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# $e$-SYMMETRIC MODULES

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**Abstract.** In this paper we introduce two classes of modules namely $e$-symmetric and $e$-reduced. Also, we study the characterizations of $e$-symmetric and $e$-reduced modules and their related properties.

## 1. Introduction

Throughout this paper, $R$ denotes an associative ring with identity, ${}_RM$ is a unitary left $R$-module, $\mathrm{Id}\,(R)$ is the set of all idempotent elements of $R$, $\mathrm{nil}\,(R)$ is the set of all nilpotent elements of $R$, $\mathrm{C}(R)$ is the center of $R$, $\mathrm{S_r}(R) = \{e \in \mathrm{Id}\,(R) : e\,R\,e = eR\}$ is the set of all right semicentral idempotent elements of $R$, and $\ell_R(M) = \{a \in R : aM = 0\}$ is the left annihilator of $M$ in $R$.

According to Lambek [3], a ring $R$ is called *symmetric* if whenever $a, b, c \in R$ such that $abc = 0$, we have $bac = 0$. An equivalent condition on a ring $R$ is that whenever a product of any number of elements is zero, any permutation of the factors still yields vanishes product. Following [5], for an idempotent $e \in R$, a ring $R$ is called *e-symmetric* if whenever $a, b, c \in R$ such that $abc = 0$, we have $acbe = 0$. Recall from [3] and [6] that a left $R$-module ${}_RM$ is called *symmetric* if whenever $a, b \in R$ and $m \in M$ such that $abm = 0$ implies $bam = 0$.

Recall that a ring $R$ is *reduced* if it has no nonzero nilpotent elements. In 2004, the reduced ring concept was extended to modules by Lee and Zhou [4] as follows: a left $R$-module ${}_RM$ is reduced if, for any $m \in M$ and any $a \in R$, $am = 0$ implies $Rm \cap aM = 0$.

This paper is organized as follows. In Section 2, we first define ${}_RM$ to be $e$-symmetric, where $e \in \mathrm{Id}(R)$, if whenever $a, b \in R$ and $m \in M$ such that $abm = 0$ implies $ebam = 0$. We discuss many properties of $e$-symmetric modules. In Sections 3 and 4, we investigate the behavior of $e$-symmetric modules with respect to matrix and polynomial extensions, respectively. In Section 5, we define ${}_RM$ to be $e$-reduced, where $e \in \mathrm{Id}(R)$, if whenever $a \in R$ and $m \in M$ such that $a^2m = 0$ implies $eaRm = 0$. We show that every $e$-reduced module is $e$-symmetric for any right semicentral idempotent $e$ in $R$.

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## 2. Modules with $e$-symmetric condition

In this section, we extend the notion of $e$-symmetric rings to modules as follows:

**Definition 2.1.** *A left $R$-module ${}_RM$ is called $e$-symmetric, where $e \in \mathrm{Id}(R)$, if whenever $a, b \in R$ and $m \in M$ such that $abm = 0$ implies $ebam = 0$.*

Obviously, $R$ is a left $e$-symmetric ring if and only if ${}_RR$ is an $e$-symmetric module.

Clearly, any symmetric module is an $e$-symmetric module for any $e \in \mathrm{Id}(R)$. In the following example we show that the class of $e$-symmetric modules is properly contains the class of symmetric modules.

**Example 2.2.** *Let $S$ be a symmetric ring, $R = \begin{pmatrix} S & S \\ 0 & S \end{pmatrix}$ and $M = \begin{pmatrix} 0 & S \\ 0 & S \end{pmatrix}$ a left $R$-module. Consider the following elements $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$ and $m = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M$. By direct computations we get that $abm = 0$ and $bam \neq 0$. So ${}_RM$ is not a symmetric left $R$-module. Consider the idempotent $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in R$, we show that ${}_RM$ is an $e$-symmetric left $R$-module. Let $a = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$, $b = \begin{pmatrix} t & u \\ 0 & v \end{pmatrix} \in R$ and $m = \begin{pmatrix} 0 & \alpha \\ 0 & \beta \end{pmatrix} \in M$ such that $abm = 0$. Hence*

$$\begin{aligned} 0 &= zv\beta, \text{ and} \\ 0 &= xt\alpha + xu\beta + yv\beta. \end{aligned}$$

*Thus*

$$ebam = \begin{pmatrix} 0 & 0 \\ 0 & vz\beta \end{pmatrix} = 0.$$

**Proposition 2.3.** *The class of $e$-symmetric modules is closed under submodules, direct products and so direct sums.*

*Proof.* The proof is straightforward depending on the definitions and algebraic structures. □

Recall that a module ${}_RM$ is called *cogenerated* by $R$ if it is embedded in a direct product of copies of $R$, and ${}_RM$ is called *faithful* if $\ell_R(M) = (0)$. The following Proposition is a direct result of definitions and Proposition 2.3.

**Proposition 2.4.** *The following conditions are equivalent for a ring $R$ :*
*(1) $R$ is an $e$-symmetric ring.*
*(2) Every cogenerated left $R$-module, by $R$, is an $e$-symmetric module.*
*(3) Every submodule of a free $R$-module is an $e$-symmetric module.*
*(4) There exists a faithful $e$-symmetric $R$-module.*

**Proposition 2.5.** *Let $R, S$ be rings, $e \in \mathrm{Id}(R)$ and $\varphi : R \to S$ be a ring homomorphism. If ${}_SM$ is a left $S$-module, then $M$ is a left $R$-module via $rm = \varphi(r)m$ for all $r \in R$ and $m \in M$. Then we get:*
*(1) If ${}_SM$ is a $\varphi(e)$-symmetric module, then ${}_RM$ is an $e$-symmetric module.*
*(2) If $\varphi$ is onto and ${}_RM$ is an $e$-symmetric module, then ${}_SM$ is a $\varphi(e)$-symmetric module.*

*Proof.* (1) Suppose that ${}_SM$ is a $\varphi(e)$-symmetric module. Let $a, b \in R$ and $m \in M$ such that $abm = 0$. Then $\varphi(ab)m = \varphi(a)\varphi(b)m = 0$. Since ${}_SM$ is $\varphi(e)$-symmetric, we have $\varphi(e)\varphi(b)\varphi(a)m = 0$. So $\varphi(eba)m = 0$, which in turn implies $ebam = 0$. Hence ${}_RM$ is an $e$-symmetric module.
(2) Suppose that $\varphi$ is onto and ${}_RM$ is an $e$-symmetric module. Let $x, y \in S$ and $m \in M$ such that $xym = 0$. Since $\varphi$ is onto, there exist $a, b \in R$ such that $x = \varphi(a)$ and $y = \varphi(b)$. Then $\varphi(a)\varphi(b)m = \varphi(ab)m = abm = 0$. Since ${}_RM$ is $e$-symmetric, implies $ebam = 0$. Hence $\varphi(eba)m = \varphi(e)\varphi(b)\varphi(a)m = 0$, and so $\varphi(e)yxm = 0$. Thus ${}_SM$ is a $\varphi(e)$-symmetric module. □

**Theorem 2.6.** *Let $R$ be a ring, $e \in \mathrm{S_r}(R)$ and ${}_RM$ is a left $R$-module. Then ${}_RM$ is an $e$-symmetric module if and only if ${}_{eR}M$ is a symmetric module.*

*Proof.* "$\Longrightarrow$" Assume that ${}_RM$ is an $e$-symmetric module. Let $a, b \in eR$ and $m \in M$ such that $abm = 0$. Since $a, b \in eR$, there exist $x, y \in R$ such that $a = ex$ and $b = ey$. So we get $(ex)(ey)m = 0$ in ${}_RM$ which implies that $e(ey)(ex)m = (ey)(ex)m = bam = 0$. Hence ${}_{eR}M$ is a symmetric module.
"$\Longleftarrow$" Assume that ${}_{eR}M$ is a symmetric module. Let $a, b \in R$ and $m \in M$ such that $abm = 0$ which implies that $eabm = 0$. Since $e \in \mathrm{S_r}(R)$, we get $(eae)bm = 0$ and then $(ea)(eb)m = 0$ in ${}_{eR}M$. Hence $(eb)(ea)m = (ebe)am = ebam = 0$. Thus ${}_RM$ is an $e$-symmetric module. □

**Corollary 2.7.** *Let $R$ be a ring, $e \in \mathrm{C}(R)$ and ${}_RM$ a left $R$-module. If ${}_{eR}M$ and ${}_{(1-e)R}M$ are symmetric modules, then ${}_RM$ is a symmetric module.*

*Proof.* We can easily check that $e \in \mathrm{C}(R)$ if and only if $(1-e) \in \mathrm{C}(R)$. From Theorem 2.6, we conclude that ${}_RM$ is both $e$-symmetric and $(1-e)$-symmetric. Let $a, b \in R$ and $m \in M$ such that $abm = 0$ which implies that $ebam = 0$ and $(1-e)bam = 0$. Therefore $bam = 0$. Thus ${}_RM$ is a symmetric module. □

**Proposition 2.8.** *Let $R$ be a ring, $e \in \mathrm{Id}(R)$ and ${}_RM$ a left $R$-module. If ${}_RM$ is an $e$-symmetric module, then $\overline{R} = R/\ell_R(M)$ is an $\overline{e}$-symmetric ring.*

*Proof.* Let $\overline{a}, \overline{b}, \overline{c} \in \overline{R}$, such that $\overline{a}\overline{b}\overline{c} = \overline{0}$. So $\overline{abc} = \overline{0}$. Hence $abc \in \ell_R(M)$, and so we have $abcm = 0$ for every $m \in M$. Since ${}_RM$ is $e$-symmetric, we get $ebacm = 0$ for every $m \in M$. Hence $ebac \in \ell_R(M)$ and so $\overline{0} = \overline{ebac} = \overline{e}\ \overline{b}\ \overline{a}\ \overline{c}$. Therefore $\overline{R}$ is an $\overline{e}$-symmetric ring. □

## 3. Matrix extensions

In this section we characterize left $e$-symmetric 2-by-2 generalized upper triangular matrix rings. Moreover, as a corollary we obtain that a ring $R$ is a left $e$-symmetric ring if and only if $T_n(R)$ is left $E$-symmetric for all positive integers $n$.

**Theorem 3.1.** *Let* $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *where* $R$ *and* $S$ *are rings, and* ${}_RM_S$ *an* $(R,S)$*-bimodule. If* $T$ *is a left* $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$*-symmetric ring, with* $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix} \in \mathrm{Id}(T)$*, then:*
*(1)* $R$ *is a left* $e$*-symmetric ring,*
*(2)* $S$ *is a left* $g$*-symmetric ring, and*
*(3)* ${}_RM$ *is a left* $e$*-symmetric* $R$*-module.*

*Proof.* Assume that $T$ is a left $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-symmetric ring, where $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix} \in \mathrm{Id}(T)$. Then by easy computations, we can check that $e \in \mathrm{Id}(R), g \in \mathrm{Id}(S)$ and $ek + kg = k$.
(1) Assume that $abc = 0$ for $a, b, c \in R$. Consider the following elements

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \in T.$$

We have

$$0 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $T$ is a left $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-symmetric ring, we get

$$0 = \begin{pmatrix} e & k \\ 0 & g \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $0 = ebac$ in $R$. Therefore $R$ is a left $e$-symmetric ring.
(2) Assume that $\alpha\beta\gamma = 0$, for $\alpha, \beta, \gamma \in S$. Consider the following elements

$$\begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix} \in T.$$

We have

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix}.$$

Since $T$ is a left $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-symmetric ring, we get

$$0 = \begin{pmatrix} e & k \\ 0 & g \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix}.$$

Hence $g\beta\alpha\gamma = 0$ in $S$. Therefore $S$ is a left $g$-symmetric ring.
(3) Let $a, b \in R$ and $m \in M$ such that $abm = 0$. Consider the following elements

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in T.$$

We have

$$0 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}.$$

Since $T$ is a left $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-symmetric ring, we get

$$0 = \begin{pmatrix} e & k \\ 0 & g \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}.$$

Hence $ebam = 0$ in ${}_RM$. Therefore ${}_RM$ is a left $e$-symmetric $R$-module. □

**Theorem 3.2.** *Let* $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *where* $R$ *and* $S$ *are rings, and* ${}_RM_S$ *an* $(R, S)$*-bimodule. If* $S$ *is a left* $g$*-symmetric ring, where* $g \in \mathrm{Id}(S)$*, then* $T$ *is a left* $\begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix}$*-symmetric ring.*

*Proof.* Assume that $S$ is a left $g$-symmetric ring, where $g \in \mathrm{Id}(S)$. Consider the following elements

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}, \begin{pmatrix} q & n \\ 0 & p \end{pmatrix}, \begin{pmatrix} u & t \\ 0 & v \end{pmatrix} \in T$$

such that

$$0 = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\begin{pmatrix} q & n \\ 0 & p \end{pmatrix}\begin{pmatrix} u & t \\ 0 & v \end{pmatrix}.$$

Thus

$$0 = \begin{pmatrix} aqu & aqt + anv + mpv \\ 0 & bpv \end{pmatrix}.$$

Hence $0 = bpv$ in $S$. Since $S$ is a left $g$-symmetric ring, we have $gpbv = 0$.
Thus

$$\begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} q & n \\ 0 & p \end{pmatrix}\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\begin{pmatrix} u & t \\ 0 & v \end{pmatrix} = 0.$$

Therefore $T$ is a left $\begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix}$-symmetric ring. □

**Corollary 3.3.** *Let* $T_n(R)$ *be the* $n$*-by-*$n$ *upper triangular matrix ring over a ring* $R$*, where* $n \geq 1$*. Then the following are equivalent:*
*(1)* $R$ *is a left* $e$*-symmetric ring, where* $e \in \mathrm{Id}(R)$*.*
*(2)* $T_2(R) = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ *is a left* $\begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}$*-symmetric ring.*

*(3)* $T_n(R)$ *is a left* $\begin{pmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & e \end{pmatrix}$*-symmetric ring for every positive integer* $n$.

*Proof.* "(3) $\Longrightarrow$ (1)" follows directly from the fact that $T_1(R) \cong R$.
"(1) $\Longrightarrow$ (2)" is clear from Theorem 3.2.
"(2) $\Longrightarrow$ (3)" Note that $T_{n+1}(R) \cong \begin{pmatrix} R & M \\ 0 & T_n(R) \end{pmatrix}$ where $M$ is the 1-by-$n$ row matrix with $R$ in every entry and 0 is the $n$-by-1 column zero matrix. So this implication is proved by using induction on $n$. □

## 4. Polynomial extensions

Recall the following extensions of a left $R$-module $_RM$ :

$$M\,[x] = \left\{ \varphi(x) = \sum_{i=0}^{n} m_i x^i : m_i \in M \right\}.$$

$M\,[x]$ is a left $R\,[x]$-module and $_{R[x]}M\,[x]$ is called *the usual polynomial extension of* $_RM$.

$$M[x, x^{-1}] = \left\{ \varphi(x) = \sum_{i=-k}^{n} m_i x^i : m_i \in M \right\}.$$

$M[x, x^{-1}]$ is a left $R[x, x^{-1}]$-module and $_{R[x,x^{-1}]}M[x, x^{-1}]$ is called *the usual Laurent polynomial extension of* $_RM$.

From ([2], Example 2.1.), we conclude that, in general, the polynomial rings over symmetric rings not be symmetric.

This motivated us to study the conditions under which the polynomial extensions of a left $R$-module $_RM$ be $e$-symmetric, for some $e \in \mathrm{Id}(R)$.

We mean by a regular element of a ring $R$, a nonzero element which is not a zero divisor.

**Theorem 4.1.** *Let* $R$ *be a ring,* $\Delta$ *be a multiplicatively closed subset of* $R$ *consisting of central regular elements,* $1 \in \Delta$ *and* $e \in \mathrm{Id}(R)$*. Then* $_RM$ *is* $e$*-symmetric if and only if* $_{(\Delta^{-1}R)}\left(\Delta^{-1}M\right)$ *is* $\left(1^{-1}e\right)$*-symmetric.*

*Proof.* Suppose that $_RM$ is $e$-symmetric. Let $a, b \in R$, $m \in M$ and $u, v, w \in \Delta$ such that $\left(u^{-1}a\right)\left(v^{-1}b\right)\left(w^{-1}m\right) = 0$ in $_{(\Delta^{-1}R)}\left(\Delta^{-1}M\right)$. Since $\Delta$ is contained in the center of $R$, we have

$$0 = \left(u^{-1}v^{-1}w^{-1}\right)(abm) = (uvw)^{-1}(abm),$$

and so $abm = 0$. Therefore $ebam = 0$, since ${}_RM$ is $e$-symmetric. So, in ${}_{(\Delta^{-1}R)}\left(\Delta^{-1}M\right)$, we have $0 = (1vuw)^{-1}(ebam) = \left(1^{-1}v^{-1}u^{-1}w^{-1}\right)(ebam)$. Thus

$$\left(1^{-1}e\right)\left(v^{-1}b\right)\left(u^{-1}a\right)\left(w^{-1}m\right) = 0.$$

Hence ${}_{(\Delta^{-1}R)}\left(\Delta^{-1}M\right)$ is $\left(1^{-1}e\right)$-symmetric.
It is clear that if ${}_{(\Delta^{-1}R)}\left(\Delta^{-1}M\right)$ is $\left(1^{-1}e\right)$-symmetric, then ${}_RM$ is $e$-symmetric. □

**Corollary 4.2.** *Let $R$ be a ring and $e \in \mathrm{Id}(R)$. Then ${}_{R[x]}M[x]$ is $e$-symmetric if and only if ${}_{R[x,x^{-1}]}M[x,x^{-1}]$ is $e$-symmetric.*

*Proof.* Consider the multiplicatively closed set $\Delta = \left\{1, x, x^2, x^3, ...\right\}$ which is clearly a subset of $R[x]$ consisting of regular elements. Since $\Delta^{-1}R[x] = R[x,x^{-1}]$ and $\Delta^{-1}M[x] = M[x,x^{-1}]$, the result follows directly from Theorem 4.1. □

Following, Anderson and Camillo [1], extended the concept of Armendariz ring to Armendariz module, as follows: A left $R$-module ${}_RM$ is called Armendariz if whenever $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x]$ and $g(x) = \sum_{j=0}^{p} a_j x^j \in R[x]$ such that $g(x)m(x) = 0$ implies $a_j m_i = 0$ for all $i$ and $j$. The Armendariz property holds for any finite product of polynomials. Clearly, $R$ is an Armendariz ring if and only if ${}_RR$ is an Armendariz $R$-module.

**Theorem 4.3.** *Let $R$ be a ring, $e \in \mathrm{Id}(R)$ and ${}_RM$ a left Armendariz $R$-module. ${}_RM$ is an $e$-symmetric module if and only if ${}_{R[x]}M[x]$ is $e$-symmetric if and only if ${}_{R[x,x^{-1}]}M[x,x^{-1}]$ is $e$-symmetric.*

*Proof.* Assume that ${}_RM$ is $e$-symmetric. Let $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{p} a_j x^j, g(x) = \sum_{k=0}^{q} b_k x^k \in R[x]$ such that $f(x)g(x)m(x) = 0$. Since ${}_RM$ is a left Armendariz $R$-module, we have $a_j b_k m_i = 0$ for all $i, j$ and $k$. Since ${}_RM$ is $e$-symmetric, we get $eb_k a_j m_i = 0$ for all $i, j$ and $k$. Thus we have $eg(x)f(x)m(x) = 0$. Therefore ${}_{R[x]}M[x]$ is $e$-symmetric. Now the required equivalence is clear from Corollary 4.2. □

## 5. $e$-reduced modules

In this section we introduce the notion of $e$-reduced modules.

**Definition 5.1.** *A left $R$-module ${}_RM$ is called $e$-reduced, where $e \in \mathrm{Id}(R)$, if whenever $a \in R$ and $m \in M$ such that $am = 0$ implies $Rm \cap eaM = 0$.*

**Proposition 5.2.** *The following conditions are equivalent for a left $R$-module ${}_RM$ :*
*(1) ${}_RM$ is an $e$-reduced module.*
*(2) The following two conditions hold:*
*(i) Whenever $a \in R$ and $m \in M$ such that $am = 0$ implies $eaRm = 0$,*
*(ii) Whenever $a \in R$ and $m \in M$ such that $a^2m = 0$ implies $eam = 0$.*
*(3) Whenever $a \in R$ and $m \in M$ such that $a^2m = 0$ implies $eaRm = 0$.*

*Proof.* "$(1) \Rightarrow (2)$" Let ${}_RM$ be an $e$-reduced module.
(i) Assume that $a \in R$ and $m \in M$ such that $am = 0$. From the given, we have $Rm \cap eaM = 0$. It is easily to check that

$$eaRm \subset Rm \text{ and } eaRm \subset eaM.$$

Therefore $eaRm \subset Rm \cap eaM = 0$. Thus $eaRm = 0$.
(ii) Assume that $a \in R$ and $m \in M$ such that $a^2m = 0$. So, we have

$$0 = e\left(a^2m\right) = ea\,(am)\,;\ ea \in R \text{ and } am \in M.$$

Therefore $R\,(am) \cap (ea)\,M = 0$. But $eam \in R\,(am) \cap (ea)\,M = 0$. Thus $eam = 0$.
"$(2) \Rightarrow (1)$" Assume that $a \in R$ and $m \in M$ such that $am = 0$. We show that $Rm \cap eaM = 0$. Let $x \in Rm \cap eaM$, so there exist $r \in R$ and $n \in M$ such that $x = rm$ and $x = ean$. Since $am = 0$, we conclude, from (2-i), that $earm = 0$. Thus $0 = eax = ea\,(ean) = (ea)^2\,n$. By using (2-ii), we get $0 = ean = x$. Therefore $Rm \cap eaM = 0$. Hence ${}_RM$ is an $e$-reduced module.
"$(2) \Rightarrow (3)$" Assume that $a \in R$ and $m \in M$ such that $a^2m = 0$. By using (2-ii), we get $0 = eam$. We conclude, from (2-i), that $0 = e\,(ea)\,Rm = eaRm$.
"$(3) \Rightarrow (2)$" (i) Assume that $a \in R$ and $m \in M$ such that $am = 0$. Hence $a^2m = 0$ which implies $0 = eaRm$.
(ii) Assume that $a \in R$ and $m \in M$ such that $a^2m = 0$. Hence $0 = eaRm$. Therefore $eam = 0$. □

A ring $R$ is $e$-reduced if and only if ${}_RR$ is an $e$-reduced module. Any reduced left $R$-module is $e$-reduced for any nontrivial idempotent $e$ in $R$.

**Lemma 5.3.** *The class of $e$-reduced modules is closed under submodules, direct products and so direct sums.*

*Proof.* The proof is straightforward depending on the definitions and algebraic structures. □

**Example 5.4.** *Consider the ring of integers modulo 12, $R = \mathbb{Z}_{12}$, as a module over itself, then by direct computations, we can conclude that:*

$$\mathrm{nil}(R) = \{0, 6\} \neq 0 \textit{ and } \mathrm{Id}(R) = \{0, 1, 4, 9\}\,.$$

*$R$ is a 4-reduced ring which is not a reduced ring. Therefore ${}_RR$ is a 4-reduced module.*

Recall from [4] that, a left $R$-module is called a *p.p.-module* if for any $m \in M$, $\ell_R(m) = gR$ where $g \in \mathrm{Id}(R)$.

**Proposition 5.5.** *Let $R$ be a ring, ${}_RM$ a left $R$-module and $e \in \mathrm{Id}(R)$. If ${}_RM$ is a p.p.-module, then ${}_RM$ is an $e$-reduced module.*

*Proof.* Assume that $a \in R$ and $m \in M$ such that $am = 0$. If $x \in Rm \cap eaM$, then $x = rm = eak$; where $r \in R$, $k \in M$. Since ${}_RM$ is p.p. and $am = 0$, we have $ea \in \ell_R(m) = gR$ and so $ea = gy$ with $g \in \mathrm{Id}(R), y \in R$. Therefore $gea = ea$. Then $x = eak = geak = gx = grm = 0$. Thus $Rm \cap eaM = 0$. Hence ${}_RM$ is $e$-reduced. □

**Proposition 5.6.** *Let $R$ be a ring and ${}_RM$ a left $R$-module. If $e \in \mathrm{S_r}(R)$ and ${}_RM$ is an $e$-reduced module, then ${}_RM$ is an $e$-symmetric module.*

*Proof.* Assume that $a, b \in R$ and $m \in M$ such that $abm = 0$. So, we have $(babb)(ab)m = 0$, which implies that $(bab)^2 m = 0$. Since ${}_RM$ is $e$-reduced, $ebabRm = 0$. Since $e \in \mathrm{S_r}(R)$, we have $ebeaebRm = 0$. Then $(ebea)^2 m = 0$. Since ${}_RM$ is $e$-reduced and $e \in \mathrm{S_r}(R)$, we get $ebam = 0$. Therefore ${}_RM$ is $e$-symmetric. □

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