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GENERALIZED PSEUDO BE-ALGEBRAS

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Abstract. In this paper, we define a new algebraic structure known as a generalized pseudo BE-algebra which is a generalization of a pseudo BE-algebra. We construct some examples in order to show the existence of the generalized pseudo BE-algebra. Moreover, we characterize different classes of generalized pseudo BE-algebras by some results.

1. Introduction

Ise'ki and Imai [9] defined two types of an abstract algebra. One is called BCK-algebra and the other is called BCI-algebra. It is obvious that every BCK-algebra is a BCI-algebra, i.e. in other words, BCI-algebra is a generalization of a BCK-algebra. Some researchers worked and defined different generalized structures and named them pseudo structures. Georgescu and Iorgulescu [5] defined the notion of pseudo MV-algebra as a generalization of a MV-algebra. Moreover, a pseudo BL-algebra [6] and a pseudo BCK-algebra [7] were introduced as well as studied by Georgescu and Iorgulescu and the said structures are an extended notion of BCK-algebras. Furthermore, Walendziak [18] worked on axioms system of pseudo BCK-algebras and explored some properties. In [10], the authors gave the notion of pseudo-homomorphism, pseudo-atom as well as pseudo-ideal in a pseudo BCI-algebra and characterized them by a number of properties. Kim and So [11] worked on minimal elements in a pseudo BCI-algebra and moreover characterized minimal elements by different properties, while the concept of a BE-algebra was initiated in [12]. Moreover in [12] the authors also gave an equivalent condition for the filters in a BE-algebra by using the concept of upper sets. BE-algebra was studied in a detail by different mathematicians [[1], [2], [3], [15], [16], [17]] and characterized it in different directions. In [13], Meng developed a method which says that a filter can be generated by a subset $\emptyset \neq \boldsymbol{A}$ of a transitive BE-algebra. Moreover, the author of [13] characterized Noetherian and Artinian BE-algebras by some properties.

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In [4], the authors deeply studied BE-algebras and defined pseudo BE-algebras. Moreover, they defined pseudo filter and pseudo sub-algebra in a pseudo BE-algebra and connected them through some results. In particular, they proved that the class of pseudo filters is a subclass of the class of pseudo sub-algebras and moreover constructed a particular example to show that the converse does not hold. They also introduced the concept of pseudo upper sets in a pseudo BE-algebra and then showed that any pseudo filter can be written as a union of pseudo upper sets.

2. Generalized Pseudo BE-algebras

In this section, we define generalized pseudo BE-algebra as a generalization of pseudo BE-algebra and construct some examples to show the existence of the generalized pseudo BE-algebra. We then explore some properties of the generalized pseudo BE-algebra.

Definition 2.1. *Let X be a non-empty set with two binary operations "$*$" and "$\diamond$" defined on X and "1" an element of X. An algebraic structure $(X; *, \diamond, 1)$ is said to be a generalized pseudo BE-algebra if it satisfies the below properties:*
(i) $a * a = 1$ *and* $a \diamond a = 1$,
(ii) $a * 1 = 1$ *and* $a \diamond 1 = 1$,
(iii) $a * (b \diamond c) = b \diamond (a * c)$,
(iv) $a * b = 1 \iff a \diamond b = 1$, *for any* $a, b, c \in X$.

Further, we have

Definition 2.2. *A binary relation "$\leq$" in a generalized pseudo BE-algebra $\boldsymbol{X}$ is defined as follows:*

$$a \leq b \iff a * b = 1 \iff a \diamond b = 1 \; \forall \; a, b \in \boldsymbol{X}.$$

The examples given below shows the existence of the generalized pseudo BE-algebra.

Example 2.3. *(i) Let $\boldsymbol{X} = \{1, x, y, z\}$ be a set with two binary operations "$*$" and "$\diamond$" defined on $\boldsymbol{X}$ in the following tables:*

$*$	1	x	y	z
1	1	x	1	z
x	1	1	1	1
y	1	x	1	z
z	1	1	1	1

$\diamond$	1	x	y	z
1	1	x	1	z
x	1	1	1	1
y	1	x	1	z
z	1	1	1	1

Then it is easy to see that $(\boldsymbol{X}; *, \diamond, 1)$ is a generalized pseudo BE-algebra.

(ii) $\boldsymbol{X} = \{1, 2, 3, 4\}$ be a set with two binary operations "$*$" and "$\diamond$" defined on $\boldsymbol{X}$ in the following tables:

$*$	1	2	3	4
1	1	2	1	4
2	1	1	3	1
3	1	2	1	4
4	1	1	1	1

$\diamond$	1	2	3	4
1	1	2	1	4
2	1	1	3	1
3	1	4	1	4
4	1	1	1	1

It is easy to see that $(\boldsymbol{X}; *, \diamond, 1)$ is a generalized pseudo BE-algebra.

Let us state and prove some basic properties. The basic properties are true for pseudo BE-algebras.

Proposition 2.4. *Let $(\boldsymbol{X}; *, \diamond, 1)$ be a generalized pseudo BE-algebra, then the below properties holds.*
(i) $a * (b \diamond a) = 1$, $a \diamond (b * a) = 1$,
(ii) $a \diamond (b \diamond a) = 1$, $a * (b * a) = 1$,
(iii) $a \diamond ((a \diamond b) * b) = 1$, $a * (a * b) \diamond b) = 1$,
(iv) $a * ((a \diamond b) * b) = 1$, $a \diamond ((a * b) \diamond b) = 1$,
(v) If $a \leq b * c$, then $b \leq a \diamond c$,
(vi) If $a \leq b \diamond c$, then $b \leq a * c$,
(vii) If $a \leq b$, then $a \leq c * b$ & $a \leq c \diamond b$,
(viii) If $a * b = c$, then $b * c = b \diamond c = 1$ & if $a \diamond b = c$, then $b * c = b \diamond c = 1$,
(ix) If $a * b = a$ & $a \neq 1$, then $a \diamond b \neq b$,
(x) If $a * b = b$ & $a \neq 1$, then $a \diamond b \neq a$,
(xi) If $a * b = a$ & $a \diamond b = c$, then $a * c = a \diamond c = 1$ & $a * (b * c) = (a * b) * (a * c)$ $= a \diamond (b * c) = (a \diamond b) * (a \diamond c) = 1$,
(xii) If $a * b = b$ & $a \diamond b = c$, then $a * c = c$ & $a * (b * c) = (a * b) * (a * c) =$ $a * (b \diamond c) = (a * b) \diamond (a * c) = 1$,
(xiii) If $a * b = c$ & $a \diamond b = z$, then $a \diamond c = a * z$ $\forall$ $a, b, c \in \boldsymbol{X}$.

Proof. (i) Let a, $b \in \boldsymbol{X}$, then

$$\begin{aligned} a * (b \diamond a) &= b \diamond (a * a) && (\because \text{ by (iii) property of Definition 2.1}) \\ &= b \diamond 1 && (\because \text{ by (i) property of Definition 2.1}) \\ &= 1 && (\because \text{ by (ii) property of Definition 2.1}) \end{aligned}$$

Also,

$$\begin{aligned} a \diamond (b * a) &= b * (a \diamond a) && (\because \text{ by (iii) property of Definition 2.1}) \\ &= b * 1 && (\because \text{ by (i) property of Definition 2.1}) \\ &= 1 && (\because \text{ by (ii) property of Definition 2.1}) \end{aligned}$$

(ii) Let a, $b \in \boldsymbol{X}$, then we need to show that $a \diamond (b \diamond a) = 1$.
By using (i) we can write $a * (b \diamond a) = 1$

$$\begin{aligned} &\Rightarrow a \leq b \diamond a && (\because \text{ by Definition 2.2}) \\ &\Rightarrow a \diamond (b \diamond a) = 1 && (\because \text{ by Definition 2.2}) \end{aligned}$$

Similarly,

$$\begin{aligned} & a \diamond (b * a) = 1 && (\because \text{ by using (i)}) \\ \Rightarrow\ & a \leq b * a && (\because \text{ by Definition 2.2}) \\ \Rightarrow\ & a * (b * a) = 1 && (\because \text{ by Definition 2.2}) \end{aligned}$$

(iii) Let $a, b \in \boldsymbol{X}$, then

$$\begin{aligned} a \diamond ((a \diamond b) * b) &= (a \diamond b) * (a \diamond b) && (\because \text{ by (iii) property of Definition 2.1}) \\ &= 1 && (\because \text{ by (i) property of Definition 2.1}) \end{aligned}$$

Similarly,

$$\begin{aligned} a * ((a * b) \diamond b) &= (a * b) \diamond (a * b) && (\because \text{ by (iii) property of Definition 2.1}) \\ &= 1 && (\because \text{ by (i) property of Definition 2.1}) \end{aligned}$$

(iv) By using (iv) property of Definition 2.1 in the above result (iii), we get the required result.

(v) Let $a, b, c \in \boldsymbol{X}$ such that $a \leq b * c$, then

$$\begin{aligned} & a \diamond (b * c) = 1 && (\because \text{ by Definition 2.2}) \\ \Rightarrow\ & b * (a \diamond c) = 1 && (\because \text{ by (iii) property of Definition 2.1}) \\ \Rightarrow\ & b \leq a \diamond c. && (\because \text{ by Definition 2.2}) \end{aligned}$$

(vi) Let $a, b, c \in \boldsymbol{X}$ such that $a \leq b \diamond c$, then

$$\begin{aligned} & a * (b \diamond c) = 1 && (\because \text{ by Definition 2.2}) \\ \Rightarrow\ & b \diamond (a * c) = 1 && (\because \text{ by (iii) property of Definition 2.1}) \\ \Rightarrow\ & b \leq a * c. && (\because \text{ by Definition 2.2}) \end{aligned}$$

(vii) Since $a \leq b \Rightarrow a * b = 1$ and $a \diamond b = 1$. Now

$$\begin{aligned} a \diamond (c * b) &= c * (a \diamond b) && (\because \text{ by (iii) property of Definition 2.1}) \\ &= c * 1 && (\because \text{ given}) \\ &= 1 && (\because \text{ by (ii) property of Definition 2.1}) \end{aligned}$$

So, $a \diamond (c * b) = 1 \Rightarrow a \leq c * b$.

Similarly,

$$\begin{aligned} a * (c \diamond b) &= c \diamond (a * b) && (\because \text{ by (iii) property of Definition 2.1}) \\ &= c \diamond 1 && (\because \text{ given}) \\ &= 1 && (\because \text{ by (ii) property of Definition 2.1}) \end{aligned}$$

So, $a * (c \diamond b) = 1 \Rightarrow a \leq c \diamond b$.

(viii) Let $a, b, c \in \boldsymbol{X}$ such that $a * b = c$, then

$$\begin{aligned} b \diamond c &= b \diamond (a * b) \\ &= a * (b \diamond b) && (\because \text{ by (iii) property of Definition 2.1}) \\ &= a * 1 && (\because \text{ by (i) property of Definition 2.1}) \\ &= 1 && (\because \text{ by (ii) property of Definition 2.1}) \end{aligned}$$

So, $b \diamond c = 1 \Rightarrow b * c = 1$ $\quad(\because$ by (iv) property of Definition 2.1)

Similarly, If $a \diamond b = c$, then

$$\begin{aligned} b * c &= b * (a \diamond b) \\ &= a \diamond (b * b) && (\because \text{ by (iii) property of Definition 2.1}) \\ &= a \diamond 1 && (\because \text{ by (i) property of Definition 2.1}) \\ &= 1 && (\because \text{ by (ii) property of Definition 2.1}) \end{aligned}$$

So, $b*c=1 \Rightarrow b\diamond c=1$ $\quad$ ($\because$ by (iv) property of Definition 2.1)
(ix) Suppose that $a*b=a$, & $a\neq 1$. Let $a\diamond b=b$, then $a*(a\diamond b)=a*b=a$. Now

$$\begin{aligned} a\diamond(a*b) &= a\diamond a && (\because \text{ given})\\ &= 1 && (\because \text{ by (iii) property of Definition 2.1}) \end{aligned}$$

So, we have $a=1$, which is impossible. Hence, $a\diamond b\neq b$.
(x) Suppose that $a*b=b$ & $a\neq 1$. Let $a\diamond b=a$, then we have $a*(a\diamond b)=a*a=1$. Now

$$\begin{aligned} a\diamond(a*b) &= a\diamond b && (\because \text{ given})\\ &= a && (\because \text{ by (iii) property of Definition 2.1}) \end{aligned}$$

So, we have $a=1$, which is impossible. Hence, $a\diamond b\neq a$.
(xi) Let us suppose that $a*b=a$ & $a\diamond b=c$, then $a*(a\diamond b)=a*c$ and $a\diamond(a*b)=a\diamond a=1$. So by (iii) property of Definition 2.1, we have $a*c=1$. Since $a\diamond b=c$, by (viii), $b*c=b\diamond c=1$. Now using (iv) property of Definition 2.1, $a*c=a\diamond c=1$. Now $a*(b*c)=a*1=1$ and $(a*b)*(a*c)=a*1=1$. So, by (iv) property of Definition 2.1, $a\diamond(b*c)=1$. Furthermore, $(a\diamond b)*(a\diamond c)=c*1=1$.
(xii) Let us suppose that $a*b=b$ and $a\diamond b=c$ then, by (viii), $b*c=b\diamond c=1$. Now $a*(a\diamond b)=a*c$ & $a\diamond(a*b)=a\diamond b=c$. By (iii) property of Definition 2.1, we have $a*c=c$. Now $a*(b\diamond c)=a*1=1$ & $a*(b*c)=a*1=1$. Now

$$\begin{aligned} (a*b)*(a*c) &= b*c && (\because \text{ given})\\ &= 1 && (\because \text{ by (viii)}) \end{aligned}$$

And so by (iv) property of Definition 2.1, $(a*b)\diamond(a*c)=1$.
(xiii) Let $a*b=c$ & $a\diamond b=z$, then

$$\begin{aligned} a\diamond c &= a\diamond(a*b)\\ &= a*(a\diamond b) && (\because \text{ by (iii) property of Definition 2.1})\\ &= a*z. \end{aligned}$$

□

3. Generalized Pseudo Sub-algebras

In this section, we define generalized pseudo sub-algebra and construct some examples to show the existence of generalized pseudo sub-algebra. We then characterize it by some results.

Definition 3.1. *Assume that $(\boldsymbol{X};*,\diamond,1)$ is a generalized pseudo BE-algebra and $\emptyset\neq \boldsymbol{S}\subseteq\boldsymbol{X}$. Then, $\boldsymbol{S}$ is known to be a generalized pseudo sub-algebra, if it satisfies $x*y\in\boldsymbol{S}$ and $x\diamond y\in\boldsymbol{S}, \forall\ x,y\in\boldsymbol{S}$.*

In order to show the existence of generalized pseudo sub-algebra, we give some examples.

Example 3.2. *In Example 2.3 (i), $\boldsymbol{S}=\{1,x\}$ is a generalized pseudo sub-algebra of $\boldsymbol{X}$. Similarly in Example 2.3 (ii), $\boldsymbol{S}=\{1,3,4\}$ is a generalized pseudo sub-algebra of $\boldsymbol{X}$.*

Let us state and prove some results. The results are true in case of pseudo BE-algebras.

Theorem 3.3. *Let $\{\boldsymbol{S}_i\}_{i\in I}$ be an arbitrary collection of generalized pseudo sub-algebras of $\boldsymbol{X}$. Then $\bigcap_{i\in I}\boldsymbol{S}_i$ is a generalized pseudo sub-algebra of $\boldsymbol{X}$ as well.*

Proof. It is clear that $\emptyset \neq \bigcap_{i\in I}\boldsymbol{S}_i$. Now let us take $a, b \in \bigcap_{i\in \boldsymbol{I}}\boldsymbol{S}_i \Rightarrow a, b \in \boldsymbol{S}_i \ \forall\ i \in \boldsymbol{I}$. As each $\boldsymbol{S}_i$ is a generalized pseudo sub-algebra of $\boldsymbol{X}$, so it follows that $a * b \in \boldsymbol{S}_i$ and $a \diamond b \in \boldsymbol{S}_i \ \forall\ i \in \boldsymbol{I}$. Thus, it follows that $a * b \in \bigcap_{i\in \boldsymbol{I}}\boldsymbol{S}_i$ and $a \diamond b \in \bigcap_{i\in \boldsymbol{I}}\boldsymbol{S}_i$. Therefore, $\bigcap_{i\in \boldsymbol{I}}\boldsymbol{S}_i$ is a generalized pseudo sub-algebra of $\boldsymbol{X}$. □

Further, we have the following remark about the union of generalized pseudo sub-algebras.

Remark 3.4. *The union of two generalized pseudo sub-algebras is not necessary to be generalized pseudo sub-algebra.*

Example 3.5. *In Example 2.3 (ii), let $\boldsymbol{S}_1 = \{1, 2\}$ and $\boldsymbol{S}_2 = \{1, 3\}$, then $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are generalized pseudo sub-algebras but $\boldsymbol{S}_1 \cup \boldsymbol{S}_2 = \{1, 2, 3\}$ is not a generalized pseudo sub-algebra because $3 \diamond 2 = 4 \notin \boldsymbol{S}_1 \cup \boldsymbol{S}_2$.*

4. Generalized Pseudo Filters

In this section, we define generalized pseudo filters. We then construct some examples to show the existence of generalized pseudo filters. We also characterize them by some results.

Definition 4.1. *Let us suppose that $(\boldsymbol{X}; *, \diamond, 1)$ is a generalized pseudo BE-algebra and let us assume that $\emptyset \neq \boldsymbol{F} \subseteq \boldsymbol{X}$, then $\boldsymbol{F}$ is said to be a generalized pseudo filter of $\boldsymbol{X}$, if it satisfies the below properties:*
($\boldsymbol{C}_1$) $1 \in \boldsymbol{F}$,
($\boldsymbol{C}_2$) $x \in \boldsymbol{F}$ *and* $x * y \in \boldsymbol{F} \Rightarrow y \in \boldsymbol{F}$.

It should be noted that a generalized pseudo filter is said to be proper if $\boldsymbol{F} \neq \boldsymbol{X}$. Also note that a proper generalized pseudo filter is maximal if it is not subset of any other proper generalized pseudo filter.

Let us give an example to show the existence of a generalized pseudo filter.

Example 4.2. *In Example 2.3 (i), $\boldsymbol{F} = \{1, x\}$ is a generalized pseudo filter of $\boldsymbol{X}$. Similarly, in Example 2.3 (ii), $\boldsymbol{F} = \{1, 2, 3\}$ is a generalized pseudo filter of $\boldsymbol{X}$.*

Let us characterize generalized pseudo filters by some results. The results are true for pseudo filters. Here, we prove them for generalized pseudo filters.

Proposition 4.3. *Let us suppose that* $\emptyset \neq \boldsymbol{F} \subseteq (\boldsymbol{X}; *, \diamond, 1)$ *and* $1 \in \boldsymbol{F}$, *then* $\boldsymbol{F}$ *is a generalized pseudo filter* $\Leftrightarrow \forall\ c, d \in \boldsymbol{X}$ *such that* $c \in \boldsymbol{F}$ & $c \diamond d \in \boldsymbol{F} \Rightarrow d \in \boldsymbol{F}$.

Proof. Let $c \in \boldsymbol{F}$ & $c \diamond d \in \boldsymbol{F}$, then by using Proposition 2.4 (iv), $c * ((c \diamond d) * d) = 1$. As $1 \in \boldsymbol{F}$, so $c * ((c \diamond d) * d) \in \boldsymbol{F}$. Now by supposition, $(c \diamond d) * d) \in \boldsymbol{F}$. Now as $c \diamond d \in \boldsymbol{F}$ and $(c \diamond d) * d \in \boldsymbol{F}$, so again by supposition, $d \in \boldsymbol{F}$.

Conversely, assume that $c \in \boldsymbol{F}$ and $c * d \in \boldsymbol{F}$. Then by using Proposition 2.4 (iv), $c \diamond ((c * d) \diamond d) = 1$. As $1 \in \boldsymbol{F}$, so $c \diamond ((c * d) \diamond d) \in \boldsymbol{F}$. Thus $(c * d) \diamond d \in \boldsymbol{F}$ and so $d \in \boldsymbol{F}$. □

Proposition 4.4. *Let* $(\boldsymbol{X}; *, \diamond, 1)$ *be a generalized pseudo BE-algebra &* $\boldsymbol{F}$ *a generalized pseudo filter of* $\boldsymbol{X}$ *and* $c, d \in \boldsymbol{X} \ni c \leq d$ *and* $c \in \boldsymbol{F}$, *then* $d \in \boldsymbol{F}$.

Proof. Assume that $c \leq d$ and $c \in \boldsymbol{F}$. Then $c * d = 1$. Now $1 \in \boldsymbol{F} \Rightarrow c * d \in \boldsymbol{F}$ but $c \in \boldsymbol{F}$ so we have that $d \in \boldsymbol{F}$. □

Theorem 4.5. *Let* $\boldsymbol{F}$ *be a generalized pseudo filter of a generalized pseudo BE-algebra* $(\boldsymbol{X}, *, \diamond, 1)$. *Then* $\boldsymbol{F}$ *must be a generalized pseudo sub-algebra of* $\boldsymbol{X}$.

Proof. Given that $\boldsymbol{F}$ is a generalized pseudo filter of $(\boldsymbol{X}; *, \diamond, 1)$ and let us assume that $p, r \in \boldsymbol{F}$. Now, by Proposition 2.4 (ii), $p * (r * p) = 1$ and this implies that $p \leq r * p$. From here, by Proposition 4.4, it follows that $r * p \in \boldsymbol{F}$. In the same way, by Proposition 2.4 (ii), we have that $p \diamond (r \diamond p) = 1 \Rightarrow p \leq r \diamond p$. From here by Proposition 4.4, $r \diamond p \in \boldsymbol{F}$ and hence $\boldsymbol{F}$ is a generalized pseudo sub-algebra. □

Here, we note the following remark.

Remark 4.6. *Every generalized pseudo sub-algebra is not necessary to be generalized pseudo filter.*

Example 4.7. *In Example 2.3 (i),* $\boldsymbol{S} = \{1, z\}$ *is a generalized pseudo sub-algebra but not a generalized pseudo filter. Similarly,in Example 2.3 (ii),* $\boldsymbol{S} = \{1, 3, 4\}$ *is a generalized pseudo sub-algebra but not a generalized pseudo filter.*

Further, we have

Theorem 4.8. *Let* $\{\boldsymbol{F}_i\}_{i \in I}$ *be a collection of generalized pseudo filters of* $\boldsymbol{X}$. *Then* $\bigcap_{i \in I} \boldsymbol{F}_i$ *is also a generalized pseudo filter of* $(\boldsymbol{X}; *, \diamond, 1)$.

Proof. As $1 \in \boldsymbol{F}_i\ \forall\ i \in \boldsymbol{I}$, it follows that $1 \in \bigcap_{i \in I} \boldsymbol{F}_i$. Thus the first condition is verified.
Let $x, y \in \boldsymbol{X}$ be $\ni x \in \bigcap_{i \in I} \boldsymbol{F}_i$ and $x * y \in \bigcap_{i \in I} \boldsymbol{F}_i$ this implies that $x \in \boldsymbol{F}_i$ $\forall\ i \in \boldsymbol{I}$ and $x * y \in \boldsymbol{F}_i\ \forall\ i \in \boldsymbol{I}$. As each $\boldsymbol{F}_i$ is a generalized pseudo filter so we have $y \in \boldsymbol{F}_i\ \forall\ i \in \boldsymbol{I} \Rightarrow y \in \bigcap_{i \in I} \boldsymbol{F}_i$. Hence $\bigcap_{i \in I} \boldsymbol{F}_i$ is a generalized pseudo filter of $\boldsymbol{X}$. □

We now have the following remark.

Remark 4.9. *The union of two generalized pseudo filters is not necessary to be a generalized pseudo filter.*

Example 4.10. *Let $\boldsymbol{X} = \{1, 2, 3, 4, 5\}$ and let "$*$" and "$\diamond$" be two binary operations defined on $\boldsymbol{X}$ in the following tables:*

$*$	1	2	3	4	5
1	1	1	1	1	1
2	1	1	3	4	5
3	1	2	1	4	5
4	1	2	3	1	3
5	1	1	1	1	1

$\diamond$	1	2	3	4	5
1	1	1	1	1	1
2	1	1	3	4	1
3	1	2	1	4	4
4	1	2	3	1	5
5	1	1	1	1	1

Then it is easy to see that $(\boldsymbol{X}; *, \diamond, 1)$ is a generalized pseudo BE-algebra. Choose $\boldsymbol{F}_1 = \{1, 3\}$ and $\boldsymbol{F}_2 = \{1, 4\}$ then one can easily see that $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are generalized pseudo filters of $\boldsymbol{X}$ but $\boldsymbol{F}_1 \cup \boldsymbol{F}_2 = \{1, 3, 4\}$ is not a generalized pseudo filter of $\boldsymbol{X}$ because $4 * 5 = 3 \in \boldsymbol{F}_1 \cup \boldsymbol{F}_2$ and $4 \in \boldsymbol{F}_1 \cup \boldsymbol{F}_2$ but $5 \notin \boldsymbol{F}_1 \cup \boldsymbol{F}_2$.

We are now going to state and prove some more properties of generalized pseudo filters.

Proposition 4.11. *Let us assume that $\boldsymbol{X}$ is a generalized pseudo BE-algebra and $\boldsymbol{F}$ a generalized pseudo filter of $\boldsymbol{X}$. Then the below holds.*
(i) If $p, q, r \in \boldsymbol{X} \ni p, q \in \boldsymbol{F}$ & $p \leq q * r$ then $r \in \boldsymbol{F}$.
(ii) If $p, q, r \in \boldsymbol{X} \ni p, q \in \boldsymbol{F}$ & $p \leq q \diamond r$ then $r \in \boldsymbol{F}$.

Proof. (i) Given that $\boldsymbol{F}$ is a generalized pseudo filter of $\boldsymbol{X}$ and $p, q, r \in \boldsymbol{X} \ni p, q \in \boldsymbol{F}$ & $p \leq q * r$, then by Definition 2.2, $p \diamond (q * r) = 1$ and $1 \in \boldsymbol{F}$ which implies that $p \diamond (q * r) \in \boldsymbol{F}$. From here we have that $q * r \in \boldsymbol{F}$. Now since $q \in \boldsymbol{F}$ & $q * r \in \boldsymbol{F}$ so we have $r \in \boldsymbol{F}$.
(ii) Given that $p, q, r \in \boldsymbol{X} \ni p, q \in \boldsymbol{F}$ & $p \leq q \diamond r$, then by Definition 2.2, $p * (q \diamond r) = 1$ and $1 \in \boldsymbol{F}$ which implies that $p * (q \diamond r) \in \boldsymbol{F}$. From here we have that $q \diamond r \in \boldsymbol{F}$. Now since $q \in \boldsymbol{F}$ & $q \diamond r \in \boldsymbol{F}$ so we have $r \in \boldsymbol{F}$. □

Proposition 4.12. *Let us suppose that $(\boldsymbol{X}; *, \diamond, 1)$ is a generalized pseudo BE-algebra and $\emptyset \neq \boldsymbol{F} \subseteq \boldsymbol{X}$ & $1 \in \boldsymbol{F}$. Then $\boldsymbol{F}$ is a generalized pseudo filter of $\boldsymbol{X} \Leftrightarrow b \leq c * a \Rightarrow a \in \boldsymbol{F}, \forall\, b, c \in \boldsymbol{F}$.*

Proof. Given that $\boldsymbol{X}$ is a generalized pseudo BE-algebra. Let us suppose that $\boldsymbol{F}$ is a generalized pseudo filter of $\boldsymbol{X} \ni b, c \in \boldsymbol{F}$ & $b \leq c * a$. Then by Definition 2.2, $b \diamond (c * a) = 1$ and $1 \in \boldsymbol{F} \Rightarrow b \diamond (c * a) \in \boldsymbol{F}$. Now as $b \in \boldsymbol{F}$ & $b \diamond (c * a) \in \boldsymbol{F} \Rightarrow c * a \in \boldsymbol{F}$. Since $c \in \boldsymbol{F}$ and $c * a \in \boldsymbol{F} \Rightarrow a \in \boldsymbol{F}$.

Conversely, given that $1 \in \boldsymbol{F}$. Let us suppose that $b, b \diamond c \in \boldsymbol{F}$ & $c \in \boldsymbol{X}$. Now by Proposition 2.4 (iii), we have $b \diamond ((b \diamond c) * c) = 1$. So by using Definition 2.2, $b \leq (b \diamond c) * c$. As $b, b \diamond c \in \boldsymbol{F}$, we get that $c \in \boldsymbol{F}$. From Proposition 4.3, we have that $\boldsymbol{F}$ is a generalized pseudo filter. □

5. Terminal Section of an Element

In this section, we define terminal section of an element of a generalized pseudo BE-algebra and give an example in order to show the existence. We then explore some properties of the said notion which are true in case of pseudo BE-algebra.

Definition 5.1. *Let x be an element of a generalized pseudo BE-algebra $(\boldsymbol{X};*,\diamond,1)$, then the terminal section of x is represented by $\boldsymbol{T}(x)$ and is defined as:*

$$\boldsymbol{T}(x)=\{y\in \boldsymbol{X}: x\leq y\}=\{y\in\ \boldsymbol{X}: x*y=1\}=\{y\in\ \boldsymbol{X}: x\diamond y=1\}.$$

Since $x*1=1\Rightarrow 1\in \boldsymbol{T}(x)$. Also $x*x=1\Rightarrow x\in \boldsymbol{T}(x)$. Hence $\boldsymbol{T}(x)$ is always a non-empty set. The example given below shows the existence of the above notion.

Example 5.2. *Consider the Example 2.3 (i), $\boldsymbol{T}(1)=\{1,y\},\boldsymbol{T}(x)=\{1,x,y,z\}$ and $\boldsymbol{T}(y)=\{1,y\}$. Similarly, in Example 2.3 (ii), $\boldsymbol{T}(1)=\{1,3\},\boldsymbol{T}(2)=\{1,2,4\}$ and $\boldsymbol{T}(4)=\{1,2,3,4\}$.*

Furthermore, we characterize terminal section of an element by the following properties. These properties hold in case of pseudo BE-algebra. Here, we prove them for generalized pseudo BE-algebra.

Theorem 5.3. *Let $(\boldsymbol{X};*,\diamond,1)$ be a generalized pseudo BE-algebra and $z\in\boldsymbol{X}$. Then the terminal section $\boldsymbol{T}(z)$ is a generalized pseudo filter $\Leftrightarrow$ below properties are satisfied.*
(i) $z\leq a*b$ & $z\leq a\Rightarrow\ z\leq b\ \forall\ a,b,z\in\boldsymbol{X}$;
(ii) $z\leq a\diamond b$ & $z\leq a\Rightarrow z\leq b\ \forall\ a,b,z\in\boldsymbol{X}$.

Proof. Let $\boldsymbol{T}(z)$ be a generalized pseudo filter of $\boldsymbol{X}$ and let $a,b,z\in\boldsymbol{X}\ni z\ \leq a*b$ & $z\leq a$ then $a*b\in\boldsymbol{T}(z)$ & $a\in\boldsymbol{T}(z)$. By assumption, $\boldsymbol{T}(z)$ is a generalized pseudo filter so, $b\in\boldsymbol{T}(z)$ which implies that $z\leq b$. In the same way, let $a,b,z\in\boldsymbol{X}\ni z\leq a\diamond b$ & $z\leq a$ then $a\diamond b\in\boldsymbol{T}(z)$ & $a\in\boldsymbol{T}(z)$. By assumption, $\boldsymbol{T}(z)$ is a generalized pseudo filter so we have, $b\in\boldsymbol{T}(z)\Rightarrow z\leq b$.

Conversely, let us take $\boldsymbol{T}(z)$, for $z\in\boldsymbol{X}$. Clearly $1\in\boldsymbol{T}(z)$. Now suppose that $a*b\in\boldsymbol{T}(z)$ and $a\in\boldsymbol{T}(z)$, i.e., in other words, $z\leq a*b$ & $z\leq a$. Now by using the hypothesis, $z\leq b$, i.e., $b\in\boldsymbol{T}(z)$. So we have that $\boldsymbol{T}(z)$ is a generalized pseudo filter of $\boldsymbol{X}$. □

Theorem 5.4. *Assume that $(\boldsymbol{X};*,\diamond,1)$ is a generalized pseudo BE-algebra and $\boldsymbol{S}$ is a generalized pseudo sub-algebra of $\boldsymbol{X}$. Then $\boldsymbol{S}$ is a generalized pseudo filter of $\boldsymbol{X}\iff p\in\boldsymbol{S},q\in\boldsymbol{X}\backslash\boldsymbol{S}$ implies that $p*q\in\boldsymbol{X}\backslash\boldsymbol{S}$ & $p\diamond q\in\boldsymbol{X}\backslash\boldsymbol{S}$.*

Proof. Let $\boldsymbol{S}$ be a generalized pseudo filter of $\boldsymbol{X}$ & $p,q\in\boldsymbol{X},\ni p\in\boldsymbol{S},q\in\boldsymbol{X}\backslash\boldsymbol{S}$. If $p*q\notin\ \boldsymbol{X}\backslash\boldsymbol{S}$, then $p*q\in\boldsymbol{S}$, i.e., $q\in\boldsymbol{S}$, which is impossible. Therefore, $p*q\in\boldsymbol{X}\backslash\boldsymbol{S}$. Similarly, if $p\diamond q\notin\ \boldsymbol{X}\backslash\boldsymbol{S}$, then $p\diamond q\in\boldsymbol{S}$, i.e., $q\in\boldsymbol{S}$, which is again impossible. Thus, it follows that $p\diamond q\in\boldsymbol{X}\backslash\boldsymbol{S}$.

Conversely, suppose that $p \in \boldsymbol{S}, q \in \boldsymbol{X} \backslash \boldsymbol{S}$ implies that $p * q \in \boldsymbol{X} \backslash \boldsymbol{S}$ & $p \diamond q \in \boldsymbol{X} \backslash \boldsymbol{S}$ is true. As $\boldsymbol{S}$ is a generalized pseudo sub-algebra, so $1 \in \boldsymbol{S}$. Also, $\forall\, p \in \boldsymbol{S}$, let us suppose that $p * q \in \boldsymbol{S}$. Let $q \notin \boldsymbol{S}$, then by hypothesis, $p * q \in \boldsymbol{X} \backslash \boldsymbol{S}$, which is impossible. Thus, we have $q \in \boldsymbol{S}$ and so $\boldsymbol{S}$ is a generalized pseudo filter of $\boldsymbol{X}$. □

6. Generalized Pseudo Homomorphisms

In this section, we define generalized pseudo homomorphism and give an example. We characterize generalized pseudo BE-algebras by the properties of generalized pseudo homomorphisms.

Definition 6.1. *Let us suppose that* $(\boldsymbol{R}; *_1, \diamond_1, 1_R)$ *and* $(\boldsymbol{S}; *_2, \diamond_2, 1_S)$ *are two generalized pseudo BE-algebras. Then* $\alpha : \boldsymbol{R} \longrightarrow \boldsymbol{S}$ *is known as generalized pseudo homomorphism if* $\alpha(x *_1 y) = \alpha(x) *_2 \alpha(y)$ & $\alpha(x \diamond_1 y) = \alpha(x) \diamond_2 \alpha(y), \forall\, x, y \in \boldsymbol{R}$.

It should to be noted that if a mapping $\alpha : \boldsymbol{R} \longrightarrow \boldsymbol{S}$ is a generalized pseudo homomorphism, then we have

$$\alpha(1_R) = \alpha(1_R *_1 1_R) = \alpha(1_R) \ *_2 \alpha(1_R) = 1_S.$$

Also note that kernel of the homomorphism $\alpha : \boldsymbol{R} \longrightarrow \boldsymbol{S}$ is denoted by Ker α and is defined as:

$$\text{Ker } \alpha = \{x \in \boldsymbol{R} : \alpha(x) = 1_S\}.$$

We are now going to state and prove some properties of generalized pseudo homomorphisms which are true in case of pseudo homomorphisms.

Theorem 6.2. *Let* $(\boldsymbol{X}; *_1, \diamond_1, 1_X)$ *and* $(\boldsymbol{Y}; *_2, \diamond_2, 1_Y)$ *be two generalized pseudo BE-algebras. Assume that* $\alpha : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ *is a generalized pseudo hmomomorphism. Then the following are true.*
(i) For a generalized pseudo filter $\boldsymbol{F}$ *of* $\boldsymbol{Y}$, α^{-1} $(\boldsymbol{F})$ *is a generalized pseudo filter of* $\boldsymbol{X}$.
(ii) If α *is onto and* $\boldsymbol{F}$ *a generalized pseudo filter of* $\boldsymbol{X} \ni$ *Ker* $\alpha \subseteq \boldsymbol{F}$, *then* α $(\boldsymbol{F})$ *is a generalized pseudo filter of* $\boldsymbol{Y}$.

Proof. (i) Suppose that $\boldsymbol{F}$ is a generalized pseudo filter of $\boldsymbol{Y}$. Clearly, $\alpha(1_X) = 1_Y$ and $1_Y \in \boldsymbol{F} \Rightarrow \alpha(1_X) \in \boldsymbol{F} \Rightarrow 1_X \in \alpha^{-1}(\boldsymbol{F})$. Let us assume that $a, a *_1 b \in \alpha^{-1}(\boldsymbol{F})$ then $\alpha(a) \in \boldsymbol{F}$ and $\alpha(a) *_2 \alpha(b) = \alpha(a *_1 b) \in \boldsymbol{F}$. It follows that $\alpha(b) \in \boldsymbol{F}$. As $\alpha(b) \in \boldsymbol{F}$ so we have $b \in \alpha^{-1}(\boldsymbol{F})$. Thus $\alpha^{-1}(\boldsymbol{F})$ is a generalized pseudo filter of $\boldsymbol{X}$.
(ii) As $1_X \in \boldsymbol{F} \Rightarrow \alpha(1_X) \in \alpha(\boldsymbol{F}) \Rightarrow 1_Y \in \alpha(\boldsymbol{F})$. Let us assume that $c, c *_2 q \in \alpha(\boldsymbol{F})$ and $q \in \boldsymbol{Y}$. As α is onto, so $\exists$ some $a \in \boldsymbol{X} \ni \alpha(a) = q$. Now by definition $\alpha(\boldsymbol{F})$ there exist $m, n \in \boldsymbol{F} \ni \alpha(m) = c$ and $\alpha(n) = c *_2 q$, then we have $\alpha(n) = c *_2 q = \alpha(m) *_2 \alpha(a) = \alpha(m *_1 a)$. Moreover, $\alpha(n *_1 (m *_1 a)) = \alpha(n) *_2 \alpha(m *_1 a) = \alpha(n) *_2 \alpha(n) = 1_Y$. Therefore, $n *_1 (m *_1 a) \in Ker\alpha \subseteq \boldsymbol{F}$. Hence $n *_1 (m *_1 a) \in \boldsymbol{F}$. As $n \in \boldsymbol{F}$, it follows that $m *_1 a \in \boldsymbol{F}$, so $a \in \boldsymbol{F}$. Thus

$q = \alpha(a) \in \alpha(\boldsymbol{F})$. Thus, it follows that $\alpha(\boldsymbol{F})$ is a generalized pseudo filter of $\boldsymbol{Y}$.

□

Further, we have the below remark.

Remark 6.3. *Let us assume that* $(\boldsymbol{X}; *_1, \diamond_1, 1_X)$ *and* $(\boldsymbol{Y}; *_2, \diamond_2, 1_Y)$ *are two generalized pseudo BE-algebras. Furthermore, let* $\alpha : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ *be generalized pseudo homomorphism and* $\boldsymbol{F}$ *generalized pseudo filter of* $\boldsymbol{X}$*. Then* $\alpha(\boldsymbol{F})$ *is not necessary to be a generalized pseudo filter of* $\boldsymbol{Y}$*.*

Example 6.4. *Let* $\boldsymbol{X} = \{1, 2, 3, 4\}$ *be a set with two binary operations "*$*_1$*" and "*$\diamond_1$*" defined on* $\boldsymbol{X}$ *in the following tables:*

$*_1$	1	2	3	4
1	1	2	1	4
2	1	1	3	1
3	1	2	1	4
4	1	1	1	1

$\diamond_1$	1	2	3	4
1	1	4	1	4
2	1	1	3	1
3	1	4	1	4
4	1	1	1	1

Then it is easy to see that $(\boldsymbol{X}; *_1, \diamond_1, 1)$ *is a generalized pseudo BE-algebra.*

Now let $\boldsymbol{Y} = \{1, 2, 3, 4\}$ *be a set with two binary operations "*$*_2$*" and "*$\diamond_2$*" defined on* $\boldsymbol{Y}$ *in the following tables:*

$*_2$	1	2	3	4
1	1	2	1	4
2	1	1	3	1
3	1	2	1	4
4	1	1	3	1

$\diamond_2$	1	2	3	4
1	1	4	1	4
2	1	1	3	1
3	1	4	1	4
4	1	1	3	1

Then $(\boldsymbol{Y}; *_2, \diamond_2, 1)$ *is also a generalized pseudo BE-algebra.*

Now define $\alpha : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ *by* $\alpha(1) = 1, \alpha(2) = 2, \alpha(3) = 1$ *and* $\alpha(4) = 4$*, then* α *is a homomorphism. It is easy to see that* $\boldsymbol{X} = \{1, 2, 3, 4\}$ *is a generalized pseudo filter of itself but* $\alpha(\boldsymbol{X}) = \{1, 2, 4\}$ *is not a generalized pseudo filter of* $\boldsymbol{Y}$*, because* $1 * 3 \in \alpha(\boldsymbol{X})$ *but* $3 \notin \alpha(\boldsymbol{X})$*.*

7. Generalized Pseudo Upper Sets in Generalized Pseudo BE-algebras

In this section, we define generalized pseudo upper set in a generalized pseudo BE-algebra and give an example to show the existence. We then explore some properties of generalized pseudo upper sets in a generalized pseudo BE-algebra. The equivalent of these properties can be found in [4].

Definition 7.1. *Let* $(\boldsymbol{X}; *, \diamond, 1)$ *be a generalized pseudo BE-algebra. Let us suppose that* $a, b \in \boldsymbol{X}$*, then the generalized pseudo upper set is denoted by* $\boldsymbol{U}(a, b)$ *and is defined as below:*

$$\boldsymbol{U}(a,b) = \{c \in \boldsymbol{X} : a * (b \diamond c) = 1\}.$$

Here, it should be noted that $1, a, b \in \boldsymbol{U}(a,b)$. In order to understand generalized pseudo upper sets, we give an example.

Example 7.2. *In Example 2.3 (i),* $\boldsymbol{U}(y,z) = \{1, x, y, z\}$. *Similarly, in Example 2.3 (ii),* $\boldsymbol{U}(1,3) = \{1,3\}$ *and* $\boldsymbol{U}(2,3) = \{1,2,3,4\}$.

Let us state and prove some properties. The idea of these properties have come from the paper [4] in which the authors do similar calculations in case of pseudo BE-algebra.

Proposition 7.3. *(i) Let us assume that* $\boldsymbol{U}(p,1)$ *is a generalized pseudo filter of a generalized pseudo BE-algebra* $(\boldsymbol{X}, *, \diamond, 1)$ *such that* $q \in \boldsymbol{U}(p,1)$. *Then we have* $\boldsymbol{U}(p,q) \subseteq \boldsymbol{U}(p,1)$.
(ii) Assume that q is an element of a generalized pseudo BE-algebra $\boldsymbol{X} \ni q * t = 1, \forall\, t \in \boldsymbol{X}$, *then* $\boldsymbol{U}(p,q) = \boldsymbol{X}, \forall\, p, q \in \boldsymbol{X}$.

Proof. (i) Let us suppose $\boldsymbol{U}(p,1)$ is a generalized pseudo filter of $\boldsymbol{X} \ni q \in \boldsymbol{U}(p,1)$ & $t \in \boldsymbol{U}(p,q)$. Then we have $p * (q \diamond t) = 1 \in \boldsymbol{U}(p,1)$ which implies that $p * (q \diamond t) \in \boldsymbol{U}(p,1)$. So by using the definition of a generalized pseudo filter $q \diamond t \in \boldsymbol{U}(p,1)$. As $\boldsymbol{U}(p,1)$ is a generalized pseudo filter & $q \in \boldsymbol{U}(p,1)$ so we get $t \in \boldsymbol{U}(p,q)$.Therefore, $\boldsymbol{U}(p,q) \subseteq \boldsymbol{U}(p,1)$.
(ii) Clearly, $\boldsymbol{U}(p,q) \subseteq \boldsymbol{X}$. Now let us suppose that $t \in \boldsymbol{X}$, then by supposition there is $q \in \boldsymbol{X} \ni q * t = 1$. Thus, we have $1 = p \diamond 1 = p \diamond (q * t)$. Hence, $t \in \boldsymbol{U}(p,q)$ and so $\boldsymbol{X} \subseteq \boldsymbol{U}(p,q)$. Consequently we have $\boldsymbol{X} = \boldsymbol{U}(p,q)$. □

Theorem 7.4. *Let us assume that* $(\boldsymbol{X}; *, \diamond, 1)$ *is a generalized pseudo BE-algebra. Then,* $\boldsymbol{F}$ *is a generalized pseudo filter of* $\boldsymbol{X} \Leftrightarrow \boldsymbol{U}(p,q) \subseteq \boldsymbol{F}, \forall\, p, q \in \boldsymbol{F}$.

Proof. Assume that $p, q \in \boldsymbol{F}$. Now let us take $r \in \boldsymbol{U}(p,q)$, then $p*(q\diamond r) = 1$ and $1 \in \boldsymbol{F}$, So $p * (q \diamond r) \in \boldsymbol{F}$. As $\boldsymbol{F}$ is a generalized pseudo filter of $\boldsymbol{X}$, so according to $(\boldsymbol{C}_2), q \diamond r \in \boldsymbol{F}$. Now by using Proposition 4.3, we have $r \in \boldsymbol{F}$. Therefore, $\boldsymbol{U}(p,q) \subseteq \boldsymbol{F}$.

Conversely, let $\boldsymbol{U}(p,q) \subseteq \boldsymbol{F}\ \forall\, p, q \in \boldsymbol{F}$. As $p*(q\diamond 1) = p*1 = 1$, we get that $1 \in \boldsymbol{U}(p,q) \subseteq \boldsymbol{F}$. Let us assume that $x, x*y \in \boldsymbol{F}$. As $1 = (x*y) \diamond (x*y) = x * ((x*y) \diamond y)$, so from here we have $y \in \boldsymbol{U}(x, x*y) \subseteq \boldsymbol{F}$. Thus, $y \in \boldsymbol{F}$. Therefore, $\boldsymbol{F}$ is a generalized pseudo filter. □

Theorem 7.5. *Let* $\boldsymbol{F}$ *be a generalized pseudo filter of a generalized pseudo BE-algebra* $(\boldsymbol{X}; *, \diamond, 1)$. *Then* $\boldsymbol{F} = \bigcup_{p\in F} \boldsymbol{U}(p,1)$.

Proof. Given that $\boldsymbol{F}$ is a generalized pseudo filter of $\boldsymbol{X}$ and let $p \in \boldsymbol{F}$. Then $p*(1\diamond p) = 1\diamond(p*p) = 1\diamond 1 = 1$, we have $p \in \boldsymbol{U}(p,1)$. Hence, $\boldsymbol{F} \subseteq \bigcup_{p\in F} \boldsymbol{U}(p,1)$.

Conversely, by using Theorem 7.4, we have $\boldsymbol{U}(p,q) \subseteq \boldsymbol{F}\ \forall\, p, q \in \boldsymbol{F}$. Then in particular $\boldsymbol{U}(p,1) \subseteq \boldsymbol{F},\ \forall\ p \in \boldsymbol{F}$, thus $\bigcup_{p\in F} \boldsymbol{U}(p,1) \subseteq \boldsymbol{F}$. Hence $\boldsymbol{F} = \bigcup_{p\in F} \boldsymbol{U}(p,1)$. □

8. Some Properties of Generalized Pseudo BE-algebras by using Congruence Relations

In this section, we define left (resp. right) congruence relation and congruence relation. We give some examples to understand the said concepts. We then prove a result which gives equivalent condition for congruence relations.

Definition 8.1. *Let* $(\boldsymbol{X};*,\diamond,1)$ *be a generalized pseudo BE-algebra. Let us take a relation* μ *on* $\boldsymbol{X}$ *then* μ *is known as left compatible if* $\forall\ p,q,r \in \boldsymbol{X}, (q,r) \in \mu$ *implies that* $(p*q, p*r), (p\diamond q, p\diamond r) \in \mu$.

In the same way, a relation μ *on* $\boldsymbol{X}$ *is called right compatible if* $\forall\ p,q,r \in \boldsymbol{X}, (q,r) \in \mu$ *implies that* $(q*p, r*p), (q\diamond p, r\diamond p) \in \mu$.

A relation μ *on* $\boldsymbol{X}$ *is called compatible if* $\forall\ a,b,x,y \in \boldsymbol{X}, (a,b),(x,y) \in \mu$ *implies that* $(a*x, b*y), (a\diamond x, b\diamond y) \in \mu$.

Note that an equivalence relation μ on $\boldsymbol{X}$ which is left compatible as well is called left congruence relation. Similarly, an equivalence relation μ on $\boldsymbol{X}$ which is right compatible as well is called right congruence relation. An equivalence relation μ on $\boldsymbol{X}$ which is compatible as well is called congruence relation.

Let us give some examples to show the existence of the above notion.

Example 8.2. *(i) Let* $\boldsymbol{X}$ *be a generalized pseudo BE-algebra. Let* $\mu_1 = \{(x,y) : x = y\}$ *and* $\mu_2 = \boldsymbol{X}\times\boldsymbol{X}$. *Then* μ_1 *and* μ_2 *are trivially left congruence relations, right congruence relations and congruence relations.*
(ii) Let $\boldsymbol{X} = \{1,2,3,4\}$ *be a set with two binary operations "$*$" and "$\diamond$" defined on* $\boldsymbol{X}$ *in the following tables:*

$*$	1	2	3	4
1	1	2	1	4
2	1	1	3	1
3	1	2	1	4
4	1	1	1	1

$\diamond$	1	2	3	4
1	1	2	1	4
2	1	1	3	1
3	1	4	1	4
4	1	1	1	1

Then it is easy to see that $(\boldsymbol{X};*,\diamond,1)$ *is a generalized pseudo BE-algebra and one can easily check that*

$$\mu = \{(1,1),(2,2),(3,3),(4,4),(1,3),(3,1)\}$$

is a congruence relation on $\boldsymbol{X}$.

Let us state and prove some properties. These properties are true in case of semigroups and we prove them for generalized pseudo BE-algebra. The equivalent can be found in [8].

Lemma 8.3. *Assume that* $(\boldsymbol{X};*,\diamond,1)$ *is a generalized pseudo BE-algebra and* $\emptyset \neq \mu \subseteq \boldsymbol{X}\times\boldsymbol{X}$. *Then (i) and (ii) are equivalent.*
(i) μ *is a congruence relation.*
(ii) μ *is left as well as right congruence relation.*

Proof. (i) $\Rightarrow$ (ii) Assume that μ is a congruence relation on the generalized pseudo BE-algebra $\boldsymbol{X}$. Let $x, p, q \in \boldsymbol{Y}$ be $\ni (p, q) \in \mu$. Now $(x, x) \in \mu \; \forall \; x \in \boldsymbol{X}$ because μ is reflexive. As μ is compatible so we have $(x*p, x*q), (x\diamond p, x\diamond q) \in \mu$. Thus μ is left compatible and is left congruence relation.

Similarly, let $x, p, q \in \boldsymbol{X}$ be such that $(p, q) \in \mu$. Now $(x, x) \in \mu \; \forall \; x \in \boldsymbol{X}$ because μ is reflexive. As μ is compatible so it follows that $(p * x, q * x)$ and $(p \diamond x, q \diamond x) \in \mu$. Thus, μ is right compatible and is right congruence relation. (ii) $\Rightarrow$ (i) Suppose μ is both right and left congruence relation. Let $x, y, p, q \in \boldsymbol{X} be \ni (x, y), (p, q) \in \mu$. As μ is right compatible, so $(x*p, y*p), (x\diamond p, y\diamond p) \in \mu$. Again as μ is left compatible, so we have $(y * p, y * q), (y \diamond p, y \diamond q) \in \mu$. By transitivity, it follows that $(x * p, y * q), (x \diamond p, y \diamond q) \in \mu$. Thus, μ is compatible, so it is a congruence relation. $\square$

Theorem 8.4. *Let $(\boldsymbol{X}; *_1, \diamond_1, 1_{\boldsymbol{X}})$ and $(\boldsymbol{Y}; *_2, \diamond_2, 1_{\boldsymbol{Y}})$ be a generalized pseudo BE-algebras and let $\alpha : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ be a generalized pseudo homomorphism from $\boldsymbol{X}$ to $\boldsymbol{Y}$, then μ defines a congruence relation on $\boldsymbol{X}$.*

Proof. Let $\mu = \{(a, b) \in \boldsymbol{X} \times \boldsymbol{X} : \alpha(a) = \alpha(b)\}$.

***Reflexive* :**
As $\alpha(a) = \alpha(a) \forall \; a \in \boldsymbol{X}$, so we have $(a, a) \in \mu \; \forall \; a \in \boldsymbol{X}$. Thus, μ is reflexive.

***Symmetric* :**
Let $a, b \in \boldsymbol{X}$ be such that $(a, b) \in \mu \Rightarrow \alpha(a) = \alpha(b) \Rightarrow \alpha(b) = \alpha(a) \Rightarrow (b, a) \in \mu$. Thus, μ is symmetric.

***Transitive* :**
Let $a, b, c \in \boldsymbol{X}$ be such that $(a, b), (b, c) \in \mu$, then $\alpha(a) = \alpha(b)$ and $\alpha(b) = \alpha(c)$. Thus, $\alpha(a) = \alpha(c)$, it follows that $(a, c) \in \mu$. Thus, μ is transitive. It follows that μ is an equivalence relation.

***Compatibility* :**
Let $a, b, c, d \in \boldsymbol{X}$ such that $(a, b), (c, d) \in \mu$, then $\alpha(a) = \alpha(b)$ and $\alpha(c) = \alpha(d)$. Now

$$\begin{aligned} \alpha(a *_1 c) &= \alpha(a) *_2 \alpha(c) && (\because \alpha \text{ is homomorphism}) \\ &= \alpha(b) *_2 \alpha(d) \\ &= \alpha(b *_1 d) && (\because \alpha \text{ is homomorphism}) \end{aligned}$$

Thus, $(a *_1 c, b *_1 d) \in \mu$.
Similarly,

$$\begin{aligned} \alpha(a \diamond_1 c) &= \alpha(a) \diamond_2 \alpha(c) && (\because \alpha \text{ is homomorphism}) \\ &= \alpha(b) \diamond_2 \alpha(d) \\ &= \alpha(b \diamond_1 d) && (\because \alpha \text{ is homomorphism}) \end{aligned}$$

Thus, $(a \diamond_1 c, b \diamond_1 d) \in \mu$.
It follows that μ is compatible and so is a congruence relation on $\boldsymbol{X}$. $\square$

Note the above relation is called Kernal of α and is denoted by $\boldsymbol{Ker\alpha}$. See the source [14] for the following result which will be used later in the construction of quotient or factor generalized pseudo BE-algebra.

Lemma 8.5. *Let us suppose that $\boldsymbol{X}$ is a set and μ is an equivalence relation on the set $\boldsymbol{X}$. Moreover, assume that $x, y \in \boldsymbol{X}$ and μ_x, μ_y are the corresponding equivalence classes. Then*

$$\mu_x = \mu_y \Leftrightarrow (x, y) \in \mu.$$

Further, we define a congruence class.

Definition 8.6. *Assume that μ is a congruence relation on a generalized pseudo BE-algebra $(\boldsymbol{X}; *_1, \diamond_1, 1)$ as discussed in Theorem 8.4. Then*

$$\mu_p = \{q \in \boldsymbol{X} : (p, q) \in \mu\}$$

is called a congruence class corresponding to the element $p \in \boldsymbol{X}$.

Let us suppose that $\boldsymbol{X}/\mu = \{\mu_p : p \in \boldsymbol{X}\}$. Our aim is to show that $\boldsymbol{X}/\mu$ is a generalized pseudo BE-algebra. For this we define "$\boldsymbol{*_1}$" and "$\boldsymbol{\diamond_1}$" in the following way:

$$\mu_p *_1 \mu_q = \mu_{p\ *_1 q} \text{ and } \mu_p \diamond_1 \mu_q = \mu_{p\ \diamond_1 q}\ \forall\ \mu_p, \mu_q \in \boldsymbol{X}/\mu.$$

Let us first show that the above binary operations are well-defined. Choose $\mu_p, \mu_u \in \boldsymbol{X}/\mu$ and $\mu_q, \mu_v \in \boldsymbol{X}/\mu$ so that $\mu_p = \mu_q$ and $\mu_u = \mu_v$
$\Rightarrow (p, q) \in \mu$ and $(u, v) \in \mu$ ($\because$ by Lemma 8.5)
$\Rightarrow (p *_1 u, q *_1 v), (p \diamond_1 u, q \diamond_1 v) \in \mu$ ($\because$ μ is compatible)
$\Rightarrow \mu_{p\ *_1 u} = \mu_{q\ *_1 v}$ and $\mu_{p\ \diamond_1 u} = \mu_{q\ \diamond_1 v}$ ($\because$ by Lemma 8.5)
$\Rightarrow \mu_p *_1 \mu_u = \mu_q *_1 \mu_v$ and $\mu_p \diamond_1 \mu_u = \mu_q \diamond_1 \mu_v$.
Further, we need to verify the following properties for all μ_a, μ_b and $\mu_c \in \boldsymbol{X}/\mu$.
(i) $\mu_a *_1 \mu_a = \mu_{a *_1 a} = \mu_1$ ($\because\ a *_1\ a = 1$)
and
$\mu_a \diamond_1 \mu_a = \mu_{a \diamond_1 a} = \mu_1$ ($\because\ a \diamond_1 a = 1$)
(ii) $\mu_a *_1 \mu_1 = \mu_{a *_1 1} = \mu_1$ ($\because a *_1 1 = 1$)
and
$\mu_a \diamond_1 \mu_1 = \mu_{a \diamond_1 1} = \mu_1$ ($\because a \diamond_1 1 = 1$)
(iii) $\mu_a *_1 (\mu_b \diamond_1 \mu_c) = \mu_a *_1 (\mu_{b \diamond_1 c}) = \mu_{a *_1 (b \diamond_1 c)} = \mu_{b \diamond_1 (a *_1 c)} = \mu_b \diamond_1 (\mu_{a *_1 c}) = \mu_b \diamond_1 (\mu_a *_1 \mu_c)$.
(iv) Let $\mu_a *_1 \mu_b = \mu_1$. Then $\mu_{a *_1 b} = \mu_1 \Rightarrow (a *_1 b, 1) \in \mu \Rightarrow (a \diamond_1 b, 1) \in \mu \Rightarrow \mu_{a\ \diamond_1 b} = \mu_1 \Rightarrow \mu_a \diamond_1 \mu_b = \mu_1$.
Similarly, let $\mu_a \diamond_1 \mu_b = \mu_1 \Rightarrow \mu_{a\ \diamond_1 b} = \mu_1 \Rightarrow (a \diamond_1 b, 1) \in \mu \Rightarrow (a *_1 b, 1) \in \mu \Rightarrow \mu_{a\ *_1 b} = \mu_1 \Rightarrow \mu_a *_1 \mu_b = \mu_1$.
It follows that, $\boldsymbol{X}/\mu$ is a generalized pseudo BE-algebra called quotient or factor generalized pseudo BE-algebra.

Further, we have the result given below which is based on congruence relations. This result is true for semigroups and the idea of this result has come from the book [8].

Theorem 8.7. *Let us suppose that* $(\boldsymbol{X}; *_1, \diamond_1, 1)$ *is a generalized pseudo BE-algebra and let* μ *be a congruence relation on* $\boldsymbol{X}$ *as discussed in Theorem 8.4, then* $\boldsymbol{X}/\mu$ *is a generalized pseudo BE-algebra under the following binary operations:*

$$\mu_a *_1 \mu_b = \mu_{a\ *_1 b} \text{ and } \mu_a \diamond_1 \mu_b = \mu_{a\ \diamond_1 b}$$

$\forall\ \mu_a, \mu_b \in \boldsymbol{X}/\mu$. *The map* $\mu^{\#} : \boldsymbol{X} \to \boldsymbol{X}/\mu$ *defined by* $\mu^{\#}(a) = \mu_a\ \forall\ a \in \boldsymbol{X}$ *is an epimorphism. Now let* $\theta : \boldsymbol{X} \to \boldsymbol{Y}$ *be a generalized pseudo homomorphism where* $(\boldsymbol{X}; *_1, \diamond_1, 1)$ *and* $(\boldsymbol{Y}; *_2, \diamond_2, 1)$ *are two generalized pseudo BE-algebras, then there is a one-one generalized pseudo homomorphism* $\alpha : \boldsymbol{X}/\mu \to \boldsymbol{Y}$ *such that ran* $\alpha =$ *ran* θ *and* $\mu^{\#}\alpha = \theta$.

Proof. It is clear from the above discussion that $\boldsymbol{X}/\mu$ is a generalized pseudo BE-algebra. Further, we prove that $\mu^{\#} : \boldsymbol{X} \longrightarrow \boldsymbol{X}/\mu$ defined by $\mu^{\#}(x) = \mu_x\ \forall\ x \in \boldsymbol{X}$ is an epimorphism. Now let $p, q \in \boldsymbol{X}$, then

$$\mu^{\#}(p *_1 q) = \mu_{p\ *_1 q} = \mu_p *_1 \mu_q = \mu^{\#}(p) *_1 \mu^{\#}(q)$$

and

$$\mu^{\#}(p \diamond_1 q) = \mu_{p\ \diamond_1 q} = \mu_p \diamond_1 \mu_q = \mu^{\#}(p) \diamond_1 \mu^{\#}(q).$$

Clearly $\mu^{\#}$ is onto because for each $\mu_x \in \boldsymbol{X}/\mu\ \exists\ x \in \boldsymbol{X}$ such that $\mu^{\#}(x) = \mu_x$. Thus, $\mu^{\#}$ is an epimorphism.

Define $\alpha : \boldsymbol{X}/\mu \to \boldsymbol{Y}$ by $\alpha(\mu_a) = \theta(a)\ \forall\ \mu_a \in \boldsymbol{X}/\mu$. We now show that α is monomorphism.

Well-defined: Let us suppose that $\mu_{a_1}, \mu_{a_2} \in \boldsymbol{X}/\mu$ such that
$\mu_{a_1} = \mu_{a_2}$
$\Rightarrow (a_1, a_2) \in \mu$ ($\because$ by Lemma 8.5)
$\Rightarrow \theta(a_1) = \theta(a_2)$
$\Rightarrow \alpha(\mu_{a_1}) = \alpha(\mu_{a_2})$.

One-One: Choose $\mu_{a_1}, \mu_{a_2} \in \boldsymbol{X}/\mu$ such that
$\alpha(\mu_{a_1}) = \alpha(\mu_{a_2})$
$\Rightarrow \theta(a_1) = \theta(a_2)$
$\Rightarrow (a_1, a_2) \in \mu$
$\Rightarrow \mu_{a_1} = \mu_{a_2}$ ($\because$ by Lemma 8.5)

Homomorphism: Let $\mu_{a_1}, \mu_{a_2} \in \boldsymbol{X}/\mu$, then
$\alpha(\mu_{a_1} *_1 \mu_{a_2}) = \alpha(\mu_{a_1 *_1 a_2})$
$= \theta(a_1 *_1 a_2) = \theta(a_1) *_2 \theta(a_2)$ ($\because \theta$ is a homomorphism)
$= \alpha(\mu_{a_1}) *_2 \alpha(\mu_{a_2})$.
Also
$\alpha(\mu_{a_1} \diamond_1 \mu_{a_2}) = \alpha(\mu_{a_1 \diamond_1 a_2})$
$= \theta(a_1 \diamond_1 a_2)$

$= \theta(a_1) \diamond_2 \theta(a_2)$ ($\because \theta$ is a homomorphism)

$= \alpha(\mu_{a_1}) \diamond_2 \alpha(\mu_{a_2})$.

Now

ran $\theta = \{\theta(a) : a \in \boldsymbol{X}\} = \{\alpha(\mu_a) : \mu_a \in \boldsymbol{X}/\mu\} =$ ran α.

Further, we need to show that $\alpha(\mu)^{\#} = \theta$. In other words we have to show that $(\alpha(\mu)^{\#})(a) = \theta(a)\ \forall\ a \in \boldsymbol{X}$. Here $(\alpha(\mu)^{\#})(a) = \alpha((\mu)^{\#}(a)) = \alpha(\mu_a) = \theta(a)$. □

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