7 + \cos u^4 \cosh u^5 \partial x_8 + \sin u^4 \cosh u^5 \partial x_9 + \sin u^4 \sinh u^5 \partial x_{10}.$$

It is clear that K is a 1-lightlike submanifold with $Rad(TK) = Span\{Z_1\}$ and the lightlike transversal bundle $ltr(TK)$ is spanned by

$$N_1 = \frac{1}{2}\{-\partial x_1 + \cos\theta \partial x_3 + \sin\theta \partial x_5\}.$$

It follows that $\tilde{J}Z_1 = Z_2 + Z_3$ and $\tilde{J}N_1 = \frac{1}{2}\{-Z_2 + Z_3\}$, which implies that $\tilde{J}Rad(TK)$ and $\tilde{J}ltr(TK)$ are distributions on N. Hence, $D = Span\{Z_4, Z_5\}$ is a slant distribution w.r.t. $\tilde{J}$ with slant angle α. Further, by direct calculations, $S(TK^\perp)$ is spanned by

$$W_1 = -\cosh u^5 \partial x_7 + \sinh u^5 \partial x_8 + \tan u^4 \sinh u^5 \partial x_9 - \tan u^4 \cosh u^5 \partial x_{10},$$

$$W_2 = -\tan u^4 \sinh u^5 \partial x_7 + \tan u^4 \cosh u^5 \partial x_8 - \cosh u^5 \partial x_9 + \sinh u^5 \partial x_{10}.$$

Therefore, K is a proper slant lightlike submanifold of R^{10}_2. Moreover, using Gauss and Weingarten formulae, we attain

$$h^l = 0,\ h^s(X, Z_1) = h^s(X, \tilde{J}Z_1) = 0,\ h^s(X, \tilde{J}N_1) = 0,\ \forall\ X \in \Gamma(TK).$$

$$h^s(Z_4, Z_4) = \frac{\cos u_4}{\sinh^2 u_5 + \cosh^2 u_5} W_1,\ h^s(Z_5, Z_5) = \frac{-\cos u_4}{\sinh^2 u_5 + \cosh^2 u_5} W_1.$$

Hence, the induced connection is a metric connection and N is not totally geodesic, but it is a proper minimal ***s.l.s.*** *of R^{10}_2.*

Theorem 6.3. *A necessary and sufficient condition for a totally umbilical proper* ***s.l.s.*** *K of $\tilde{K}$ to be minimal is that, trace $A_{W_l}|_D = 0$ for $W_l \in \Gamma(S(TK^\perp))$, where $l \in \{1, 2, ..., t\}$.*

Proof. According to Definition 3.1, K is minimal, if and only if,

$$h^s(\xi_i, \xi_j) = 0$$

and

$$\sum_{i=1}^{r} h(\tilde{J}\xi_i, \tilde{J}\xi_i) + \sum_{i=1}^{r} h(\tilde{J}N_i, \tilde{J}N_i) + \sum_{j=1}^{q} h(e_j, e_j) = 0,$$

where $\{\xi_i\}_{i=1}^r$, $\{N_i\}_{i=1}^r$ and $\{e_j\}_{j=1}^q$ are bases of $Rad(TK)$, $ltr(TK)$ and D respectively. From Eq. (42), we note that $h(\xi_i, \xi_j) = h(\tilde{J}\xi_i, \tilde{J}\xi_i) = h(\tilde{J}N_i, \tilde{J}N_i) = 0$. Therefore, we get $h^s = 0$ on $Rad(TK)$. On the other hand, by using Eq. (54), we conclude that $h^l = 0$. Thus K is minimal, if and only if, $\sum_{j=1}^q h^s(e_j, e_j) = 0$, where

$$\sum_{j=1}^{q} h^s(e_j, e_j) = \sum_{j=1}^{q} \left\{ \frac{1}{t} \sum_{k=1}^{t} \tilde{g}(h^s(e_j, e_j), W_l) W_l \right\}.$$

Then using Eq. (4), the above equation reduces to

$$\sum_{j=1}^{q} h^s(e_j, e_j) = \sum_{j=1}^{q} \left\{ \frac{1}{t} \sum_{k=1}^{t} g(A_{W_l} e_j, e_j), W_l \right\}.$$

Hence, the result follows. □

Theorem 6.4. *Consider an irrotational* **s.l.s.** K *of* $\tilde{K}$*. Then* K *is minimal, if and only if,*

$$trace\ A_{W_l}|_{S(TK)} = 0 \ and\ trace\ A^*_{\xi_j}|_{S(TK)} = 0,$$

where $W_l \in \Gamma(S(TK^{\perp}))$, $\xi_j \in \Gamma(Rad(TK))$, $l \in \{1, 2, ..., t\}$ *and* $j \in \{1, 2, ..., r\}$.

Proof. Since K is irrotational, therefore $h^s = 0$ on $Rad(TK)$. Moreover, we have

$$trace\ h|_{S(TK)} = \sum_{i=1}^{r} h(\tilde{J}\xi_i, \tilde{J}\xi_i) + \sum_{i=1}^{r} h(\tilde{J}N_i, \tilde{J}N_i) + \sum_{p=1}^{q} h(e_p, e_p).$$

Then using Eqs. (4) and (6), we obtain

(63)
$$\sum_{i=1}^{r} h(\tilde{J}\xi_i, \tilde{J}\xi_i) = \sum_{i=1}^{r} \left\{ \frac{1}{r} \sum_{j=1}^{r} g(A^*_{\xi_j} \tilde{J}\xi_i, \tilde{J}\xi_i) N_j + \frac{1}{t} \sum_{l=1}^{t} g(A_{W_l} \tilde{J}\xi_i, \tilde{J}\xi_i) W_l \right\}.$$

Similarly,

(64)
$$\sum_{i=1}^{r} h(\tilde{J}N_i, \tilde{J}N_i) = \sum_{i=1}^{r} \left\{ \frac{1}{r} \sum_{j=1}^{r} g(A^*_{\xi_j} \tilde{J}N_i, \tilde{J}N_i) N_j + \frac{1}{t} \sum_{l=1}^{t} g(A_{W_l} e_i, e_i) W_l \right\}$$

and

$$\sum_{p=1}^{q} h(e_p, e_p) = \sum_{p=1}^{q} \left\{ \frac{1}{r} \sum_{j=1}^{r} g(A^*_{\xi_j} e_p, e_p) N_j + \frac{1}{t} \sum_{l=1}^{t} g(A_{W_l} e_p, e_p) W_l \right\}. \quad (65)$$

Thus, we conclude that $trace\ h|_{S(TK)} = 0$, if and only if, $trace\ A_{W_l} = 0$ and $trace\ A^*_{\xi_j} = 0$.

Hence, the result follows from Eqs. (63)–(65). □

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