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CERTAIN k-FRACTIONAL CALCULUS OPERATORS AND IMAGE FORMULAS OF GENERALIZED k-BESSEL FUNCTION

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Abstract. In this paper, the Saigo's k-fractional integral and derivative operators involving k-hypergeometric function in the kernel are applied to the generalized k-Bessel function; results are expressed in term of k-Wright function, which are used to present image formulas of integral transforms including beta transform. Also special cases related to fractional calculus operators and Bessel functions are considered.

1. Introduction and Preliminaries

Recently, a series research publications in respect of generalized classical fractional calculus operators, Mubeen and Habibullah [17] were introduced k-fractional integral of the Riemann-Liouville type and its application, Dorrego [7] was introduced an alternative definition for the k-Riemann-Liouville fractional derivative, Gupta and Parihar [14] were introduced Saigo k-fractional calculus operators. Since the k-Gamma function and the k-Pochhammer symbol introduced by Díaz et al. [4, 5, 6] the fractional calculus has been significant development through the generalization of the so-called special functions, like the k-Beta and k-Gamma function, the k-Mittag-Leffler function, the k-Wright function, the k-Bessel functions, the k-Struve function. Mubeen and Rehman [19] have studied extension of k-gamma and Pochhamer k-symbol, Mubeen and Habibullah [18] introduced an integral representation of k-hypergeometric functions within Pochhammer k-symbols, k-gamma and k-beta functions. A comprehensive account of k- Bessel function along with their properties and applications can be found in [1, 2, 3, 16, 20, 26] and so forth.

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k-Bessel function: The generalized k-Bessel function defined in Mondal [15] as:

$$w_{v,c}^{k}(z)=\sum_{n=0}^{\infty}\frac{(-c)^n}{\Gamma_k(nk+v+k)n\,!}\left(\frac{z}{2}\right)^{2n+\frac{v}{k}}, \tag{1}$$

where $k>0$, $v>-1$, and $c\in\mathbb{R}$ and $\Gamma_k(z)$ is the k-gamma function defined in Díaz and Pariguan [5] as:

$$\Gamma_k(z)=\int_0^{\infty}t^{z-1}e^{-\frac{t^k}{k}}\,dt,\ \ z\in\mathbb{C}. \tag{2}$$

By inspection the following relation holds:

$$\Gamma_k(z+k)=z\,\Gamma_k(z), \tag{3}$$

and

$$\Gamma_k(z)=k^{(z/k)-1}\,\Gamma\left(\frac{z}{k}\right). \tag{4}$$

If $k\to 1$ and $c=1$, then the generalized k-Bessel function defined in (1) reduces to the well-known classical Bessel function J_v defined in Erdélyi [8]. For further detail about k-Bessel function and its properties (see [9, 10, 12]).

k-Beta function: The k-beta function [5] is defined as

$$B_k\left(g,h\right)=\frac{1}{k}\int_0^1 t^{\frac{g}{k}-1}\left(1-t\right)^{\frac{h}{k}-1}dt,\ g>0,h>0. \tag{5}$$

and they have the following important identities

$$B_k\left(g,h\right)=\frac{1}{k}B\left(\frac{g}{k},\frac{h}{k}\right)=\frac{\Gamma_k\left(g\right)\Gamma_k\left(h\right)}{\Gamma_k\left(g+h\right)}. \tag{6}$$

k-hypergeometric function: The k-hypergeometric function F_k is defined by Mubeen and Habibullah [18] in a power series form as:

$$F_k\left(\left(\tau,k\right);\left(\gamma,k\right);x\right)=\sum_{n=0}^{\infty}\frac{\tau_{n,k}x^n}{\left(\gamma\right)_{n,k}n!},\ k\in\mathbb{R}^+,\tau,\gamma\in\mathbb{C},\Re\left(\tau\right)>0,\Re\left(\gamma\right)>0. \tag{7}$$

and its integral representation can be determined as follows:

$${}_1F_1\left(\left(\tau,k\right);\left(\gamma,k\right);x\right)=\frac{\Gamma_k\left(\gamma\right)}{k\Gamma_k\left(\tau\right)\Gamma_k\left(\gamma-\tau\right)}\int_0^1 t^{\frac{\tau}{k}-1}\left(1-t\right)^{\frac{\gamma-\tau}{k}-1}e^{xt}dt, \tag{8}$$

Also, if $\Re\left(\gamma\right)>\Re\left(\tau\right)>0,k>0,m\geq 0,m\in\mathbb{Z}^+$ and $|x|<1$, then

$$\begin{aligned}&{}_{m+1}F_{m,k}\left[\begin{array}{c}\left(\varepsilon,k\right),\left(\frac{\tau}{m},k\right),\left(\frac{\tau+k}{m},k\right),\cdots,\left(\frac{\gamma+(m-1)k}{m},k\right);\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\cdots,\left(\frac{\gamma+(m-1)k}{m},k\right);\end{array}x\right]\\ &=\frac{\Gamma_k\left(\gamma\right)}{k\Gamma_k\left(\tau\right)\Gamma_k\left(\gamma-\tau\right)}\int_0^1 t^{\frac{\tau}{k}-1}\left(1-t\right)^{\frac{\gamma-\tau}{k}-1}\left(1-kxt\right)^{-\frac{\varepsilon}{k}}dt.\end{aligned} \tag{9}$$

and if $\Re(\gamma) > \Re(\tau) > 0$ and $|x| < 1$, then

$$ {}_2F_{1,k}\left((\varepsilon,k),(\tau,k);(\gamma,k);x\right) = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\tau)\Gamma_k(\gamma-\tau)} \tag{10} $$

$$ \times \int_0^1 t^{\frac{\tau}{k}-1}(1-t)^{\frac{\gamma-\tau}{k}-1}(1-kxt)^{-\frac{\varepsilon}{k}}\,dt. $$

k-Wright function: Gehlot and Prajapati [11] introduced the generalized k-wright function ${}_p\Psi_q^k(z)$ defined for $k \in \mathbb{R}^+; z, a_i, b_j \in \mathbb{C}, A_i, B_j \in \mathbb{R}(A_i, B_j \neq 0)$ where $i = 1, 2, ..., p$; $j = 1, 2, ..., q$ and $(a_i + A_i n), (b_j + B_j n) \in \mathbb{C}\backslash k\mathbb{Z}^-$

$$ {}_p\Psi_q^k(z) = {}_p\Psi_q^k\left[\begin{array}{c} (a_i, A_i)_{1,p} \\ (b_j, B_j)_{1,q} \end{array}\middle| z\right] = \sum_{n=0}^{\infty} \frac{\prod\limits_{i=1}^{p}\Gamma_k(a_i + A_i n)}{\prod\limits_{j=1}^{q}\Gamma_k(b_j + B_j n)} \frac{z^n}{n!}, \tag{11} $$

satisfies the following condition

$$ \sum_{j=1}^{q}\frac{B_j}{k} - \sum_{i=1}^{p}\frac{A_i}{k} > -1. \tag{12} $$

2. Saigo k-fractional integration in term of k-Wright function

Here, we establish k-fractional integral formulas defined by Gupta and Parihar [14] (see also; [24, 25]) for the generalized k-Bessel function. For our purpose, we recall the following pair of Saigo k-hypergeometric fractional integral operators. Let $x \in \mathbb{R}^+$, $\varepsilon, \tau, \gamma \in \mathbb{C}$ with $Re(\varepsilon) > 0, k > 0$, we have

$$ \left(I_{0+,k}^{\varepsilon,\tau,\gamma} f\right)(x) = \frac{x^{\frac{-\varepsilon-\tau}{k}}}{k\Gamma_k(\varepsilon)} \int_0^x (x-t)^{\frac{\varepsilon}{k}-1} \tag{13} $$

$$ \times {}_2F_{1,k}\left((\varepsilon+\tau,k),(-\gamma,k);(\varepsilon,k);\left(1-\tfrac{t}{x}\right)\right) f(t)\,dt $$

$$ \left(I_{-,k}^{\varepsilon,\tau,\gamma} f\right)(x) = \frac{1}{k\Gamma_k(\varepsilon)} \int_x^\infty (t-x)^{\frac{\varepsilon}{k}-1} t^{\frac{-\varepsilon-\tau}{k}} \tag{14} $$

$$ \times {}_2F_{1,k}\left((\varepsilon+\tau,k),(-\gamma,k);(\varepsilon,k);\left(1-\frac{x}{t}\right)\right) f(t)\,dt $$

where ${}_2F_{1,k}((\varepsilon,k),(\tau,k);(\gamma,k);x)$ defined by [6] for $x \in \mathbb{C}, |x| < 1$, $\Re(\gamma) > \Re(\tau) > 0$ as:

$$ {}_2F_{1,k}\left((\varepsilon,k),(\tau,k);(\gamma,k)\;;x\right) = \sum_{n=0}^{\infty} \frac{(\varepsilon)_{n,k}(\tau)_{n,k}\,x^n}{(\gamma)_{n,k}\,n!} \tag{15} $$

Remark 1: When we set $k = 1$ in Eqs. (13) and (14), operators reduces in to Saigo's fractional integral operators stated in [22].

Now, we need to recall the following formulas in Lemmas 1 and 2 (see [14]).

Lemma 1. Let $\varepsilon,\tau,\gamma,\rho\in\mathbb{C}$ and $\Re(\varepsilon)>0$, $k\in\mathbb{R}^+(0,\infty)$ such that $\Re(\rho)>\max[0,\,\Re(\tau-\gamma)]$, then

$$\left(I_{0+,k}^{\varepsilon,\tau,\gamma}t^{\frac{\rho}{k}-1}\right)(x)=\sum_{n=0}^{\infty}k^n\frac{\Gamma_k(\rho)\,\Gamma_k(\rho-\tau+\gamma)}{\Gamma_k(\rho-\tau)\,\Gamma_k(\rho+\varepsilon+\gamma)}x^{\frac{\rho-\tau}{k}-1}. \tag{16}$$

Lemma 2. Let $\varepsilon,\tau,\gamma,\rho\in\mathbb{C}$ and $\Re(\varepsilon)>0$, $k\in\mathbb{R}^+(0,\infty)$ such that $\Re(\rho)>\max[\Re(-\tau),\Re(-\gamma)]$, then

$$\left(I_{-,k}^{\varepsilon,\tau,\gamma}t^{-\frac{\rho}{k}}\right)(x)=\sum_{n=0}^{\infty}k^n\frac{\Gamma_k(\rho+\tau)\,\Gamma_k(\rho+\gamma)}{\Gamma_k(\rho)\,\Gamma_k(\rho+\varepsilon+\tau+\gamma)}x^{\frac{-\rho-\tau}{k}}. \tag{17}$$

Now we are ready to present our main results asserted by Theorems 3 and 4.

Theorem 1. Let $\varepsilon,\tau,\gamma,\rho,\xi\in\mathbb{C}$, $\Re(v)>-1$, $k\in\mathbb{R}^+$ such that $\Re(\varepsilon)>0$, $\Re(\rho)>\max[0,\,\Re(\tau-\gamma)]$, If condition (12) is satisfied and $I_{0+,k}^{\varepsilon,\tau,\gamma}$ be the left sided operator of the generalized k-fractional integration involving k-Gauss hypergeometric function, then the following result true:

$$\left(I_{0+,k}^{\varepsilon,\tau,\gamma}\left(t^{\frac{\rho}{k}-1}w_{v,\,c}^{k}\left[a\,t^{\frac{\xi}{k}}\right]\right)\right)(x)=x^{\frac{\rho-\tau}{k}+\frac{v\xi}{k^2}-1}(a/2)^{\frac{v}{k}} \tag{18}$$
$$\times{}_2\Psi_3^k\left[\begin{matrix}\left(\rho+\frac{v\xi}{k},2\xi\right),\left(\rho+\frac{v\xi}{k}-\tau+\gamma,2\xi\right)\\(v+k,k),\left(\rho+\frac{v\xi}{k}-\tau,\,2\xi\right),\left(\rho+\frac{v\xi}{k}+\varepsilon+\gamma,2\xi\right)\end{matrix}\middle|\frac{-cka^2x^{\frac{2\xi}{k}}}{4}\right].$$

Proof. By applying (1) on the left side of (18), let I_1, we have

$$I_1=I_{0+,k}^{\varepsilon,\tau,\gamma}\left(t^{\frac{\rho}{k}-1}\sum_{n=0}^{\infty}\frac{(-c)^n}{\Gamma_k(nk+v+k)n!}\left(\frac{at^{\frac{\xi}{k}}}{2}\right)^{2n+\frac{v}{k}}\right)(x), \tag{19}$$
$$=\sum_{n=0}^{\infty}\frac{(-c)^n}{\Gamma_k(nk+v+k)n!}\left(\frac{a}{2}\right)^{2n+\frac{v}{k}}I_{0+,k}^{\varepsilon,\tau,\gamma}\left(t^{\frac{\rho+2n\xi+v\xi/k}{k}-1}\right)(x),$$

which upon Lemma 1, yields

$$I_1=x^{\frac{\rho-\tau}{k}+\frac{v\xi}{k^2}-1}\sum_{n=0}^{\infty}\frac{(-c)^n\,k^n}{\Gamma_k(nk+v+k)n!}\left(\frac{a}{2}\right)^{2n+\frac{v}{k}} \tag{20}$$
$$\times\frac{\Gamma_k(\rho+(v\xi/k)+2n\xi)\,\Gamma_k(\rho+(v\xi/k)-\tau+\gamma+2n\xi)}{\Gamma_k(\rho+(v\xi/k)-\tau+2n\xi)\,\Gamma_k(\rho+(v\xi/k)+\varepsilon+\gamma+2n\xi)}x^{\frac{2n\xi}{k}},$$

Using the definition of (11) in the right-hand side of (20), we arrive at the result (18).

Theorem 2. Let $\varepsilon,\tau,\gamma,\rho,\xi\in\mathbb{C},\Re(v)>-1$, $k\in\mathbb{R}^+$ such that $\Re(\varepsilon)>0$, $\Re(\varepsilon+\rho)>\max[-\Re(\tau),-\Re(\gamma)]$. If condition (12) is satisfied and $I_{0+,k}^{\varepsilon,\tau,\gamma}$ be the right sided operator of the generalized k-fractional integration involving k-Gauss hypergeometric function, then the following result true:

$$\left(I_{-,k}^{\varepsilon,\tau,\gamma}\left(t^{\frac{-\varepsilon-\rho}{k}}w_{v,\,c}^{k}\left[a\,t^{-\frac{\xi}{k}}\right]\right)\right)(x)=x^{\frac{-\varepsilon-\rho-\tau}{k}-\frac{v\xi}{k^2}}\left(\frac{a}{2}\right)^{\frac{v}{k}} \tag{21}$$

$$\times {}_2\Psi_3^k \left[\begin{array}{c} \left(\varepsilon+\rho+\gamma+\frac{v\xi}{k},\, 2\xi\right), \left(\varepsilon+\rho+\frac{v\xi}{k}+\tau,\, 2\xi\right) \\ (v+k,k), \left(\varepsilon+\rho+\frac{v\xi}{k},\, 2\xi\right), \left(\rho+2\varepsilon+\gamma+\tau+\frac{v\xi}{k}, 2\xi\right) \end{array} \middle| \frac{-c\,k\,a^2 x^{\frac{-2\xi}{k}}}{4} \right].$$

Proof. The proof is parallel to that of Theorem 1. Therefore we omit the details.

The results given in (18) and (21), being very general, can yield a large number of special cases by assigning some suitable values to the involved parameters. Now, we demonstrate some Corollaries as below.

If we take $k = 1$ and $c = 1$ in (18) and (21), we obtain the following two formulas in Corollaries 1 and 2.

Corollary 1. Let $\varepsilon, \tau, \gamma, \rho, \xi \in \mathbb{C}, \Re(v) > -1$, such that $\Re(\varepsilon) > 0$ and $\Re(\rho) > \max[0, \Re(\tau-\gamma)]$, then the following result true:

$$\left(I_{0+,k}^{\varepsilon,\tau,\gamma}\left(t^{\rho-1} J_v\left[at^{\xi}\right]\right)\right)(x) = x^{\rho-\tau+v\xi-1}\left(\frac{a}{2}\right)^v \tag{22}$$

$$\times {}_2\Psi_3 \left[\begin{array}{c} (\rho+v\xi, 2\xi), (\rho+v\xi-\tau+\gamma, 2\xi) \\ (v+1,1), (\rho+v\xi-\tau, 2\xi), (\rho+v\xi+\varepsilon+\gamma, 2\xi) \end{array} \middle| \frac{-\left(a\,x^{\xi}\right)^2}{4} \right].$$

Corollary 2. Let $\varepsilon, \tau, \gamma, \rho, \xi \in \mathbb{C}, \Re(v) > -1$ such that $\Re(\varepsilon) > 0$ and $\Re(\varepsilon+\rho) > \max[-\Re(\tau), -\Re(\gamma)]$, then the following result true:

$$\left(I_{-}^{\varepsilon,\tau,\gamma}\left(t^{-\varepsilon-\rho} J_v\left(ct^{-\xi}\right)\right)\right)(x) = x^{-\varepsilon-\rho-\tau-v\xi}\left(\frac{a}{2}\right)^v \tag{23}$$

$$\times {}_2\Psi_3 \left[\begin{array}{c} (\varepsilon+\rho+\gamma+v\xi,\, 2\xi), (\varepsilon+\rho+\tau+v\xi, 2\xi) \\ (v+1,1), (\varepsilon+\rho+v\xi, 2\xi), (\rho+2\varepsilon+\gamma+\tau+v\xi, 2\xi) \end{array} \middle| \frac{-\left(ax^{-\xi}\right)^2}{4} \right].$$

If we substitute $\tau = -\varepsilon$ in equations (18) and (21), Saigo k-fractional integral operators reduce to k-Riemann-Liouville integral operators as follows:

Corollary 3. Let $\varepsilon, \gamma, \rho, \xi \in \mathbb{C}, \Re(v) > -1$, $k \in \mathbb{R}^+$ such that $\Re(\varepsilon) > 0$, then the following result true:

$$\left(I_{0+,k}^{\varepsilon}\left(t^{\frac{\rho}{k}-1} w_{v,c}^k\left[a\,t^{\frac{\xi}{k}}\right]\right)\right)(x) \tag{24}$$

$$= x^{\frac{\rho+\varepsilon}{k}+\frac{v\xi}{k^2}-1}(a/2)^{\frac{v}{k}}\, {}_1\Psi_2^k \left[\begin{array}{c} \left(\rho+\frac{v\xi}{k}, 2\xi\right) \\ (v+k,k), \left(\rho+\frac{v\xi}{k}+\varepsilon, 2\xi\right) \end{array} \middle| \frac{-cka^2x^{\frac{2\xi}{k}}}{4} \right].$$

Corollary 4. Let $\varepsilon, \gamma, \rho, \xi \in \mathbb{C}, \Re(v) > -1$, $k \in \mathbb{R}^+$ such that $\Re(\varepsilon) > 0$, then the following result true:

$$\left(I_{-,k}^{\varepsilon}\left(t^{\frac{-\varepsilon-\rho}{k}} w_{v,c}^k\left[a\,t^{-\frac{\xi}{k}}\right]\right)\right)(x) \tag{25}$$

$$= x^{\frac{-\rho}{k}-\frac{v\xi}{k^2}}\left(\frac{a}{2}\right)^{\frac{v}{k}} {}_1\Psi_2^k \left[\begin{array}{c} \left(\rho+\frac{v\xi}{k}, 2\xi\right) \\ (v+k,k), \left(\varepsilon+\rho+\frac{v\xi}{k}, 2\xi\right) \end{array} \middle| \frac{-cka^2x^{\frac{-2\xi}{k}}}{4} \right].$$

3. Saigo k-fractional differentiation in term of k-Wright function

Here, we establish k-fractional derivative formulas defined by Gupta and Parihar [14] for the generalized k-Bessel function. For this, we recall the following involving Saigo k-hypergeometric fractional derivative operators. We have

(26)

$$\left(D_{0+,k}^{\varepsilon,\tau,\gamma} f\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{0+,k}^{-\varepsilon+n,-\tau-n,\varepsilon+\gamma-n} f\right)(x)\,,\ n = [\Re(\varepsilon)+1]\,,$$

$$= \left(\frac{d}{dx}\right)^n \frac{x^{\frac{\varepsilon+\tau}{k}}}{k\Gamma_k(-\varepsilon+n)} \int_0^x (x-t)^{\frac{\varepsilon}{k}+n-1}$$

$$\times {}_2F_{1,k}\left((-\varepsilon-\tau,k),(-\gamma-\varepsilon+n,k);(-\varepsilon+n,k);\left(1-\tfrac{t}{x}\right)\right) f(t)\,dt,$$

(27)

$$\left(D_{-,k}^{\varepsilon,\tau,\gamma} f\right)(x) = \left(-\frac{d}{dx}\right)^n \left(I_{-,k}^{-\varepsilon+n,-\tau-n,\varepsilon+\gamma} f\right)(x)\,,\quad n = [Re(\varepsilon)+1]\,,$$

$$= \left(-\frac{d}{dx}\right)^n \frac{1}{k\Gamma_k(-\varepsilon+n)} \int_x^\infty (t-x)^{-\frac{\varepsilon+n}{k}-1}\, t^{\frac{\varepsilon+\tau}{k}}$$

$$\times {}_2F_{1,k}\left((-\varepsilon-\tau,k),(-\gamma-\varepsilon,k);(-\varepsilon+n,k);\left(1-\tfrac{x}{t}\right)\right) f(t)\,dt,$$

where $x>0, \varepsilon\in\mathbb{C}, \Re(\varepsilon)>0, k>0$ and $[\Re(\varepsilon)]$ is the integer part of $\Re(\varepsilon)$.

Remark 2: For $k=1$, Eqs.(26) and (27) reduces in to Saigo's fractional derivative operators stated in [22].

Here, we recall the following formulas in Lemmas 3 and 4 (see [14]).

Lemma 3. Let $\varepsilon,\tau,\gamma,\rho\in\mathbb{C},\ n=(\Re(\varepsilon))+1, k\in\mathbb{R}^+(0,\infty)$ such that $\Re(\rho)>\max[0,\ \Re(-\varepsilon-\tau-\gamma)]$, then

$$\left(D_{0+,k}^{\varepsilon,\tau,\gamma} t^{\frac{\rho}{k}-1}\right)(x) = \sum_{n=0}^{\infty} \frac{\Gamma_k(\rho)\,\Gamma_k(\rho+\tau+\gamma+\varepsilon)}{\Gamma_k(\rho+\gamma)\,\Gamma_k(\rho+\tau+n-nk)} x^{\frac{\rho+\tau+n}{k}-n-1}. \tag{28}$$

Lemma 4. Let $\varepsilon,\tau,\gamma,\rho\in\mathbb{C},\ n=(\Re(\varepsilon))+1, k\in\mathbb{R}^+(0,\infty)$ such that $\Re(\rho)>\max[\,\Re(-\varepsilon-\gamma),\Re(\tau-nk+n)]$, then

$$\left(D_{-,k}^{\varepsilon,\tau,\gamma} t^{-\frac{\rho}{k}}\right)(x) = \sum_{n=0}^{\infty} \frac{\Gamma_k(\rho-\tau-n+nk)\,\Gamma_k(\rho+\varepsilon+\gamma)}{\Gamma_k(\rho)\,\Gamma_k(\rho-\tau+\gamma)} x^{\frac{-\rho+\tau+n}{k}-n}. \tag{29}$$

Theorem 3. Let $\varepsilon,\tau,\gamma,\rho,\xi\in\mathbb{C},\ \Re(v)>-1,\ k\in\mathbb{R}^+$ be such that $\Re(\varepsilon)>0$, $\Re(\rho)>\max[0,\ \Re(-\varepsilon-\tau-\gamma)]$, If condition (12) is satisfied and $D_{0+,k}^{\varepsilon,\tau,\gamma}$ be the left sided operator of the generalized k- fractional differentiation involving k-Gauss hypergeometric function, then following result true:

$$\left(D_{0+,k}^{\varepsilon,\tau,\gamma}\left(t^{\frac{\rho}{k}-1} w_{v,\,c}^{k}\left[a\,t^{\frac{\xi}{k}}\right]\right)\right)(x) = x^{\frac{\rho+\tau}{k}+\frac{v\xi}{k^2}-1}\left(\frac{a}{2}\right)^{\frac{v}{k}} \tag{30}$$

$$\times {}_2\Psi_3^k \left[\begin{array}{c} \left(\rho + \frac{v\xi}{k}, 2\xi\right), \left(\rho + \frac{v\xi}{k} + \tau + \gamma + \varepsilon, 2\xi\right) \\ (v+k,k), \left(\rho + \frac{v\xi}{k} + \gamma,\, 2\xi\right), \left(\rho + \frac{v\xi}{k} + \tau, 2\xi - k + 1\right) \end{array} \middle| \frac{-c\,a^2 x^{\frac{2\xi+1}{k}-1}}{4} \right].$$

Proof. By applying equation (1) in the left-side of (30), let I_2, we get

$$I_2 = D_{0+,k}^{\varepsilon,\tau,\gamma} \left(t^{\frac{\rho}{k}-1} \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k\,(nk+v+k)\, n\,!} \left(\frac{a t^{\frac{\xi}{k}}}{2} \right)^{2n+\frac{v}{k}} \right)(x)\,, \tag{31}$$

$$= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k\,(nk+v+k)\, n!} \left(\frac{a}{2}\right)^{2n+\frac{v}{k}} \left(D_{0+,k}^{\varepsilon,\tau,\gamma} t^{\frac{\rho+2n\xi}{k}+\frac{v\xi}{k^2}-1} \right)(x)\,,$$

Using Lemma 3, in the above equation can be written as

$$I_2 = x^{\frac{\rho+\tau}{k}+\frac{v\xi}{k^2}-1} \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k\,(nk+v+k)\, n!} \left(\frac{a}{2}\right)^{2n+\frac{v}{k}} \sum_{n=0}^{\infty} \frac{1}{\Gamma_k\,(nk+v+k)} \tag{32}$$

$$\times \frac{\Gamma_k\,(\rho + (v\xi/k) + 2n\xi)\ \Gamma_k\,(\rho + (v\xi/k) + \tau + \gamma + \varepsilon + 2n\xi)}{\Gamma_k\,(\rho + (v\xi/k) + \gamma + 2n\xi)\ \Gamma_k\,(\rho + (v\xi/k) + \tau + 2n\xi + n - nk)}$$

$$\times x^{\frac{2n\xi+n}{k}-n},$$

Using the definition of (11) in the right-hand side of (32), we arrive at the result (30).

Theorem 4. Let $\varepsilon, \tau, \gamma, \rho, \xi \in \mathbb{C}, \Re(v) > -1$, and $k \in \mathbb{R}^+$ be such that $\Re(\varepsilon) > 0, \Re(\rho) > \max\left[\Re(-\varepsilon-\gamma), \Re(\tau - nk + n)\right]$, where $(n = [\Re(\varepsilon+1)])$ and $D_{-,k}^{\varepsilon,\tau,\gamma}$ be the left sided operator of the generalized k-fractional differentiation then the following formula holds true:

$$\left(D_{-,k}^{\varepsilon,\tau,\gamma} \left(t^{\frac{\varepsilon-\rho}{k}} w_{v,c}^{k} \left[a t^{-\frac{\xi}{k}} \right] \right) \right)(x) = x^{\frac{\varepsilon-\rho-\tau}{k}-\frac{v\xi}{k^2}} \left(\frac{a}{2}\right)^{\frac{v}{k}} \tag{33}$$

$$\times {}_2\Psi_3^k \left[\begin{array}{c} \left(\rho - \varepsilon + \frac{v\xi}{k} - \tau,\, 2\xi + k - 1\right), \left(\rho + \frac{v\xi}{k} + \gamma, 2\xi\right) \\ (v+k,k), \left(\rho - \varepsilon + \frac{v\xi}{k}, 2\xi\right), \left(\rho - \varepsilon + \frac{v\xi}{k} - \tau + \gamma, 2\xi\right) \end{array} \middle| \frac{-c a^2 x^{\frac{-2\xi+1}{k}-1}}{4} \right].$$

Proof. The proof is parallel to that of Theorem 3. Therefore, we omit the details.

If we take $k = 1$, $c = 1$ in (30) and (33), we obtain the following two formulas as:

Corollary 5. Let $\varepsilon, \tau, \gamma, \rho, \xi \in \mathbb{C}, \Re(v) > -1$, such that $\Re(\varepsilon) > 0$, and $\Re(\rho) > \max\left[0,\, \Re(-\varepsilon-\tau-\gamma)\right]$, then the following result true:

$$\left(D_{0+}^{\varepsilon,\tau,\gamma} \left(t^{\rho-1} J_v \left(a t^{\xi} \right) \right) \right)(x) = x^{\rho+\tau+v\xi-1} \left(\frac{a}{2}\right)^{v} \tag{34}$$

$$\times {}_2\Psi_3 \left[\begin{array}{c} (\rho + v\xi, 2\xi), (\rho + v\xi + \tau + \gamma + \varepsilon,\, 2\xi) \\ (v+1,1), (\rho + v\xi + \gamma,\, 2\xi), (\rho + v\xi + \tau, 2\xi) \end{array} \middle| \frac{-a^2 x^{2\xi}}{4} \right].$$

Corollary 6. Let $\varepsilon,\tau,\gamma,\rho,\xi\in\mathbb{C},\Re(v)>-1$ such that $\Re(\varepsilon)>0,\Re(\rho)>\max[\Re(-\varepsilon-\gamma),\Re(\tau-nk+n)]$, then the following result true:

$$(35)\qquad \left(D_{-}^{\varepsilon,\tau,\gamma}\left(t^{\varepsilon-\rho}J_v\left(ct^{-\xi}\right)\right)\right)(x)=x^{\varepsilon-\rho-\tau-v\xi}\left(\frac{a}{2}\right)^{v}$$

$$\times{}_2\Psi_3\left[\begin{array}{c}(-\varepsilon+\rho+v\xi-\tau,2\xi),(\rho+v\xi+\gamma,\,2\xi)\\(v+1,1),(-\varepsilon+\rho+v\xi,\,2\xi),(-\varepsilon+\rho+v\xi-\tau+\gamma,2\,\xi)\end{array}\left|\frac{-a^2x^{-2\xi}}{4}\right.\right].$$

If we substitute $\tau=-\varepsilon$ in equations (30) and (33), Saigo k-fractional derivative operators reduce to k-Riemann-Liouville derivative operators as follows:

Corollary 7. Let $\varepsilon,\gamma,\rho,\xi\in\mathbb{C},\Re(v)>-1,\ k\in\mathbb{R}^+$ be such that $\Re(\varepsilon)>0$, then following result true:

$$(36)\qquad \left(D_{0+,k}^{\varepsilon}\left(t^{\frac{\rho}{k}-1}w_{v,c}^{k}\left[a\,t^{\frac{\xi}{k}}\right]\right)\right)(x)=x^{\frac{\rho-\varepsilon}{k}+\frac{v\xi}{k^2}-1}\left(\frac{a}{2}\right)^{\frac{v}{k}}$$

$$\times{}_1\Psi_2^k\left[\begin{array}{c}\left(\rho+\frac{v\xi}{k},2\xi\right)\\(v+k,k),\left(\rho+\frac{v\xi}{k}-\varepsilon,2\xi-k+1\right)\end{array}\left|\frac{-ca^2x^{\frac{2\xi+1}{k}-1}}{4}\right.\right].$$

Corollary 8. Let $\varepsilon,\gamma,\rho,\xi\in\mathbb{C},\Re(v)>-1$, and $k\in\mathbb{R}^+$ be such that $\Re(\varepsilon)>0$, then the following formula holds true:

$$(37)\qquad \left(D_{-,k}^{\varepsilon}\left(t^{\frac{\varepsilon-\rho}{k}}w_{v,c}^{k}\left[at^{-\frac{\xi}{k}}\right]\right)\right)(x)=x^{\frac{2\varepsilon-\rho}{k}-\frac{v\xi}{k^2}}\left(\frac{a}{2}\right)^{\frac{v}{k}}$$

$$\times{}_1\Psi_2^k\left[\begin{array}{c}\left(\rho+\frac{v\xi}{k},\,2\xi+k-1\right)\\(v+k,k),\left(\rho-\varepsilon+\frac{v\xi}{k},2\xi\right)\end{array}\left|\frac{-ca^2x^{\frac{-2\xi+1}{k}-1}}{4}\right.\right].$$

4. Image formulas associated with integral transforms

In this section, we establish some theorems involving the results obtained in previous sections pertaining with the integral transform.
Here, we are defined k-beta function as

$$(38)\qquad B_k\left(f(z);g,h\right)=\frac{1}{k}\int_0^1 t^{\frac{g}{k}-1}(1-t)^{\frac{h}{k}-1}f(t)dt,\ g>0,h>0.$$

Theorem 5. Let $\varepsilon,\tau,\gamma,\rho,\xi\in\mathbb{C},\Re(v)>-1,\ k\in\mathbb{R}^+$ such that $\Re(\varepsilon)>0$, $\Re(\rho)>\max[0,\,\Re(\tau-\gamma)]$, then the following fractional integral holds true:

$$(39)\qquad B_k\left(\left(I_{0+,k}^{\varepsilon,\tau,\gamma}\left(t^{\frac{\rho}{k}-1}w_{v,c}^{k}\left[a\,t^{\frac{\xi}{k}}\right]\right)\right)(x);\,g,\,h\right)=x^{\frac{\rho-\tau}{k}+\frac{v\xi}{k^2}-1}\Gamma_k(h)\left(\frac{a}{2}\right)^{\frac{v}{k}}$$

$$\times{}_3\Psi_4^k\left[\begin{array}{c}\left(\rho+\frac{v\xi}{k},2\xi\right),\left(\rho+\frac{v\xi}{k}-\tau+\gamma,2\xi\right),\\(v+k,k),\left(\rho+\frac{v\xi}{k}-\tau,2\xi\right),\left(\rho+\frac{v\xi}{k}+\varepsilon+\gamma,2\xi\right),\end{array}\right.$$

$$\left.\begin{matrix} \left(g+\frac{v\xi}{k},2\xi\right) \\ \left(g+h+\frac{v\xi}{k},2\xi\right) \end{matrix} \right| \left.\frac{-cka^2x^{\frac{2\xi}{k}}}{4}\right].$$

Proof. Let ℓ be the left-hand side of (39) and using (38), we have

$$\ell=\frac{1}{k}\int_0^1 z^{\frac{g}{k}-1}(1-z)^{\frac{h}{k}-1}\left(I_{0+,k}^{\varepsilon,\tau,\gamma}\left(t^{\frac{\rho}{k}-1}w_{v,c}^{k}\left[at^{\frac{\xi}{k}}\right]\right)\right)(x)dz, \tag{40}$$

which, using (1) and changing the order of integration and summation, which is valid under the conditions of Theorem 1, yields

$$\ell=\sum_{n=0}^{\infty}\frac{(-c)^n}{\Gamma_k(nk+v+k)\,n\,!}\left(\frac{a}{2}\right)^{2n+\frac{v}{k}}\left(I_{0+,k}^{\varepsilon,\tau,\gamma}\left(t^{\frac{\rho+2n\xi+v\xi/k}{k}-1}\right)\right)(x) \tag{41}$$
$$\times\frac{1}{k}\int_0^1 z^{\frac{g+2n\xi+v\xi/k}{k}-1}(1-z)^{\frac{h}{k}-1}dz,$$

which upon Lemma 1 and Eq. (6) in (41), we get

$$\ell=x^{\frac{\rho-\tau}{k}+\frac{v\xi}{k^2}-1}\sum_{n=0}^{\infty}\frac{(-c)^n k^n}{\Gamma_k(nk+v+k)n!}\left(\frac{a}{2}\right)^{2n+\frac{v}{k}} \tag{42}$$
$$\times\frac{\Gamma_k\left(\rho+(v\xi/k)+2n\xi\right)\Gamma_k\left(\rho+(v\xi/k)-\tau+\gamma+2n\xi\right)}{\Gamma_k\left(\rho+(v\xi/k)-\tau+2n\xi\right)\Gamma_k\left(\rho+(v\xi/k)+\varepsilon+\gamma+2n\xi\right)}$$
$$\times\frac{\Gamma_k\left(g+2n\xi+v\xi/k\right)\Gamma_k(h)}{\Gamma_k\left(g+h+2n\xi+v\xi/k\right)}(x)^{\frac{2n\xi}{k}},$$

Using the definition of (11) in the right-hand side of (42), we arrive at the result (39).

Theorem 6. Let $\varepsilon,\tau,\gamma,\rho,\xi\in\mathbb{C},\Re(v)>-1,\ k\in\mathbb{R}^+$ such that $\Re(\varepsilon)>0$, $\Re(\varepsilon+\rho)>\max\left[-\Re(\tau),-\Re(\gamma)\right]$, then the following fractional integral holds true:

(43)

$$B_k\left(\left(I_{-,k}^{\varepsilon,\tau,\gamma}\left(t^{\frac{-\varepsilon-\rho}{k}}w_{v,c}^{k}\left[a\,t^{-\frac{\xi}{k}}\right]\right)\right)(x)\,;\,g,h\right)=x^{\frac{-\varepsilon-\rho-\tau}{k}-\frac{v\xi}{k^2}}\Gamma_k(h)\left(\frac{a}{2}\right)^{\frac{v}{k}}$$
$$\times{}_3\Psi_4^k\left[\begin{matrix}\left(\varepsilon+\rho+\gamma+\frac{v\xi}{k},2\xi\right),\left(\varepsilon+\rho+\frac{v\xi}{k}+\tau,2\xi\right),\\ (v+k,k),\left(\varepsilon+\rho+\frac{v\xi}{k},\ 2\xi\right),\left(\rho+2\varepsilon+\gamma+\tau+\frac{v\xi}{k},2\xi\right),\end{matrix}\right.$$
$$\left.\begin{matrix} \left(g+\frac{v\xi}{k},2\xi\right) \\ \left(g+h+\frac{v\xi}{k},2\xi\right) \end{matrix} \right| \left.\frac{-cka^2x^{\frac{-2\xi}{k}}}{4}\right].$$

Proof. The proof is similar of Theorem 5. Therefore we omit the details.

Theorem 7. Let $\varepsilon,\tau,\gamma,\rho,\xi\in\mathbb{C},\Re(v)>-1,\ k\in\mathbb{R}^+$ be such that $\Re(\varepsilon)>0$, $\Re(\rho)>\max\left[0,\Re(-\varepsilon-\tau-\gamma)\right]$, then the following fractional derivative holds true:

$$B_k\left(\left(D_{0+,k}^{\varepsilon,\tau,\gamma}\left(t^{\frac{\rho}{k}-1}w_{v,c}^{k}\left[at^{\frac{\xi}{k}}\right]\right)\right)(x)\,;\,g,\,h\right)=x^{\frac{\rho+\tau}{k}+\frac{v\xi}{k^2}-1}\Gamma_k(h)\left(\frac{a}{2}\right)^{\frac{v}{k}} \tag{44}$$

$$\times {}_3\Psi_4^k \left[\begin{array}{c} \left(\rho+\frac{v\xi}{k},2\xi\right),\left(\rho+\frac{v\xi}{k}+\tau+\gamma+\varepsilon,\,2\xi\right), \\ (v+k,k),\left(\rho+\frac{v\xi}{k}+\gamma,\,2\xi\right),\left(\rho+\frac{v\xi}{k}+\tau,\,2\xi-k+1\right), \end{array} \right.$$

$$\left. \begin{array}{c} \left(g+\frac{v\xi}{k},2\xi\right) \\ \left(g+h+\frac{v\xi}{k},2\xi\right) \end{array} \middle| \frac{-c\,a^2x^{\frac{2\xi+1}{k}-1}}{4} \right].$$

Proof. Let $\Im$ be the left-hand side of (44) and using the definition of Beta transform, we have

$$\Im = \frac{1}{k}\int_0^1 z^{\frac{g}{k}-1}(1-z)^{\frac{h}{k}-1}\left(D_{0+,k}^{\varepsilon,\tau,\gamma}\left(t^{\frac{\rho}{k}-1}w_{v,c}^k\left[at^{\frac{\xi}{k}}\right]\right)\right)(x)dz, \tag{45}$$

which, using (1) and changing the order of integration and summation, which is valid under the conditions of Theorem 3, yields

$$\Im = \sum_{n=0}^{\infty}\frac{(-c)^n}{\Gamma_k(nk+v+k)n!}\left(\frac{a}{2}\right)^{2n+\frac{v}{k}}\left(D_{0+,k}^{\varepsilon,\tau,\gamma}\left(t^{\frac{\rho+2n\xi+v\xi/k}{k}-1}\right)\right)(x) \tag{46}$$

$$\times\frac{1}{k}\int_0^1 z^{\frac{g+2n\xi+v\xi/k}{k}-1}(1-z)^{\frac{h}{k}-1}dz,$$

which upon Lemma 1 and Eq.(6) in (46), we get

$$\Im = x^{\frac{\rho+\tau}{k}+\frac{v\xi}{k^2}-1}\sum_{n=0}^{\infty}\frac{(-c)^n\,k^n}{\Gamma_k(nk+v+k)n!}\left(\frac{a}{2}\right)^{2n+\frac{v}{k}}\sum_{n=0}^{\infty}\frac{1}{\Gamma_k\,(nk+v+k)} \tag{47}$$

$$\times\frac{\Gamma_k\left(\rho+(v\xi/k)+2n\xi\right)\Gamma_k\left(\rho+(v\xi/k)+\tau+\gamma+\varepsilon+2n\xi\right)}{\Gamma_k\left(\rho+(v\xi/k)+\gamma+2n\xi\right)\Gamma_k\left(\rho+(v\xi/k)+\tau+2n\xi+n-nk\right)}$$

$$\times\frac{\Gamma_k\left(g+2n\xi+v\xi/k\right)\Gamma_k\left(h\right)}{\Gamma_k\left(g+h+2n\xi+v\xi/k\right)}(x)^{\frac{2n\xi+n}{k}-n},$$

Using the definition of (11) in the right-hand side of (47), we arrive at the result (44).

Theorem 8. Let $\varepsilon,\tau,\gamma,\rho,\xi\in\mathbb{C},\Re(v)>-1$, and $k\in\mathbb{R}^+$ be such that $\Re(\varepsilon)>0$, $\Re(\rho)>\max\left[\Re(-\varepsilon-\gamma),\Re(\tau-nk+n)\right]$, then the following formula holds true:

(48)

$$B_k\left(\left(D_{-,k}^{\varepsilon,\tau,\gamma}\left(t^{\frac{\varepsilon-\rho}{k}}w_{v,c}^k\left[at^{-\frac{\xi}{k}}\right]\right)\right)(x)\,;\,g,h\right) = x^{\frac{\varepsilon-\rho-\tau}{k}-\frac{v\xi}{k^2}}\Gamma_k\left(h\right)\left(\frac{a}{2}\right)^{\frac{v}{k}}$$

$$\times {}_3\Psi_4^k \left[\begin{array}{c} \left(\rho-\varepsilon+\frac{v\xi}{k}-\tau,2\xi+k-1\right),\left(\rho+\frac{v\xi}{k}+\gamma,2\xi\right), \\ (v+k,k),\left(\rho-\varepsilon+\frac{v\xi}{k},2\xi\right),\left(\rho-\varepsilon+\frac{v\xi}{k}-\tau+\gamma,2\xi\right), \end{array} \right.$$

$$\left. \begin{array}{c} \left(g+\frac{v\xi}{k},2\xi\right) \\ \left(g+h+\frac{v\xi}{k},2\xi\right) \end{array} \middle| \frac{-ca^2x^{\frac{-2\xi+1}{k}-1}}{4} \right].$$

Proof. The proof is parallel to that of Theorem 7. Therefore, we omit the details.

Setting $k = 1$, $c = 1$ in (39), (43), (44) and (48), we obtain the following new formulas as:

Corollary 9. Let $\varepsilon, \tau, \gamma, \rho, \xi \in \mathbb{C}, \Re(v) > -1$, such that $\Re(\varepsilon) > 0$, $\Re(\rho) > \max[0, \Re(\tau - \gamma)]$, then

$$B\left(\left(I_{0+}^{\varepsilon,\tau,\gamma}\left(t^{\rho-1}J_v\left[at^{\xi}\right]\right)\right)(x)\,;\,g,\,h\right) \tag{49}$$
$$= x^{\rho-\tau+v\xi-1}\,(a/2)^v\,\Gamma(h)\,{}_3\Psi_4\left[\begin{array}{c}(\rho+v\xi,2\xi),\\(v+1,1),(\rho+v\xi-\tau,2\xi),\end{array}\right.$$
$$\left.\begin{array}{c}(\rho+v\xi-\tau+\gamma,2\xi),(g+v\xi,2\xi)\\(\rho+v\xi+\varepsilon+\gamma,2\xi),(g+h+v\xi,2\xi)\end{array}\right|\left.\frac{-\left(ax^{\xi}\right)^2}{4}\right].$$

Corollary 10. Let $\varepsilon, \tau, \gamma, \rho, \xi \in \mathbb{C}, \Re(v) > -1$, such that $\Re(\varepsilon) > 0, \Re(\varepsilon+\rho) > \max[-\Re(\tau), -\Re(\gamma)]$, then

$$B\left(\left(I_{-}^{\varepsilon,\tau,\gamma}\left(t^{-\varepsilon-\rho}J_v\left[a\,t^{-\xi}\right]\right)\right)(x)\,;\,g,h\right) \tag{50}$$
$$= x^{-(\varepsilon+\rho+\tau+v\xi)}\,(a/2)^v\,\Gamma(h)\,{}_3\Psi_4\left[\begin{array}{c}(\varepsilon+\rho+\gamma+v\xi,2\xi),\\(v+1,1),(\varepsilon+\rho+v\xi,2\xi),\end{array}\right.$$
$$\left.\begin{array}{c}(\varepsilon+\rho+v\xi+\tau,2\xi),(g+v\xi,2\xi)\\(\rho+2\varepsilon+\gamma+\tau+v\xi,2\xi),\,(g+h+v\xi,2\xi)\end{array}\right|\left.\frac{-c(ax^{-\xi})^2}{4}\right].$$

Corollary 11. Let $\varepsilon, \tau, \gamma, \rho, \xi \in \mathbb{C}, \Re(v) > -1$, be such that $\Re(\varepsilon) > 0$, $\Re(\rho) > \max[0, \Re(-\varepsilon-\tau-\gamma)]$, then

$$B\left(\left(D_{0+}^{\varepsilon,\tau,\gamma}\left(t^{\rho-1}J_v\left[a\,t^{\xi}\right]\right)\right)(x)\,;\,g,\,h\right) \tag{51}$$
$$= x^{\rho+\tau+v\xi-1}\,(a/2)^v\,\Gamma(h)\,{}_3\Psi_4\left[\begin{array}{c}(\rho+v\xi,2\xi),\\(v+1,1),(\rho+v\xi+\gamma,2\xi),\end{array}\right.$$
$$\left.\begin{array}{c}(\rho+v\xi+\tau+\gamma+\varepsilon,2\xi),(g+v\xi,2\xi)\\(\rho+v\xi+\tau,2\xi-k+1),(g+h+v\xi,2\xi)\end{array}\right|\left.\frac{-(ax^{\xi})^2}{4}\right].$$

Corollary 12. Let $\varepsilon, \tau, \gamma, \rho, \xi \in \mathbb{C}, \Re(v) > -1$, be such that $\Re(\varepsilon) > 0$, $\Re(\rho) > \max[\Re(-\varepsilon-\gamma), \Re(\tau-nk+n)]$, then the following formula holds true:

$$B\left(\left(D_{-}^{\varepsilon,\tau,\gamma}\left(t^{\varepsilon-\rho}J_v\left[at^{-\xi}\right]\right)\right)(x)\,;\,g,h\right) \tag{52}$$
$$= x^{\varepsilon-\rho-\tau-v\xi}\,(a/2)^v\,\Gamma(h)\,{}_3\Psi_4\left[\begin{array}{c}(\rho-\varepsilon+v\xi-\tau,2\xi+k-1),\\(v+1,1),(\rho-\varepsilon+v\xi,2\xi),\end{array}\right.$$
$$\left.\begin{array}{c}(\rho+v\xi+\gamma,2\xi),(g+v\xi,2\xi)\\(\rho-\varepsilon+v\xi-\tau+\gamma,2\xi),(g+h+v\xi,2\xi)\end{array}\right|\left.\frac{-(ax^{-\xi})^2}{4}\right].$$

Similarly, If we put $\tau=-\varepsilon$ in equations (39), (43), (44) and (48), Saigo k-fractional calculus operators reduce to k-Riemann-Liouville calculus operators as follows:

Corollary 13. Let $\varepsilon,\gamma,\rho,\xi\in\mathbb{C},\Re(v)>-1,\ k\in\mathbb{R}^{+}$ such that $\Re(\varepsilon)>0$, then

$$B_k\left(\left(I_{0+}^{\varepsilon}\left(t^{\frac{\rho}{k}-1}w_{v,\,c}^{k}\left[a\,t^{\frac{\xi}{k}}\right]\right)\right)(x)\,;\,g,\ h\right)=x^{\frac{\rho+\varepsilon}{k}+\frac{v\xi}{k^2}-1}\,(a/2)^{\frac{v}{k}}\,\Gamma_k\,(h) \tag{53}$$

$$\times{}_2\Psi_3^k\left[\begin{array}{c}\left(\rho+\frac{v\xi}{k},2\xi\right),\left(g+\frac{v\xi}{k},\,2\xi\right)\\(v+k,k)\,,\left(\rho+\frac{v\xi}{k}-\tau,\,2\xi\right),\left(g+h+\frac{v\xi}{k},2\xi\right)\end{array}\left|\frac{-c\,k\,\left(ax^{\xi/k}\right)^2}{4}\right.\right].$$

Corollary 14. Let $\varepsilon,\gamma,\rho,\xi\in\mathbb{C},\Re(v)>-1$, $k\in\mathbb{R}^{+}$ such that $\Re(\varepsilon)>0$, then

$$B_k\left(\left(I_{-,k}^{\varepsilon}\left(t^{\frac{-\varepsilon-\rho}{k}}w_{v,\,c}^{k}\left[a\,t^{-\frac{\xi}{k}}\right]\right)\right)(x)\,;\,g,h\right)=x^{\frac{-\rho}{k}-\frac{v\xi}{k^2}}\,(a/2)^{\frac{v}{k}}\,\Gamma_k\,(h) \tag{54}$$

$$\times{}_2\Psi_3^k\left[\begin{array}{c}\left(\varepsilon+\rho+\frac{v\xi}{k}+\tau,2\xi\right)\left(g+\frac{v\xi}{k},2\xi\right)\\(v+k,k)\,,\left(\varepsilon+\rho+\frac{v\xi}{k},2\xi\right),\,\left(g+h+\frac{v\xi}{k},2\xi\right)\end{array}\left|\frac{-c\,k\,\left(ax^{-\xi/k}\right)^2}{4}\right.\right].$$

Corollary 15. Let $\varepsilon,\gamma,\rho,\xi\in\mathbb{C},\Re(v)>-1,\ k\in\mathbb{R}^{+}$ be such that $\Re(\varepsilon)>0$, then

$$B_k\left(\left(D_{0+,k}^{\varepsilon,}\left(t^{\frac{\rho}{k}-1}w_{v,\,c}^{k}\left[a\,t^{\frac{\xi}{k}}\right]\right)\right)(x)\,;\,g,\ h\right)=x^{\frac{\rho-\varepsilon}{k}+\frac{v\xi}{k^2}-1}\,(a/2)^{\frac{v}{k}}\,\Gamma_k\,(h) \tag{55}$$

$$\times{}_2\Psi_3^k\left[\begin{array}{c}\left(\rho+\frac{v\xi}{k},2\xi\right),\left(g+\frac{v\xi}{k},2\xi\right)\\(v+k,k)\,,\left(\rho+\frac{v\xi}{k}-\varepsilon,\,2\xi-k+1\right),\left(g+h+\frac{v\xi}{k},2\xi\right)\end{array}\left|\frac{-c\,a^2x^{\frac{2\xi+1}{k}-1}}{4}\right.\right].$$

Corollary 16. Let $\varepsilon,\gamma,\rho,\xi\in\mathbb{C},\Re(v)>-1$, and $k\in\mathbb{R}^{+}$ be such that $\Re(\varepsilon)>0$, then

$$B_k\left(\left(D_{-,k}^{\varepsilon}\left(t^{\frac{\varepsilon-\rho}{k}}w_{v,c}^{k}\left[at^{-\frac{\xi}{k}}\right]\right)\right)(x)\,;\,g,h\right)=x^{\frac{2\varepsilon-\rho}{k}-\frac{v\xi}{k^2}}\,(a/2)^{\frac{v}{k}}\,\Gamma_k\,(h) \tag{56}$$

$$\times{}_2\Psi_3^k\left[\begin{array}{c}\left(\rho+\frac{v\xi}{k},\,2\xi+k-1\right),\,\left(g+\frac{v\xi}{k},2\xi\right)\\(v+k,k)\,,\left(\rho-\varepsilon+\frac{v\xi}{k},2\xi\right),\left(g+h+\frac{v\xi}{k},2\xi\right)\end{array}\left|\frac{-ca^2x^{\frac{-2\xi+1}{k}-1}}{4}\right.\right].$$

5. Concluding remark

The generalized k-fractional calculus operators have advantage that it generalizes Saigo's fractional integral and derivative operators, therefore, many authors called this a general operator. So, we conclude this paper by emphasizing that many other interesting image formulas can be derived as the specific cases of our leading results Theorems 1 to 4, involving familiar k-fractional integral and derivative operators as above said. Some special cases of k-fractional calculus involving k-Bessel function have been explored in the literature by a numeral of authors ([13, 21, 23, 26]) with different arguments. Therefore, results presented in this paper are easily converted in terms of a comparable type of novel interesting integrals with diverse arguments after various suitable parametric replacements.

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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