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MODULAR TRANSFORMATION FORMULAE COMING FROM GENERALIZED NON-HOLOMORPHIC EISENSTEIN SERIES AND INFINITE SERIES IDENTITIES

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Abstract. B. C. Berndt has found modular transformation formulae for a large class of functions coming from generalized Eisenstein series. Using those formulae, he established a lot of infinite series identities, some of which explain many infinite series identities given by Ramanujan. Continuing his work, the author proved a lot of new infinite series identities. Moreover, recently the author found transformation formulae for a class of functions coming from generalized non-holomorphic Eisenstein series. In this paper, using those formulae, we evaluate a few new infinite series identities which generalize the author's previous results.

1. Introduction

B. C. Berndt has found modular transformation formulae for a large class of functions which stem from generalized Eisenstein series [2], [3]. Using those formulae, he established a lot of infinite series identities [2], [3], some of which explain many infinite series identities given by Ramanujan. Continuing his work, the author proved a lot of new infinite series identities [4]. For example, the following Theorem 1.1 shows a generalized form of Ramanujan's formula([6], pp. 319-320) which is famous for the arithmetic series relation for $\zeta(2n+1)$, where $\zeta(s)$ is the Riemann zeta function and n is a positive integer. Let $B_n(x)$ denote the n-th Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\ (|t|<2\pi).$$

The n-th Bernoulli number B_n, $n\geq 0$, is defined by $B_n = B_n(0)$. For a real number x, $[x]$ denotes the greatest integer less than or equal to x and $\{x\} := x-[x]$. Put $\bar{B}_n(x)=B_n(\{x\}),\ n\geq 0$.

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Theorem 1.1. [4]. *Let α and β be positive real numbers with $\alpha\beta = \pi^2$. Let c denote a positive integer. Then, for any integer n,*

$$\alpha^{-n}\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2k(\alpha-i\pi)/c}-1} = (-\beta)^{-n}\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2k(\beta+i\pi)/c}-1}$$
$$-2^{2n}\sum_{j=1}^{c}\sum_{k=0}^{2n+2}\frac{B_k(j/c)\bar{B}_{2n+2-k}(j/c)}{k!(2n+2-k)!}\alpha^{n-k+1}(-i\pi)^k + I_0(n),$$

where

$$I_0(n) := \begin{cases} \frac{1}{2}\left((-\beta)^{-n} - \alpha^{-n}\right)\zeta(1+2n), & if\ n \neq 0, \\ -\frac{1}{4}(\log\beta - \log\alpha) + i\frac{\pi}{4}, & if\ n = 0. \end{cases}$$

If $c = 1$ and $n > 0$, then Theorem 1.1 gives Ramanujan's formula([6], pp. 319-320). Let $n = -N$ for any positive integer N. Then the sum containing Bernoulli polynomials is valid only for $N = 1$. Thus we have the following formula.

$$\alpha^{N}\sum_{k=1}^{\infty}\frac{k^{2N-1}}{e^{2k(\alpha-i\pi)/c}-1} + \frac{\alpha^N}{2}\zeta(1-2N)$$
$$= (-\beta)^{N}\sum_{k=1}^{\infty}\frac{k^{2N-1}}{e^{2k(\beta+i\pi)/c}-1} + \frac{(-\beta)^N}{2}\zeta(1-2N) - \nu_N(c), \tag{1.1}$$

where

$$\nu_N(c) = \begin{cases} \frac{c}{4}, & N = 1, \\ 0, & N \geq 2. \end{cases}$$

Recently the author found modular transformation formulae for a class of functions which stem from generalized non-holomorphic Eisenstein series [5]. In this paper, using those formulae, we obtain a class of new identities about infinite series. In some sense, the results contain the equation (1.1). It is noteworthy that several identities have symmetric relation for given values.

2. Notations

Unless otherwise stated, we choose the branch of the argument for a complex z by $-\pi \leq \arg z < \pi$. For non-negative integer n, let $(x)_n$ denote the rising factorial defined by

$$(x)_n := x(x+1)\cdots(x+n-1) \text{ for } n > 0,\ (x)_0 = 1.$$

Let $\Gamma(s)$ denote the gamma function. Then we see that

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}.$$

Let ${}_2F_1(\alpha,\beta;\gamma;z)$ be a hypergeometric function defined by

$$ {}_2F_1(\alpha,\beta;\gamma;z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n. $$

The function $\frac{1}{\Gamma(\gamma)}\ {}_2F_1(\alpha,\beta;\gamma;z)$ can be analytically continued to all $\alpha,\ \beta,\ \gamma \in \mathbb{C}$ with $|z| < 1$([1], p. 65). The confluent hypergeometric function of the first kind ${}_1F_1(\alpha;\beta;z)$ is defined by

$$ {}_1F_1(\alpha;\beta;z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n n!} z^n $$

and the confluent hypergeometric function of the second kind $U(\alpha,\beta,z)$ is defined to be

$$ U(\alpha,\beta,z) := \frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}\ {}_1F_1(\alpha;\beta;z) + \frac{\Gamma(\beta-1)}{\Gamma(\alpha)} z^{1-\beta}\ {}_1F_1(1+\alpha-\beta;2-\beta;z). $$

We see that $U(\alpha,\beta,z)$ can be analytically continued to all values of $\alpha,\ \beta$ and z real or complex [7]. Let $r = (r_1, r_2)$ and $h = (h_1, h_2)$ for real $r_i,\ h_i,\ i = 1, 2$. For brevity, let $e(w) := e^{2\pi i w}$. Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, the upper half-plane. For $\tau \in \mathbb{H}$ and $s_1,\ s_2 \in \mathbb{C}$, define

$$ \mathcal{A}(\tau,s_1,s_2;r,h) := \sum_{m+r_1>0}\sum_{n-h_2>0} \frac{e\left(mh_1 + ((m+r_1)\tau + r_2)(n-h_2)\right)}{(n-h_2)^{1-s}} \times U(s_2;s;4\pi(m+r_1)(n-h_2)\mathrm{Im}(\tau)) $$

and

$$ \bar{\mathcal{A}}(\tau,s_1,s_2;r,h) := \sum_{m+r_1>0}\sum_{n+h_2>0} \frac{e\left(mh_1 - ((m+r_1)\bar{\tau} + r_2)(n+h_2)\right)}{(n+h_2)^{1-s}} \times U(s_1;s;4\pi(m+r_1)(n+h_2)\mathrm{Im}(\tau)), $$

where $s = s_1 + s_2$. Two functions $\mathcal{A}(\tau,s_1,s_2;r,h)$ and $\bar{\mathcal{A}}(\tau,s_1,s_2;r,h)$ are well-defined for all $s_1,\ s_2 \in \mathbb{C}$. Let

$$ \mathcal{H}(\tau,s_1,s_2;r,h) := \mathcal{A}(\tau,s_1,s_2;r,h) + e^{\pi i s}\mathcal{A}(\tau,s_1,s_2;-r,-h) $$

and

$$ \bar{\mathcal{H}}(\tau,s_1,s_2;r,h) := \bar{\mathcal{A}}(\tau,s_1,s_2;r,h) + e^{\pi i s}\bar{\mathcal{A}}(\tau,s_1,s_2;-r,-h). $$

Let

$$ \mathbf{H}(\tau,\bar{\tau},s_1,s_2;r,h) := \frac{1}{\Gamma(s_1)}\mathcal{H}(\tau,s_1,s_2;r,h) + \frac{1}{\Gamma(s_2)}\bar{\mathcal{H}}(\tau,s_1,s_2;r,h). $$

For real x, α and $t \in \mathbb{C}$ with Re $t > 1$, let

$$ \psi(x,\alpha,t) := \sum_{n+\alpha>0} \frac{e(nx)}{(n+\alpha)^t} $$

and

$$ \Psi(x,\alpha,t) := \psi(x,\alpha,t) + e^{\pi i t}\psi(-x,-\alpha,t), $$

$$\Psi_{-1}(x,\alpha,t) := \psi(x,\alpha,t-1) + e^{\pi i t}\psi(-x,-\alpha,t-1).$$

Let λ denote the characteristic function of the integers. Let

$$V\tau = \frac{a\tau+b}{c\tau+d}$$

denote a modular transformation with $c>0$ for $\tau \in \mathbb{C}$. We define R and H as

$$R = (R_1,R_2) = (ar_1+cr_2, br_1+dr_2)$$

and

$$H = (H_1,H_2) = (dh_1-bh_2, -ch_1+ah_2).$$

The following theorem is the modular transformation formulae which originally come from generalized non-holomorphic Eisenstein series. We shall use this formulae to obtain our main results.

Theorem 2.1. [5]. *Let $Q = \{\tau \in \mathbb{H} \mid \operatorname{Re}\tau > -d/c\}$ and $\varrho = c\{R_2\}-d\{R_1\}$. Let $s_1, s_2 \in \mathbb{C}$ with $s = s_1+s_2$. We assume that if $h_2 \in \mathbb{Z}$ and $s_1 \notin \mathbb{Z}$, then s is not an integer less than or equal to 1. Then, for $\tau \in Q$,*

$$\begin{aligned}
z^{-s_1}\bar{z}^{-s_2}\mathbf{H}(V\tau, V\bar{\tau}, s_1, s_2; r, h) &= \mathbf{H}(\tau,\bar{\tau},s_1,s_2;R,H)\\
&+\lambda(R_1)e(-R_1H_1)(2\pi i)^{-s}e^{-\pi i s_2}\Psi(-H_2,-R_2,s)\\
&-\lambda(r_1)e(-r_1h_1)(2\pi i)^{-s}e^{\pi i s_1}z^{-s_1}\bar{z}^{-s_2}\Psi(h_2,r_2,s)\\
&+\lambda(H_2)(4\pi\mathrm{Im}(\tau))^{1-s}\frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)}\Psi_{-1}(H_1,R_1,s)\\
&-\lambda(h_2)(4\pi\mathrm{Im}(\tau))^{1-s}\frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)}z^{s_2-1}\bar{z}^{s_1-1}\Psi_{-1}(h_1,r_1,s)\\
&+\frac{(2\pi i)^{-s}e^{-\pi i s_2}}{\Gamma(s_1)\Gamma(s_2)}\mathbf{L}(\tau,\bar{\tau},s_1,s_2;R,H),
\end{aligned}$$

where $z = c\tau+d$ and

$$\begin{aligned}
&\mathbf{L}(\tau,\bar{\tau},s_1,s_2;R,H)\\
&=\sum_{j=1}^{c} e(-H_1(j+[R_1]-c)-H_2([R_2]+1+[(jd+\varrho)/c]-d))\\
&\quad\times\int_0^1 v^{s_1-1}(1-v)^{s_2-1}\int_C u^{s-1}\frac{e^{-(zv+\bar{z}(1-v))(j-\{R_1\})u/c}}{e^{-(zv+\bar{z}(1-v))u}-e(cH_1+dH_2)}\\
&\qquad\times\frac{e^{\{(jd+\varrho)/c\}u}}{e^u-e(-H_2)}\,dudv,
\end{aligned}$$

where C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u=0$ is the only zero of

$$(e^{-(zv+\bar{z}(1-v))u}-e(cH_1+dH_2))(e^u-e(-H_2))$$

lying inside the loop, and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of u^s with $0<\arg u<2\pi$.

Remark 2.2. Let $s = s_1 + s_2 = -n$ for an integer n. By the residue theorem, we find that

$$\int_C u^{-n-1} \frac{e^{-(zv+\bar{z}(1-v))(j-\{R_1\})u/c}}{e^{-(zv+\bar{z}(1-v))u}-1} \frac{e^{\{(\varrho+jd)/c\}u}}{e^u-1}\, dudv$$
$$= 2\pi i \sum_{k=0}^{n+2} \frac{B_k((j-\{R_1\})/c)\bar{B}_{n+2-k}((\varrho+jd)/c)}{k!(n+2-k)!}(-(z-\bar{z})-\bar{z})^{k-1}.$$

Thus, for $s = s_1 + s_2 = -n$,

$$\int_0^1 v^{s_1-1}(1-v)^{s_2-1} \int_C u^{-n-1} \frac{e^{-(zv+\bar{z}(1-v))(j-\{R_1\})u/c}}{e^{-(zv+\bar{z}(1-v))u}-1} \frac{e^{\{(\varrho+jd)/c\}u}}{e^u-1}\, dudv$$
$$= 2\pi i \sum_{k=0}^{n+2} \frac{B_k((j-\{R_1\})/c)\bar{B}_{n+2-k}((\varrho+jd)/c)}{k!(n+2-k)!}(-z)^{k-1}$$
$$\times \int_0^1 (1-v)^{s_1-1} v^{s_2-1} \left(1 - \frac{z-\bar{z}}{z}v\right)^{k-1} dv$$
$$= 2\pi i \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(-n)} \sum_{k=0}^{n+2} \frac{B_k((j-\{R_1\})/c)\bar{B}_{n+2-k}((\varrho+jd)/c)}{k!(n+2-k)!}(-z)^{k-1}$$
$$\times\ {}_2F_1\left(s_2, 1-k; -n; \frac{z-\bar{z}}{z}\right).$$

We now see that

$$\frac{\mathbf{L}(\tau,\bar{\tau},s_1,s_2,;R,H)}{\Gamma(s_1)\Gamma(s_2)}$$

vanishes for $s = -n > 2$.

3. A class of infinite series identities

In this section, we obtain a class of new infinite series identities from Theorem 2.1. The Eulerian number $E(n,j)$ is defined to be the number of permutations of numbers from 1 to n such that exactly j numbers are greater than the previous elements. Note that $E(n,j) = E(n,n-j-1)$. For any integer n, the polylogarithm function $\mathrm{Li}_n(z)$ is defined by

$$\mathrm{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n},$$

where $z \in \mathbb{C}$ and $|z| < 1$. For $n > 0$, we see that

$$\mathrm{Li}_{-n}(z) = \frac{1}{(1-z)^{n+1}} \sum_{j=0}^{n-1} E(n,j) z^{n-j}.$$

From now on, we set

$$V\tau = \frac{\tau - 1}{c\tau - c + 1}.$$

For brevity, let

$$\hat{k} := \left[\frac{k-1}{2}\right] \quad \text{and} \quad \mu_k = \begin{cases} 0, & k \text{ odd}, \\ 1, & k \text{ even} \end{cases}$$

for any integer $k \geq 1$.

Theorem 3.1. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any integers $B \geq 0$ and $N \geq 1$,*

$$\begin{aligned}
&(-1)^B \alpha^N \sum_{k=1}^{B} \binom{B}{k} \frac{(-2\alpha/c)^k}{(2N+k-1)!} \sum_{j=0}^{\hat{k}}{}' E(k, \hat{k}-j) \\
&\qquad \times \sum_{m=1}^{\infty} \frac{\cosh(m(2j+\mu_k)(\alpha - i\pi)/c)}{m^{1-2N-k} \sinh^{k+1}(m(\alpha - i\pi)/c)} \\
&+ \frac{(-1)^B \alpha^N}{(2N-1)!} \left(\sum_{m=1}^{\infty} \frac{m^{2N-1}}{e^{2m(\alpha - i\pi)/c} - 1} + \frac{1}{2}\zeta(1-2N) \right) \\
&= (-\beta)^N \sum_{k=1}^{B} \binom{B}{k} \frac{(-2\beta/c)^k}{(2N+k-1)!} \sum_{j=0}^{\hat{k}}{}' E(k, \hat{k}-j) \\
&\qquad \times \sum_{m=1}^{\infty} \frac{\cosh(m(2j+\mu_k)(\beta + i\pi)/c)}{m^{1-2N-k} \sinh^{k+1}(m(\beta + i\pi)/c)} \\
&\quad + \frac{(-\beta)^N}{(2N-1)!} \left(\sum_{m=1}^{\infty} \frac{m^{2N-1}}{e^{2m(\beta + i\pi)/c} - 1} + \frac{1}{2}\zeta(1-2N) \right) - \delta_N(B, c),
\end{aligned}$$

where $'$ means that if k is odd, then the term with $j = 0$ is multiplied by $\frac{1}{2}$ and

$$\delta_N(B, c) = \begin{cases} \dfrac{c}{8(B+1)}(1 + (-1)^B), & \text{if } N = 1, \\ 0, & \text{if } N \geq 2. \end{cases}$$

Proof. Let $s_1 = A \geq 1$, $s_2 = -B \leq 0$ and $s = 2N$, $N \geq 1$. Here A, B and N are integers. Put $r = h = (0,0)$ and $z = c\tau - c + 1 = \frac{\pi}{\alpha} i$ in Theorem 2.1. For $s_2 = -B \leq 0$, $\frac{1}{\Gamma(s_2)} = \frac{1}{\Gamma(-B)} = 0$. Then

$$\begin{aligned}
&\mathbf{H}(V\tau, V\bar{\tau}, s_1, s_2; r, h) \\
&= \frac{1}{\Gamma(A)} \big(\mathcal{A}(V\tau, s_1, s_2; r, h) + e^{\pi i s} \mathcal{A}(V\tau, s_1, s_2; -r, -h) \big) \\
&= \frac{2}{(A-1)!} \mathcal{A}(V\tau, A, -B; (0,0), (0,0)) \\
&= \frac{2}{(A-1)!} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{e(mnV\tau)}{n^{1-2N}} U(-B; 2N; 4\pi mn \mathrm{Im}(V\tau)).
\end{aligned}$$

Let $w := 4\pi mn \mathrm{Im}(V\tau) = 4mn\alpha/c$. By definition, we see that

$$U(-B; 2N; w) = \frac{\Gamma(1-2N)}{\Gamma(1-A)} \sum_{k=0}^{\infty} \frac{(-B)_k}{(2N)_k} \frac{w^k}{k!}.$$

Observe that

$$\begin{aligned}\frac{\Gamma(1-2N)}{\Gamma(1-A)} &= \frac{\Gamma(1-A+B)}{\Gamma(1-A)}\\ &= (1-A)_B\\ &= (1-A)(1-A+1)\cdots(1-A+B-1)\\ &= (-1)^B(A-1)(A-2)\cdots(A-B)\\ &= (-1)^B\frac{(A-1)!}{(2N-1)!},\end{aligned}$$

$$\begin{aligned}(-B)_k &= (-B)(-B+1)\cdots(-B+k-1)\\ &= (-1)^k B(B-1)\cdots(B-k+1)\\ &= \begin{cases}(-1)^k\frac{B!}{(B-k)!}, & k\le B,\\ 0, & k>B\end{cases}\end{aligned}$$

and

$$(2N)_k = \frac{\Gamma(2N+k)}{\Gamma(2N)} = \frac{(2N-1+k)!}{(2N-1)!}.$$

Then we have

$$\begin{aligned}U(-B,2N,w) &= (-1)^B\frac{(A-1)!}{(2N-1)!}\sum_{k=0}^{B}(-1)^k\frac{B!}{(B-k)!}\frac{(2N-1)!}{(2N-1+k)!}\frac{w^k}{k!}\\ &= (-1)^B(A-1)!\sum_{k=0}^{B}\binom{B}{k}\frac{(-w)^k}{(2N-1+k)!}. \qquad (3.1)\end{aligned}$$

Thus, employing (3.1), we obtain that

$$\begin{aligned}\mathbf{H}(V\tau,V\bar{\tau},s_1,s_2;r,h) &= 2(-1)^B\sum_{k=0}^{B}\binom{B}{k}\frac{(-4\alpha/c)^k}{(2N+k-1)!}\\ &\quad\times\sum_{n=1}^{\infty}\frac{1}{n^{1-2N-k}}\sum_{m=1}^{\infty}\frac{(e^{-2n(\alpha-i\pi)/c})^m}{m^{-k}}\\ &= 2(-1)^B\sum_{k=0}^{B}\binom{B}{k}\frac{(-4\alpha/c)^k}{(2N+k-1)!}\\ &\quad\times\sum_{n=1}^{\infty}\frac{\mathrm{Li}_{-k}(e^{-2n(\alpha-i\pi)/c})}{n^{1-2N-k}}. \qquad (3.2)\end{aligned}$$

Let $q = n(\alpha - i\pi)/c$ and $k > 0$. Short calculations show that

$$\begin{aligned}\mathrm{Li}_{-k}(e^{-2q}) &= \frac{1}{(1-e^{-2q})^{k+1}}\sum_{j=0}^{k-1}E(k,j)e^{-2q(k-j)}\\ &= \frac{e^{q(k+1)}}{(e^{q}-e^{-q})^{k+1}}\sum_{j=0}^{k-1}E(k,j)e^{-2q(k-j)}\end{aligned}$$

$$= \frac{2^{-k-1}}{\sinh^{k+1}(q)} \sum_{j=0}^{k-1} E(k,j) e^{-q(k-2j-1)}. \tag{3.3}$$

Using $E(k,j) = E(k,k-j-1)$, we find that, for k even,

$$\begin{aligned}
\sum_{j=0}^{k-1} E(k,j) e^{-q(k-2j-1)} &= \sum_{j=0}^{k/2-1} E(k,j)(e^{-q(k-2j-1)} + e^{-q(-k+2j+1)}) \\
&= \sum_{j=0}^{k/2-1} E(k,k/2-1-j)(e^{-q(2j+1)} + e^{q(2j+1)}) \\
&= 2\sum_{j=0}^{k/2-1} E(k,k/2-1-j)\cosh(q(2j+1)) \qquad (3.4)
\end{aligned}$$

and, for k odd,

$$\begin{aligned}
\sum_{j=0}^{k-1} E(k,j) e^{-q(k-2j-1)} &= \sum_{j=0}^{(k-3)/2} E(k,j)(e^{-q(k-2j-1)} + e^{-q(-k+2j+1)}) \\
&\qquad + E(k,(k-1)/2) \\
&= {\sum_{j=0}^{(k-1)/2}}' E(k,(k-1)/2-j)(e^{-q(2j)} + e^{q(2j)}) \\
&= 2{\sum_{j=0}^{(k-1)/2}}' E(k,(k-1)/2-j)\cosh(q(2j)), \qquad (3.5)
\end{aligned}$$

where $'$ means that if k is odd, then the term with $j = 0$ is multiplied by $\frac{1}{2}$. Thus, putting (3.4), (3.5) in (3.3), we have that

$$\mathrm{Li}_{-k}(e^{-2q}) = \frac{2^{-k}}{\sinh^{k+1}(q)} {\sum_{j=0}^{\hat{k}}}' E(k,\hat{k}-j)\cosh(q(2j+\mu_k)). \tag{3.6}$$

For $k = 0$, it is easy to see that

$$\mathrm{Li}_0(e^{-2q}) = \sum_{m=1}^{\infty} e^{-2qm} = \frac{1}{e^{2q}-1}. \tag{3.7}$$

Now substitute (3.6), (3.7) in (3.2) to obtain that

$$\begin{aligned}
\mathbf{H}(V\tau, V\bar{\tau}, s_1, s_2; r, h) &= 2(-1)^B \sum_{k=0}^{B} \binom{B}{k} \frac{(-2\alpha/c)^k}{(2N+k-1)!} {\sum_{j=0}^{\hat{k}}}' E(k,\hat{k}-j) \\
&\qquad \times \sum_{n=1}^{\infty} \frac{\cosh(n(\alpha-i\pi)(2j+\mu_k))/c}{n^{1-2N-k}\sinh^{k+1}(n(\alpha-i\pi)/c} \\
&\quad + \frac{2(-1)^B}{(2N-1)!} \sum_{n=1}^{\infty} \frac{n^{2N-1}}{e^{2n(\alpha-i\pi)/c}-1}. \qquad (3.8)
\end{aligned}$$

Similarly we see that

$$\mathbf{H}(\tau,\bar\tau,s_1,s_2;R,H) = 2(-1)^B \sum_{k=0}^{B} \binom{B}{k} \frac{(-2\beta/c)^k}{(2N+k-1)!} \sideset{}{'}\sum_{j=0}^{\hat k} E(k,\hat k - j) \times \sum_{n=1}^{\infty} \frac{\cosh(n(\beta+i\pi)(2j+\mu_k))/c}{n^{1-2N-k}\sinh^{k+1}(n(\beta+i\pi)/c} + \frac{2(-1)^B}{(2N-1)!}\sum_{n=1}^{\infty}\frac{n^{2N-1}}{e^{2n(\beta+i\pi)/c}-1}. \tag{3.9}$$

Since $r_1 = R_1 = h_2 = H_2 = 0$, $\lambda(r_1) = \lambda(R_1) = \lambda(h_2) = \lambda(H_2) = 1$. It is easy to see that, using $\zeta(1-2N) = (-1)^N(2N-1)!2^{1-2N}\pi^{-2N}\zeta(2N)$,

$$\begin{aligned}\frac{e^{-\pi i s_2}}{(2\pi i)^s}\Psi(-H_2,-R_2,s) &= \frac{(-1)^{B+N}}{(2\pi)^{2N}}(\psi(0,0,2N)+e^{2\pi i N}\psi(0,0,2N)) \\ &= (-1)^{B+N}(2\pi)^{-2N}\sum_{k=1}^{\infty}\frac{2}{k^{2N}} \\ &= (-1)^{B+N}2^{1-2N}\pi^{-2N}\zeta(2N) \\ &= \frac{(-1)^B}{(2N-1)!}\zeta(1-2N)\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\frac{e^{\pi i s_1}}{(2\pi i)^s}z^{-s_1}\bar z^{-s_2}\Psi(h_2,r_2,s) &= \frac{e^{i\pi A}}{(2\pi i)^{2N}}\left(\frac{\pi i}{\alpha}\right)^{-A}\left(-\frac{\pi i}{\alpha}\right)^{B}\Psi(0,0,2N) \\ &= (-1)^N 2^{1-2N}\pi^{-2N}\left(\frac{\pi i}{\alpha}\right)^{-2N}\zeta(2N) \\ &= 2^{1-2N}\beta^{-2N}\zeta(2N) \\ &= \frac{\alpha^N(-\beta)^{-N}}{(2N-1)!}\zeta(1-2N).\end{aligned} \tag{3.11}$$

Recall $\frac{1}{\Gamma(s_2)} = \frac{1}{\Gamma(-B)} = 0$ to obtain that

$$(4\pi \mathrm{Im}(\tau))^{1-s}\frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)}\Psi_{-1}(H_1,R_1,s) = 0 \tag{3.12}$$

and

$$(4\pi \mathrm{Im}(\tau))^{1-s}\frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)}z^{s_2-1}\bar z^{s_1-1}\Psi_{-1}(h_1,r_1,s) = 0. \tag{3.13}$$

To evaluate $\frac{L(\tau,\bar\tau,s_1,s_2;R,H)}{\Gamma(s_1)\Gamma(s_2)}$, we employ Remark 2.2 which says

$$\frac{L(\tau,\bar\tau,s_1,s_2;R,H)}{\Gamma(s_1)\Gamma(s_2)} = \frac{2\pi i}{\Gamma(s)}\sum_{j=1}^{c}\sum_{k=0}^{-s+2}\frac{B_k(j/c)\bar B_{-s+2-k}(j(1-c)/c)}{k!(-s+2-k)!} \times \left(-\frac{i\pi}{\alpha}\right)^{k-1} {}_2F_1(s_2,1-k;s;2).$$

Since $s = 2N \geq 2$, the sum over k in the above equation is valid only for $s = 2$. Then

$$\begin{aligned} 2\pi i \sum_{j=1}^{c} \left(-\frac{i\pi}{\alpha}\right)^{-1} {}_2F_1(-B, 1; 2; 2) &= -2c\alpha \sum_{k=0}^{\infty} \frac{(-B)_k (1)_k}{(2)_k k!} 2^k \\ &= -2c\alpha \sum_{k=0}^{\infty} \frac{(-B)_k}{(2)_k} 2^k. \end{aligned}$$

Observe that

$$(-B)_k = \begin{cases} (-1)^k \dfrac{B!}{(B-k)!}, & k \leq B, \\ 0, & k > B \end{cases}$$

and

$$(2)_k = 2(2+1)\cdots(2+k-1) = (k+1)!.$$

We easily deduce that

$$\sum_{k=0}^{\infty} \frac{(-B)_k}{(2)_k} 2^k = \sum_{k=0}^{B} \frac{B!}{(B-k)!(k+1)!} (-2)^k = \sum_{k=0}^{B} \binom{B}{k} \frac{(-2)^k}{k+1}.$$

Now consider that

$$(x+1)^B = \sum_{k=0}^{B} \binom{B}{k} x^k. \tag{3.14}$$

Taking integral for both sides of (3.14), we find that

$$x \sum_{k=0}^{B} \binom{B}{k} \frac{x^k}{k+1} = \frac{1}{B+1}(x+1)^{B+1} - \frac{1}{B+1}. \tag{3.15}$$

Putting $x = -2$ in (3.15),

$$\sum_{k=0}^{B} \binom{B}{k} \frac{(-2)^k}{k+1} = \frac{1}{2(B+1)}(1 + (-1)^B).$$

Finally we obtain that, for s=2,

$$\begin{aligned} \frac{(2\pi i)^{-s} e^{-\pi i s_2}}{\Gamma(s_1)\Gamma(s_2)} L(\tau, \bar{\tau}, s_1, s_2; R, H) &= \frac{e^{\pi i B}}{(2\pi i)^2} \frac{-2c\alpha}{2(B+1)} (1 + (-1)^B) \\ &= \frac{c}{4\beta(B+1)} (1 + (-1)^B). \end{aligned} \tag{3.16}$$

Combining (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.16), we complete the proof. □

If $B = 0$ in Theorem 3.1, then Theorem 1.1 with $n < 0$, i,e., the equation (1.1) follows. We can divide the identity in Theorem 3.1 into two identities according to the parity of B. Let B be even. Applying Theorem 1.1, we find that

$$\begin{aligned}
&\alpha^N \sum_{k=1}^{B} \binom{B}{k} \frac{(-2\alpha/c)^k}{(2N+k-1)!} \sum_{j=0}^{\hat{k}}{}' E(k, \hat{k}-j) \\
&\qquad\qquad \times \sum_{m=1}^{\infty} \frac{\cosh(m(2j+\mu_k)(\alpha - i\pi)/c)}{m^{1-2N-k} \sinh^{k+1}(m(\alpha - i\pi)/c)} \\
&= (-\beta)^N \sum_{k=1}^{B} \binom{B}{k} \frac{(-2\beta/c)^k}{(2N+k-1)!} \sum_{j=0}^{\hat{k}}{}' E(k, \hat{k}-j) \\
&\qquad\qquad \times \sum_{m=1}^{\infty} \frac{\cosh(m(2j+\mu_k)(\beta + i\pi)/c)}{m^{1-2N-k} \sinh^{k+1}(m(\beta + i\pi)/c)} \\
&\quad + \frac{2^{-2N}}{(2N-1)!} \sum_{j=1}^{c} \sum_{k=0}^{-2N+2} \frac{B_k(j/c)\bar{B}_{2N+2-2k}(j/c)}{k!(2N+2-k)!} \frac{(-i\pi)^k}{\alpha^{N+k-1}} - \delta_N(B, c).
\end{aligned}$$

Here, the sum of Bernoulli polynomials, i.e.,

$$\frac{2^{-2N}}{(2N-1)!} \sum_{j=1}^{c} \sum_{k=0}^{-2N+2} \frac{B_k(j/c)\bar{B}_{2N+2-2k}(j/c)}{k!(2N+2-k)!} \alpha^{-N-k+1} (-i\pi)^k$$

is valid only for $N = 1$ and it is $\frac{c}{4}$. Thus we obtain the following theorem.

Theorem 3.2. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any even integr $B \geq 0$ and any integer $N \geq 1$,*

$$\begin{aligned}
&\alpha^N \sum_{k=1}^{B} \binom{B}{k} \frac{(-2\alpha/c)^k}{(2N+k-1)!} \sum_{j=0}^{\hat{k}}{}' E(k, \hat{k}-j) \\
&\qquad\qquad \times \sum_{m=1}^{\infty} \frac{\cosh(m(2j+\mu_k)(\alpha - i\pi)/c)}{m^{1-2N-k} \sinh^{k+1}(m(\alpha - i\pi)/c)} \\
&= (-\beta)^N \sum_{k=1}^{B} \binom{B}{k} \frac{(-2\beta/c)^k}{(2N+k-1)!} \sum_{j=0}^{\hat{k}}{}' E(k, \hat{k}-j) \\
&\qquad\qquad \times \sum_{m=1}^{\infty} \frac{\cosh(m(2j+\mu_k)(\beta + i\pi)/c)}{m^{1-2N-k} \sinh^{k+1}(m(\beta + i\pi)/c)} \\
&\quad -\delta_N(B, c) + \nu_N(c),
\end{aligned}$$

where

$$\nu_N(c) = \begin{cases} \dfrac{c}{4}, & N = 1, \\ 0, & N \geq 2. \end{cases}$$

For B odd, applying Theorem 1.1 again, we also have the following theorem.

Theorem 3.3. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any odd integer $B \geq 1$ and any integer $N \geq 1$,*

$$\begin{aligned}
&\alpha^N \sum_{k=1}^{B} \binom{B}{k} \frac{(-2\alpha/c)^k}{(2N+k-1)!} {\sum_{j=0}^{\hat{k}}}{}' E(k,\hat{k}-j) \\
&\qquad\qquad\qquad \times \sum_{m=1}^{\infty} \frac{\cosh(m(2j+\mu_k)(\alpha-i\pi)/c)}{m^{1-2N-k}\sinh^{k+1}(m(\alpha-i\pi)/c)} \\
&= -(-\beta)^N \sum_{k=1}^{B} \binom{B}{k} \frac{(-2\beta/c)^k}{(2N+k-1)!} {\sum_{j=0}^{\hat{k}}}{}' E(k,\hat{k}-j) \\
&\qquad\qquad\qquad \times \sum_{m=1}^{\infty} \frac{\cosh(m(2j+\mu_k)(\beta+i\pi)/c)}{m^{1-2N-k}\sinh^{k+1}(m(\beta+i\pi)/c)} \\
&\quad -\frac{2(-\beta)^N}{(2N-1)!}\left(\sum_{m=1}^{\infty} \frac{m^{2N-1}}{e^{2m(\beta+i\pi)/c}-1} + \frac{1}{2}\zeta(1-2N)\right) - \nu_N(c).
\end{aligned}$$

Corollary 3.4. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any integers $B \geq 0$ and $N \geq 1$, which have the same parity,*

$$\begin{aligned}
&\alpha^N \sum_{k=1}^{B} \binom{B}{k} \frac{(-2\alpha)^k}{(2N+k-1)!} {\sum_{j=0}^{\hat{k}}}{}' E(k,\hat{k}-j) \sum_{m=1}^{\infty} \frac{\cosh(m(2j+\mu_k)\alpha)}{m^{1-2N-k}\sinh^{k+1}(m\alpha)} \\
&+\frac{\alpha^N}{(2N-1)!}\left(\sum_{m=1}^{\infty} \frac{m^{2N-1}}{e^{2m\alpha}-1} + \frac{1}{2}\zeta(1-2N)\right) \\
&= \beta^N \sum_{k=1}^{B} \binom{B}{k} \frac{(-2\beta)^k}{(2N+k-1)!} {\sum_{j=0}^{\hat{k}}}{}' E(k,\hat{k}-j) \sum_{m=1}^{\infty} \frac{\cosh(m(2j+\mu_k)\beta)}{m^{1-2N-k}\sinh^{k+1}(m\beta)} \\
&\quad +\frac{\beta^N}{(2N-1)!}\left(\sum_{m=1}^{\infty} \frac{m^{2N-1}}{e^{2m\beta}-1} + \frac{1}{2}\zeta(1-2N)\right).
\end{aligned}$$

Proof. Let $c = 1$ in Theorem 3.1. Assume that B and N have the same parity. Then $(-1)^{B+N} = 0$. Applying

$$e^{in\pi\mu_k} = \begin{cases} 1, & k \text{ odd}, \\ (-1)^n, & k \text{ even}, \end{cases}$$

we see that

$$\begin{aligned}
&\cosh(n(\alpha-i\pi)(2j+\mu_k)) \\
&= \frac{1}{2}\left(e^{n\alpha(2j+\mu_k)-in\pi(2j+\mu_k)} + e^{-n\alpha(2j+\mu_k)+in\pi(2j+\mu_k)}\right) \\
&= \begin{cases} \cosh(n\alpha(2j+\mu_k)), & k \text{ odd}, \\ (-1)^n\cosh(n\alpha(2j+\mu_k)), & k \text{ even}. \end{cases}
\end{aligned}$$

Since $e^{in\pi} = (-1)^n$,

$$\begin{aligned}\sinh^{k+1}(n(\alpha - i\pi)) &= \frac{1}{2^{k+1}}\left(e^{n\alpha - in\pi} - e^{-n\alpha + in\pi}\right)^{k+1}\\ &= \begin{cases}\sinh^{k+1}(n\alpha), & k \text{ odd},\\ (-1)^n \sinh^{k+1}(n\alpha), & k \text{ even}.\end{cases}\end{aligned}$$

Note that B and N have the same parity. If $N = 1$, then B is odd. Thus we have

$$\delta_1(B, c) = \frac{c}{8(B+1)}(1 + (-1)^B) = 0.$$

The desired result now follows. □

Corollary 3.5. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any even integers $B \geq 0$ and $N \geq 2$,*

$$\begin{aligned}&\alpha^N \sum_{k=1}^{B} \binom{B}{k} \frac{(-2\alpha)^k}{(2N+k-1)!} {\sum_{j=0}^{\hat{k}}}' E(k, \hat{k} - j) \sum_{m=1}^{\infty} \frac{\cosh(m(2j + \mu_k)\alpha)}{m^{1-2N-k}\sinh^{k+1}(m\alpha)}\\ &= \beta^N \sum_{k=1}^{B} \binom{B}{k} \frac{(-2\beta)^k}{(2N+k-1)!} {\sum_{j=0}^{\hat{k}}}' E(k, \hat{k} - j) \sum_{m=1}^{\infty} \frac{\cosh(m(2j + \mu_k)\beta)}{m^{1-2N-k}\sinh^{k+1}(m\beta)}.\end{aligned}$$

Proof. Let N be even and $c = 1$ in Theorem 3.2. Then the proof is similar with the proof of Corollary 3.4. □

Corollary 3.6. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. Then we have*

$$\begin{aligned}&\alpha \sum_{m=1}^{\infty} \frac{me^{-m\alpha}}{\sinh(m\alpha)} - \alpha^2 \sum_{m=1}^{\infty} \frac{m^2}{\sinh^2(m\alpha)} - \frac{1}{12}\alpha\\ &= \beta \sum_{m=1}^{\infty} \frac{me^{-m\beta}}{\sinh(m\beta)} - \beta^2 \sum_{m=1}^{\infty} \frac{m^2}{\sinh^2(m\beta)} - \frac{1}{12}\beta.\end{aligned}$$

Proof. Put $B = 1$ and $N = 1$ in corollary 3.4 and apply

$$\sum_{m=1}^{\infty} \frac{m}{e^{2m\alpha} - 1} = \sum_{m=1}^{\infty} \frac{me^{-m\alpha}}{e^{m\alpha} - e^{-m\alpha}} = \frac{1}{2} \sum_{m=1}^{\infty} \frac{me^{-m\alpha}}{\sinh(m\alpha)}.$$

□

Corollary 3.7. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any integer $N \geq 1$,*

$$\begin{aligned}&\alpha^{2N+1} \sum_{m=1}^{\infty} \frac{m^{4N}}{\sinh^2(m\alpha)} - \frac{2\alpha^{2N+2}}{4N+1} \sum_{m=1}^{\infty} \frac{m^{4N+1}\cosh(m\alpha)}{\sinh^3(m\alpha)}\\ &= \beta^{2N+1} \sum_{m=1}^{\infty} \frac{m^{4N}}{\sinh^2(m\beta)} - \frac{2\beta^{2N+2}}{4N+1} \sum_{m=1}^{\infty} \frac{m^{4N+1}\cosh(m\beta)}{\sinh^3(m\beta)}.\end{aligned}$$

Proof. Put $B = 2$ in Corollary 3.5 and repalce N by $2N$. □

It is noteworthy that above four corollaries show symmetric identities for α and β.

Corollary 3.8. *Let $B \geq 0$ and $N \geq 1$ be integers which have the different parity. Then we have*

$$\sum_{k=1}^{B} \binom{B}{k} \frac{(-2\pi)^k}{(2N+k-1)!} {\sum_{j=0}^{\hat{k}}}' E(k, \hat{k}-j) \sum_{m=1}^{\infty} \frac{\cosh(m(2j+\mu_k)\pi)}{m^{1-2N-k}\sinh^{k+1}(m\pi)}$$
$$= -\frac{1}{(2N-1)!}\left(\sum_{m=1}^{\infty} \frac{m^{2N-1}}{e^{2m\pi}-1} + \frac{1}{2}\zeta(1-2N)\right) - (-1)^B (2\pi)^{-1}\delta_N(B,1).$$

Proof. Let $c = 1$ and $\alpha = \beta = \pi$ in Theorem 3.1. Employing

$$e^{im\mu_k\pi} = \begin{cases} 1, & k \text{ odd}, \\ (-1)^m, & k \text{ even} \end{cases}$$
$$= (-1)^{m(k+1)},$$

we see that

$$\cosh(m(2j+\mu_k)(\pi \pm i\pi)) = (-1)^{m(k+1)} \cosh(m(2j+\mu_k)\pi).$$

And it is easy to see that

$$\sinh^{k+1}(m(\pi \pm i\pi)) = (-1)^{m(k+1)} \sinh^{k+1}(m\pi).$$

Using the above two equations, the proof is done. □

Corollary 3.9. *Let $B \geq 0$ and $N \geq 1$ be integers. Then we have*

$$\sum_{k=1}^{2B} \binom{2B}{k} \frac{(-2\pi)^k}{(4N+k-3)!} {\sum_{j=0}^{\hat{k}}}' E(k, \hat{k}-j) \sum_{m=1}^{\infty} \frac{\cosh(m(2j+\mu_k)\pi)}{m^{3-4N-k}\sinh^{k+1}(m\pi)}$$
$$= \begin{cases} \dfrac{B}{4\pi(2B+1)}, & N = 1, \\ 0, & N > 1. \end{cases}$$

Proof. Replace N by $2N-1$ and B by $2B$ in Corollary 3.8. We find from (1.1) that

$$\sum_{m=1}^{\infty} \frac{m^{4N-3}}{e^{2m\pi}-1} + \frac{1}{2}\zeta(3-4N) = -\frac{1}{2\pi}\nu_N(1). \tag{3.17}$$

Next, apply

$$\frac{1}{2\pi}\nu_1(1) - 2\pi\delta_1(2B,1) = \frac{B}{4\pi(2B+1)}$$

to complete the proof. □

Corollary 3.10. *For any integer $N \geq 1$, we have*

$$\frac{\pi}{N}\sum_{m=1}^{\infty}\frac{m^{4N}}{\sinh^2(m\pi)}-\sum_{m=1}^{\infty}\frac{m^{4N-1}e^{-m\pi}}{\sinh(m\pi)}=\zeta(1-4N)$$

Proof. Let $B=1$ in Corollary 3.8. Then N is even and so $\delta_N(1,1)=0$. Replace N by $2N$ and use

$$\frac{1}{e^{2m\pi}-1}=\frac{e^{-m\pi}}{e^{m\pi}-e^{-m\pi}}=\frac{1}{2}\frac{e^{-m\pi}}{\sinh(m\pi)}.$$

□

Corollary 3.11. *For any integer $N \geq 1$, we have*

$$\frac{2\pi}{4N-1}\sum_{m=1}^{\infty}\frac{m^{4N-1}\cosh(m\pi)}{\sinh^3(m\pi)}-\sum_{m=1}^{\infty}\frac{m^{4N-2}}{\sinh^2(m\pi)}=\begin{cases}\dfrac{1}{12\pi^2}, & N=1,\\ 0, & N>1.\end{cases}$$

Proof. Let $B=1$ in Corollary 3.9. Using (3.17), the desired result follows. □

Let $c=2$ in Theorem 3.1. Dividing the sum over m into two parts, we have that

$$\sum_{m=1}^{\infty}\frac{\cosh(m(2j+\mu_k)(\alpha-i\pi)/2)}{m^{1-2N-k}\sinh^{k+1}(m(\alpha-i\pi)/2)}=\sum_{m=1}^{\infty}\frac{\cosh(m(2j+\mu_k)(\alpha-i\pi))}{(2m)^{1-2N-k}\sinh^{k+1}(m(\alpha-i\pi))}$$
$$+\sum_{m=0}^{\infty}\frac{\cosh((2m+1)(2j+\mu_k)(\alpha-i\pi)/2)}{(2m+1)^{1-2N-k}\sinh^{k+1}((2m+1)(\alpha-i\pi)/2)}$$

and

$$\sum_{m=1}^{\infty}\frac{m^{2N-1}}{e^{m(\alpha-i\pi)}-1}=2^{2N-1}\sum_{m=1}^{\infty}\frac{m^{2N-1}}{e^{2m(\alpha-i\pi)}-1}+\sum_{m=0}^{\infty}\frac{(2m+1)^{2N-1}}{e^{(2m+1)(\alpha-i\pi)}-1}$$
$$=2^{2N-1}\sum_{m=1}^{\infty}\frac{m^{2N-1}}{e^{2m\alpha}-1}-\sum_{m=0}^{\infty}\frac{(2m+1)^{2N-1}}{e^{(2m+1)\alpha}+1}.$$

It is easy to see that

$$\frac{\cosh((2m+1)(2j+\mu_k)(\alpha-i\pi)/2)}{\sinh^{k+1}((2m+1)(\alpha-i\pi)/2)}=\frac{(-1)^{j+\hat{k}+1}g_k(m,j,\alpha)}{\cosh^{k+1}((2m+1)\alpha/2)},$$

where

$$g_k(m,j,x)=\begin{cases}\cosh((2m+1)jx), & k\text{ odd},\\ \sinh((2m+1)(2j+1)x/2), & k\text{ even}.\end{cases}$$

Now, using the identities obtained from Theorem 3.1 with $c=1$ and $c=2$, we obtain the following theorem.

Theorem 3.12. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any integers $B \geq 0$ and $N \geq 1$,*

$$
\begin{aligned}
&(-1)^B \alpha^N \sum_{k=1}^{B} \binom{B}{k} \frac{(-\alpha)^k}{(2N+k-1)!} {\sum_{j=0}^{\hat{k}}}' E(k, \hat{k}-j) \\
&\qquad \times \sum_{m=0}^{\infty} \frac{(-1)^{j+\hat{k}+1} g_k(m,j,\alpha)}{(2m+1)^{1-2N-k} \cosh^{k+1}((2m+1)\alpha/2)} \\
&- \frac{(-1)^B \alpha^N}{(2N-1)!} \left(\sum_{m=0}^{\infty} \frac{(2m+1)^{2N-1}}{e^{(2m+1)\alpha}+1} - (2^{-1} - 2^{2N-2})\zeta(1-2N) \right) \\
&= (-\beta)^N \sum_{k=1}^{B} \binom{B}{k} \frac{(-\beta)^k}{(2N+k-1)!} {\sum_{j=0}^{\hat{k}}}' E(k, \hat{k}-j) \\
&\qquad \times \sum_{m=0}^{\infty} \frac{(-1)^{j+\hat{k}+1} g_k(m,j,\beta)}{(2m+1)^{1-2N-k} \cosh^{k+1}((2m+1)\beta/2)} \\
&\quad - \frac{(-\beta)^N}{(2N-1)!} \left(\sum_{m=0}^{\infty} \frac{(2m+1)^{2N-1}}{e^{(2m+1)\beta}+1} - (2^{-1} - 2^{2N-2})\zeta(1-2N) \right).
\end{aligned}
$$

Theorem 3.12 includes Corollary 4.12 in [3], which shows the following formula; Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any integer $N \geq 1$,

$$
\begin{aligned}
&\alpha^N \left(\sum_{m=0}^{\infty} \frac{(2m+1)^{2N-1}}{e^{(2m+1)\alpha}+1} - (2^{-1} - 2^{2N-2})\zeta(1-2N) \right) \\
&= (-\beta)^N \left(\sum_{m=0}^{\infty} \frac{(2m+1)^{2N-1}}{e^{(2m+1)\beta}+1} - (2^{-1} - 2^{2N-2})\zeta(1-2N) \right).
\end{aligned}
$$

Corollary 3.13. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any integers $B \geq 0$ and $N \geq 1$,*

$$
\begin{aligned}
&\alpha^N \sum_{k=1}^{2B} \binom{2B}{k} \frac{(-\alpha)^k}{(2N+k-1)!} {\sum_{j=0}^{\hat{k}}}' E(k, \hat{k}-j) \\
&\qquad \times \sum_{m=0}^{\infty} \frac{(-1)^{j+\hat{k}+1} g_k(m,j,\alpha)}{(2m+1)^{1-2N-k} \cosh^{k+1}((2m+1)\alpha/2)} \\
&= (-\beta)^N \sum_{k=1}^{2B} \binom{2B}{k} \frac{(-\beta)^k}{(2N+k-1)!} {\sum_{j=0}^{\hat{k}}}' E(k, \hat{k}-j) \\
&\qquad \times \sum_{m=0}^{\infty} \frac{(-1)^{j+\hat{k}+1} g_k(m,j,\beta)}{(2m+1)^{1-2N-k} \cosh^{k+1}((2m+1)\beta/2)}.
\end{aligned}
$$

Proof. Replace B by $2B$ in Theorem 3.12 and apply Corollary 4.12. in [3]. □

Corollary 3.14. *For any integers $B \geq 0$ and $N \geq 1$,*

$$\sum_{k=1}^{2B} \binom{2B}{k} \frac{(-\pi)^k}{(4N+k-3)!} {\sum_{j=0}^{\hat{k}}}' E(k, \hat{k}-j) \times \sum_{m=0}^{\infty} \frac{(-1)^{j+\hat{k}+1} g_k(m,j,\pi)}{(2m+1)^{3-4N-k} \cosh^{k+1}((2m+1)\pi/2)} = 0.$$

Proof. Let $\alpha = \beta = \pi$ in Corollary 3.13 and replace N by $2N-1$. □

Corollary 3.15. *Let $B \geq 0$ and $N \geq 1$ be integers with the different parity. Then*

$$\sum_{k=1}^{B} \binom{B}{k} \frac{(-\pi)^k}{(2N+k-1)!} {\sum_{j=0}^{\hat{k}}}' E(k, \hat{k}-j) \times \sum_{m=0}^{\infty} \frac{(-1)^{j+\hat{k}+1} g_k(m,j,\pi)}{(2m+1)^{1-2N-k} \cosh^{k+1}((2m+1)\pi/2)}$$
$$= \frac{1}{(2N-1)!} \left(\sum_{m=0}^{\infty} \frac{(2m+1)^{2N-1}}{e^{(2m+1)\pi}+1} - (2^{-1} - 2^{2N-2})\zeta(1-2N) \right).$$

Proof. Put $\alpha = \beta = \pi$ in Theorem 3.12. □

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