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ON ASYMPTOTICALLY LACUNARY STATISTICAL EQUIVALENT TRIPLE SEQUENCES VIA IDEALS AND ORLICZ FUNCTION

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Abstract. In the present paper, we introduce the concepts of $\mathcal{I}$-asymptotically statistical $\widetilde{\phi}$-equivalence and $\mathcal{I}$-asymptotically lacunary statistical $\widetilde{\phi}$-equivalence for triple sequences. Moreover, we give the relations between these new notions.

1. Introduction

In 1980, Pobyvanets gave the definitions for asymptotically equivalent sequences and asymptotic regular matrices. Marouf [20] and Li [19] studied the relationships between the asymptotic equivalence of two sequences in summability theory and presented some variations of asymptotic equivalence. In 2003, Patterson [25] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. In [26], asymptotically lacunary statistical equivalence is given as a natural combination of the definitions for asymptotically equivalence, statistical convergence and lacunary sequences.

Fast [8] presented a generalization of the usual concept of sequential limit which is called statistical convergence. Šalát [30] gave some basic properties of statistical convergence. Recently, Mursaleen and Edely [23] presented the idea of statistical convergence for multiple sequences, and there are several papers on the statistical and ideal convergence of double and triple sequences (see literature [2, 11, 17]). The study on the statistical convergence of triple sequences due to Şahiner et al. [28]. Fridy and Orhan [10] defined the concept of lacunary statistical convergence. Various applications of this concept can be found in [9, 12, 13, 15, 16, 21, 31, 33]. The idea is based on the notion of natural density of subsets of $\mathbb{N}$, the set of all positive integers which is defined as follows: The natural density of a subset A of $\mathbb{N}$ denoted as $\delta(A)$

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is defined by $\delta(A) = \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in A\}|$. Generalizing the concept of statistical convergence, Kostyrko et al. introduced the idea of $\mathcal{I}$-convergence in [18]. More investigations in this direction and more applications of ideals can be found in [11, 14, 24, 32]. In another direction, a new type of convergence, called $\mathcal{I}$-statistical convergence, was introduced in [5].

Since sequence convergence plays a very important role in the fundamental theory of mathematics, there are many convergence concepts in summability theory, in classical measure theory, approximation theory and probability theory, and the relationships between them are discussed. The interested reader may consult Gürdal and Huban [11], Hazarika et al. [15], Mohiuddine and Alamri [21] and Acar and Mohiuddine [1], the monographs [3] and [22] for the background on the sequence spaces and related topics. Inspired by this, in this paper, a further investigation into the mathematical properties of triple sequences will be made. Section 2 recalls some definitions and theorems in summability theory. In Section 3, we study $\mathcal{I}$-asymptotically lacunary statistical $\widetilde{\phi}$-equivalence of triple sequences and discuss the relationships among them. We shall use asymptotically equivalent, lacunary sequence and Orlicz function $\widetilde{\phi}$ to introduce the concepts $\mathcal{I}$-asymptotically statistical $\widetilde{\phi}$-equivalence and $\mathcal{I}$-asymptotically lacunary statistical $\widetilde{\phi}$-equivalence of triple sequences. In addition to these definitions, natural inclusion theorems shall also be presented.

2. Definitions and Notations

First we recall some of the basic concepts which will be used in this paper. By $\mathbb{N}$ and $\mathbb{R}$, we mean the sets of all natural and real numbers, respectively.

The notion of statistical convergence depends on the density of the subsets of the set $\mathbb{N}$ of natural numbers. If K is a subset of $\mathbb{N}$, then K_n denotes the set $\{k \in K : k \le n\}$ and $|K_n|$ also denotes the cardinality of the set K_n. The natural density of K given by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |K_n|,$$

It is said that a sequence $x = (x_k)_{k\in\mathbb{N}}$ is statistically convergent to a point L, which provided that

$$\delta(\{k \in \mathbb{N} : |x_k - L| \ge \varepsilon\}) = 0,$$

for every $\varepsilon > 0$. If $(x_k)_{k\in\mathbb{N}}$ is statistically convergent to L and is written as $L = st\text{-}\lim x_k$.

In the literature, statistical convergence of any real sequence is defined relatively to absolute value. While, we know that the absolute value of real numbers is a special case of an Orlicz function [27], i.e. a function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ which satisfies the following criterion:

(a) $\widetilde{\phi}$ is an even function

(b) $\widetilde{\phi}$ is non-decreasing on $\mathbb{R}^+$

(c) $\widetilde{\phi}(x) = 0$ if and only if $x = 0$

(d) $\widetilde{\phi}$ is continuous on $\mathbb{R}$

(e) $x \to \infty$ implies $\widetilde{\phi}(x) \to \infty$

Further, an Orlicz function $\widetilde{\phi}$ is said to satisfy the condition $\triangle_2$, if there exists an positive real number K satisfying the condition $\widetilde{\phi}(2x) \leq K.\widetilde{\phi}(x)$, $\forall x \in \mathbb{R}^+$. In [27], Rao and Ren describes some important applications of Orlicz functions in many areas such as economics, stochastic problems etc. The reader can also refer to the paper [6] and recent monograph [4] related with various ways to generalize Orlicz sequence spaces systematically and investigate several structural properties of such spaces. Few examples of Orlicz functions are given below:

Example 2.1. *(i) For a fixed $r \in \mathbb{N}$, the function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ defined as $\widetilde{\phi}(x) = |x|^r$ is an Orlicz function.*

(ii) The function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ defined as $\widetilde{\phi}(x) = x^2$ is an Orlicz function satisfying the $\triangle_2$ condition.

(iii) The function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ defined as $\widetilde{\phi}(x) = e^{|x|} - |x| - 1$ is an Orlicz function not satisfying the $\triangle_2$ condition.
(iv) The function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ defined as $\widetilde{\phi}(x) = x^3$ is not an Orlicz function.

Definition 2.2. ([31]) *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A sequence $x = (x_n)$ is said to be statistically $\widetilde{\phi}$-convergent to L if for each $\varepsilon > 0$,*

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \widetilde{\phi}\left(x_k - L\right) \geq \varepsilon \right\} \right| = 0.$$

The notion of statistical convergence was further generalized in the paper [18] using the notion of an ideal of subsets of the set $\mathbb{N}$. We say that a non-empty family of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is an ideal on $\mathbb{N}$ if $\mathcal{I}$ is hereditary (i.e. $B \subset A \in \mathcal{I} \Rightarrow B \in \mathcal{I}$) and additive (i.e. $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$). An ideal $\mathcal{I}$ on $\mathbb{N}$ for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is called a proper ideal. A proper ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$. If not otherwise stated in the sequel $\mathcal{I}$ will denote an admissible ideal. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal. A class $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \backslash A\}$ called the filter associated with the ideal $\mathcal{I}$, is a filter on $\mathbb{N}$.

Recall the generalization of statistical convergence from [18]. Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$ and $x = (x_k)$ be a real sequence. We say that the sequence x is $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I}$. Take for $\mathcal{I}$ the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence.

For more information about $\mathcal{I}$-convergent, see the references in [24]. We also recall that the concept of $\mathcal{I}$-statistically convergent is studied in [5].

A sequence (x_k) is said to be $\mathcal{I}$-statistically convergent to L if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ k \leq n : |x_k - L| \geq \varepsilon \} \right| \geq \delta \right\} \in \mathcal{I}.$$

In this case, L is called $\mathcal{I}$-statistical limit of the sequence (x_k) and we write $\mathcal{I}\text{-}st - \lim_{k\to\infty} x_k = L$.

We now recall the following basic concepts from [28, 29] which will be needed throughout the paper.

A function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) is called a real (complex) triple sequence. A triple sequence (x_{jkl}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $|x_{jkl} - L| < \varepsilon$ whenever $j, k, l \geq n_0$. A triple sequence (x_{jkl}) is said to be bounded if there exists $M > 0$ such that $|x_{jkl}| < M$ for all $j, k, l \in \mathbb{N}$. We denote the space of all bounded triple sequences by ℓ^3_∞.

Definition 2.3. *A subset K of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is said to have natural density $\delta(K)$ if*

$$\delta(K) = P - \lim_{n,k,l\to\infty} \frac{|K_{nkl}|}{nkl}$$

exists, where the vertical bars denote the number of (n, k, l) in K such that $p \leq n$, $q \leq k$, $r \leq l$. Then, a real triple sequence $x = (x_{nkl})$ is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\delta\left(\{(n, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{nkl} - L| \geq \varepsilon\}\right) = 0.$$

Throughout the paper we consider the ideals of $\mathcal{P}(\mathbb{N})$ by $\mathcal{I}$; the ideals of $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ by $\mathcal{I}_2$ and the ideals of $\mathcal{P}(\mathbb{N} \times \mathbb{N} \times \mathbb{N})$ by $\mathcal{I}_3$.

Definition 2.4. *A real triple sequence (x_{nkl}) is said to be $\mathcal{I}$-convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,*

$$\{(n, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{nkl} - L| \geq \varepsilon\} \in \mathcal{I}_3.$$

In this case, one writes $\mathcal{I}_3$-$\lim x_{nkl} = L$.

We define the asymptotically equivalence of single sequences as follows (see [20]) :

Definition 2.5. *Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if*

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$).

Patterson [25] presented a natural combination of the concepts of statistical convergence and asymptotically equivalence to introduce the concept of asymptotically statistically equivalence, as follows:

Definition 2.6. *Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$,*

$$\lim_n \frac{1}{n} \left\{ \text{the number of } k < n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} = 0$$

(denoted by $x \overset{S_L}{\sim} y$) and simply asymptotically statistical equivalent if $L = 1$.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, ...$, where $k_0 = 0$, we shall mean an increasing sequence of nonnegative integers with $k_r - k_{r-1}$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

Definition 2.7. ([26]) *Let θ be a lacunary sequence; the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$*

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0$$

(denoted by $x \overset{S_\theta^L}{\sim} y$) and simply asymptotically lacunary statistical equivalent if $L = 1$.

Definition 2.8. *Let θ be a lacunary sequence. A sequence $x = (x_k)$ is said to be strongly $\mathcal{I}$-lacunary convergent to L or $N_\theta(\mathcal{I})$-convergent to L if, for any $\varepsilon > 0$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| \geq \varepsilon \right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L\left(N_\theta(\mathcal{I})\right)$. The class of all strongly $\mathcal{I}$-lacunary statistically convergent sequences will be denoted by $N_\theta(\mathcal{I})$.

3. Main Results

In this section, we present the notion of triple sequences and study $\mathcal{I}$-asymptotically statistical $\widetilde{\phi}$-equivalence and $\mathcal{I}$-asymptotically lacunary statistical $\widetilde{\phi}$-equivalence for these sequences.

Definition 3.1. ([7]) *The triple sequence $\theta_3 = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ is called triple lacunary sequence if there exist three increasing sequences of integers such that*

$$j_0 = 0,\ h_r = j_r - j_{r-1} \to \infty \text{ as } r \to \infty,$$
$$k_0 = 0,\ h_s = k_s - k_{s-1} \to \infty \text{ as } s \to \infty,$$

and

$$l_0 = 0,\ h_t = l_t - l_{t-1} \to \infty \text{ as } t \to \infty.$$

Let $k_{r,s,t} = j_r k_s l_t, h_{r,s,t} = h_r h_s h_t$ *and* $\theta_{r,s,t}$ *is determined by*

$$I_{r,s,t} = \{(j,k,l) : j_{r-1} < j \leq j_r, k_{s-1} < k \leq k_s \text{ and } l_{t-1} < l \leq l_t\},$$

$$q_r = \frac{j_r}{j_{r-1}}, q_s = \frac{k_s}{k_{s-1}}, q_t = \frac{l_t}{l_{t-1}} \text{ and } q_{r,s,t} = q_r q_s q_t.$$

Let $D \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. *The number*

$$\delta_3^{\theta}(D) = \lim_{r,s,t} \frac{1}{h_{r,s,t}} |\{(j,k,l) \in I_{r,s,t} : (j,k,l) \in D\}|$$

is said as the $\theta^3_{r,s,t}$*-density of* D*, provided the limit exists.*

We define the following :

Definition 3.2. *Let* $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ *be an Orlicz function. Two number triple sequences* $x = (x_{jkl})_{j,k,l\in\mathbb{N}}$ and $y = (y_{jkl})_{j,k,l\in\mathbb{N}}$ *are said to be* $\mathcal{I}$*-asymptotically statistical* $\widetilde{\phi}$*-equivalent of multiple* L *provided that for every* $\varepsilon > 0, \delta > 0$,

$$\left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \tfrac{1}{rst}\left|\left\{j \leq r, k \leq s, l \leq t : \widetilde{\phi}\left(\tfrac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_3$$

(denoted by $x \overset{S^L(\mathcal{I}_3-\widetilde{\phi})}{\sim} y$ *) and simply* $\mathcal{I}$*-asymptotically statistical* $\widetilde{\phi}$*-equivalent if* $L = 1$.

Definition 3.3. *Let* $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ *be an Orlicz function and* $\theta_3 = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ *be a lacunary triple sequence. Two number triple sequences* $x = (x_{jkl})$ *and* $y = (y_{jkl})$ *are said to be* $\mathcal{I}$*-asymptotically lacunary statistical* $\widetilde{\phi}$*-equivalent of multiple* L *if*

$$\left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \tfrac{1}{h_{r,s,t}}\left|\left\{(j,k,l) \in I_{r,s,t} : \widetilde{\phi}\left(\tfrac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_3$$

for every $\varepsilon > 0, \delta > 0$ *(denoted by* $x \overset{S^L_{\theta_3}(\mathcal{I}_3-\widetilde{\phi})}{\sim} y$ *) and simply* $\mathcal{I}$*-asymptotically lacunary statistical* $\widetilde{\phi}$*-equivalent if* $L = 1$.

Definition 3.4. *Let* $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ *be an Orlicz function. Two number triple sequences* $x = (x_{jkl})_{j,k,l\in\mathbb{N}}$ and $y = (y_{jkl})_{j,k,l\in\mathbb{N}}$ *are said to be Cesáro* $\mathcal{I}$*-asymptotically* $\widetilde{\phi}$*-equivalent of multiple* L *if for every* $\varepsilon > 0$

$$\left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \widetilde{\phi}\left(\frac{1}{rst}\sum_{j,k,l=1}^{r,s,t} \frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\} \in \mathcal{I}_3$$

denoted by $x \overset{(\sigma_1)^L(\mathcal{I}_3-\widetilde{\phi})}{\sim} y$*) and simply Cesáro* $\mathcal{I}$*-asymptotically* $\widetilde{\phi}$*-equivalent if* $L = 1$.

Definition 3.5. *Let* $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ *be an Orlicz function. Two number triple sequences* $x = (x_{jkl})_{j,k,l\in\mathbb{N}}$ and $y = (y_{jkl})_{j,k,l\in\mathbb{N}}$ *are said to be strongly Cesáro*

$\mathcal{I}$-asymptotically $\widetilde{\phi}$-equivalent of multiple L if for every $\varepsilon > 0$

$$\left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{rst}\sum_{j,k,l=1}^{r,s,t} \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\} \in \mathcal{I}_3$$

(denoted by $x \overset{\left(\widetilde{\phi}(\sigma_1)\right)^L\left(\mathcal{I}_3-\widetilde{\phi}\right)}{\sim} y$) and simply strongly Cesáro $\mathcal{I}$-asymptotically $\widetilde{\phi}$-equivalent if $L = 1$.

Definition 3.6. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ be a lacunary triple sequence. Two number triple sequences $x = (x_{jkl})_{j,k,l\in\mathbb{N}}$ and $y = (y_{jkl})_{j,k,l\in\mathbb{N}}$ are said to be strongly $\mathcal{I}$-asymptotically lacunary $\widetilde{\phi}$-equivalent of multiple L if*

$$\left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s,t}}\sum_{(j,k,l)\in I_{r,s,t}} \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\} \in \mathcal{I}_3$$

for every $\varepsilon > 0$ (denoted by $x \overset{N_{\theta_3}^L\left(\mathcal{I}_3-\widetilde{\phi}\right)}{\sim} y$) and strongly simply $\mathcal{I}$-asymptotically lacunary $\widetilde{\phi}$-equivalent if $L = 1$.

Theorem 3.7. *Let $\mathcal{I}_3 \subset \mathcal{P}(\mathbb{N}\times\mathbb{N}\times\mathbb{N})$ be a non-trivial ideal. Let $\theta_3 = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ be a lacunary triple sequence. If $(x_{jkl}), (y_{jkl}) \in \ell_\infty^3$ and $x \overset{S^L\left(\mathcal{I}_3-\widetilde{\phi}\right)}{\sim} y$, then $x \overset{(\sigma_1)^L\left(\mathcal{I}_3-\widetilde{\phi}\right)}{\sim} y$.*

Proof. Suppose that $(x_{jkl}), (y_{jkl}) \in \ell_\infty^3$ and $x \overset{S^L\left(\mathcal{I}_3-\widetilde{\phi}\right)}{\sim} y$. Then we can assume that

$$\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \leq M \text{ for almost all } j,k,l.$$

Let $\varepsilon > 0$. Then we have

$$\begin{aligned}
&\widetilde{\phi}\left(\frac{1}{rst}\sum_{j,k,l=1,1,1}^{r,s,t} \frac{x_{jkl}}{y_{jkl}} - L\right) \\
&\leq \frac{1}{rst}\sum_{j,k,l=1,1,1}^{r,s,t} \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \\
&\leq \frac{1}{rst}\sum_{\substack{j,k,l \\ \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)\geq\varepsilon}} \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) + \frac{1}{rst}\sum_{\substack{j,k,l \\ \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)<\varepsilon}} \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \\
&\leq \frac{M}{rst}\left\{j \leq r, k \leq s, l \leq t : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\} + \frac{1}{rst}.rst.\varepsilon.
\end{aligned}$$

Consequently, if $\delta > \varepsilon > 0$, δ and ε are independent, put $\delta_1 = \delta - \varepsilon > 0$, we have

$$\left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \widetilde{\phi}\left(\tfrac{1}{rst}\sum_{j,k,l=1,1,1}^{r,s,t} \tfrac{x_{jkl}}{y_{jkl}} - L\right) \geq \delta\right\}$$
$$\subseteq \left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \tfrac{1}{rst}\left|\left\{j \leq r, k \leq s, l \leq t) : \widetilde{\phi}\left(\tfrac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\}\right| \geq \tfrac{\delta_1}{M}\right\}$$
$$\in \mathcal{I}_3.$$

This shows that $x \overset{(\sigma_1)^L(\mathcal{I}_3 - \widetilde{\phi})}{\sim} y$. ∎

As an immediate consequence of Theorem 3.7, we have the following result.

Corollary 3.8. *Let $\mathcal{I}_3 \subset \mathcal{P}(\mathbb{N}\times\mathbb{N}\times\mathbb{N})$ be a non-trivial ideal. If*

$$(x_{jkl}), (y_{jkl}) \in \ell_\infty^3 \quad \text{and} \quad x \overset{S^L(\mathcal{I}_3 - \widetilde{\phi})}{\sim} y,$$

then $x \overset{(\widetilde{\phi}(\sigma_1))^L(\mathcal{I}_3 - \widetilde{\phi})}{\sim} y$.

Theorem 3.9. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,s,t} = \{(j_r, k_s, l_t)\}$ be a lacunary triple sequence. Then, the following statements hold:*

(i) If $x \overset{N_{\theta_3}^L(\mathcal{I}_3 - \widetilde{\phi})}{\sim} y$ then $x \overset{S_{\theta_3}^L(\mathcal{I}_3 - \widetilde{\phi})}{\sim} y$;

(ii) $x \overset{N_{\theta_3}^L(\mathcal{I}_3 - \widetilde{\phi})}{\sim} y$ is a proper subset of $x \overset{S_{\theta_3}^L(\mathcal{I}_3 - \widetilde{\phi})}{\sim} y$;

(iii) If $x, y \in \ell_\infty^3$ and $x \overset{S_{\theta_3}^L(\mathcal{I}_3 - \widetilde{\phi})}{\sim} y$ then $x \overset{N_{\theta_3}^L(\mathcal{I}_3 - \widetilde{\phi})}{\sim} y$;

(iv) $x \overset{S_{\theta_3}^L(\mathcal{I}_3 - \widetilde{\phi})}{\sim} y \cap \ell_\infty^3 = x \overset{N_{\theta_3}^L(\mathcal{I}_3 - \widetilde{\phi})}{\sim} y \cap \ell_\infty^3$.

Proof. *(i)* If $\varepsilon > 0$ and $x \overset{N_{\theta_3}^L(\mathcal{I}_3 - \widetilde{\phi})}{\sim} y$, then

$$\sum_{(j,k,l)\in I_{r,s,t}} \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \sum_{\substack{(j,k,l)\in I_{r,s,t} \\ \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon}} \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right)$$
$$\geq \varepsilon \left|\left\{(j,k,l) \in I_{r,s,t} : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\}\right|$$

and so

$$\frac{1}{\varepsilon h_{r,s,t}} \sum_{(j,k,l)\in I_{r,s,t}} \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \frac{1}{h_{r,s,t}} \left|\left\{(j,k,l) \in I_{r,s,t} : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\}\right|.$$

Then, for any $\delta > 0$,

$$\left\{(r,s,t)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s,t}}\left|\left\{(j,k,l)\in I_{r,s,t}:\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)\geq\varepsilon\right\}\right|\geq\delta\right\}$$
$$\subseteq\left\{(r,s,t)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s,t}}\sum_{(j,k,l)\in I_{r,s,t}}\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)\geq\varepsilon.\delta\right\}\in\mathcal{I}_3.$$

Hence, we have $x \overset{S^L_{\theta_3}(\mathcal{I}_3-\widetilde{\phi})}{\sim} y$

(ii) Suppose that $x \overset{N^L_{\theta_3}(\mathcal{I}_3-\widetilde{\phi})}{\sim} y \subset x \overset{S^L_{\theta_3}(\mathcal{I}_3-\widetilde{\phi})}{\sim} y$. Let $x=(x_{jkl})$ and $y=(y_{jkl})$ be two sequences defined as follows :

$$x_{jkl}=\begin{cases} jkl, & \text{if } j_{r-1}<j\leq j_{r-1}+\left[\sqrt{h_r}\right], k_{s-1}<k\leq k_{s-1}+\left[\sqrt{h_s}\right] \\ & , l_{t-1}<l\leq l_{t-1}+\left[\sqrt{h_t}\right],\ r,s,t=1,2,... \\ 0, & \text{otherwise.}\end{cases}$$

$y_{jkl}=1$ for all $j,k,l\in\mathbb{N}$. These sequences satisfy the following $x \overset{S^L_{\theta_3}(\mathcal{I}_3-\widetilde{\phi})}{\sim} y$, but the following fails $x \overset{N^L_{\theta_3}(\mathcal{I}_3-\widetilde{\phi})}{\sim} y$.

(iii) Suppose that $x=(x_{jkl})$ and $y=(y_{jkl})$ belongs to the space ℓ^3_∞ and $x \overset{S^L_{\theta_3}(\mathcal{I}_3-\widetilde{\phi})}{\sim} y$. Then we can assume that

$$\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)\leq M \text{ for all } j,k \text{ and } l.$$

Given $\varepsilon>0$, we have

$$\frac{1}{h_{r,s,t}}\sum_{(j,k,l)\in I_{r,s,t}}\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)$$
$$=\frac{1}{h_{r,s,t}}\sum_{\substack{(j,k,l)\in I_{r,s,t}\\ \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)\geq\varepsilon}}\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)$$
$$+\frac{1}{h_{r,s,t}}\sum_{\substack{(j,k,l)\in I_{r,s,t}\\ \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)<\varepsilon}}\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)$$
$$\leq\frac{M}{h_{r,s,t}}\left|\left\{(j,k,l)\in I_{r,s,t}:\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)\geq\frac{\varepsilon}{2}\right\}\right|+\frac{\varepsilon}{2}.$$

Consequently, we have

$$\left\{(r,s,t)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s,t}}\sum_{(j,k,l)\in I_{r,s,t}}\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)\geq\varepsilon\right\}$$
$$\subseteq\left\{(r,s,t)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s,t}}\left|\left\{(j,k,l)\in I_{r,s,t}:\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)\geq\frac{\varepsilon}{2}\right\}\right|\right.$$
$$\left.\geq\frac{\varepsilon}{2M}\right\}\in\mathcal{I}_3.$$

Therefore, $x\overset{N_{\theta_3}^{L}\left(\mathcal{I}_3-\widetilde{\phi}\right)}{\sim}y$.

(iv) Follows from *(i)*, *(ii)* and *(iii)*. □

Theorem 3.10. *Let $\mathcal{I}_3\subset\mathcal{P}(\mathbb{N}\times\mathbb{N}\times\mathbb{N})$ be an admissible ideal. Suppose that for given $\delta>0$ and every $\varepsilon>0$ such that*

$$\left\{(r,s,t)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{rst}\left|\left\{0\leq j\leq r-1,0\leq k\leq s-1,0\leq l\leq t-1:\right.\right.\right.$$
$$\left.\left.\left.\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)\geq\varepsilon\right\}\right|<\delta\right\}\in\mathcal{F}_3,$$

then $x\overset{S^{L}\left(\mathcal{I}_3-\widetilde{\phi}\right)}{\sim}y$.

Proof. Let $\delta>0$ be given. For every $\varepsilon>0$, choose r_1,s_1,t_1 such that

(1)
$$\frac{1}{rst}\left|\left\{0\leq j\leq r-1,0\leq k\leq s-1,0\leq l\leq t-1:\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)\geq\varepsilon\right\}\right|<\frac{\delta}{2},$$

for all $r\geq r_1,s\geq s_1,t\geq t_1$. It is sufficient to show that there exist r_2,s_2,t_2 such that for $r\geq r_2,s\geq s_2,t\geq t_2$,

(2)
$$\frac{1}{rst}\left|\left\{0\leq j\leq r-1,0\leq k\leq s-1,0\leq l\leq t-1:\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)\geq\varepsilon\right\}\right|<\frac{\delta}{2}.$$

Let $r_0=\max\{r_1,r_2\}$, $s_0=\max\{s_1,s_2\}$, $t_0=\max\{t_1,t_2\}$. The relation (1) will be true for $r>r_0$, $s>s_0$ and $t>t_0$. If p_0,q_0,m_0 chosen fixed, then we get

$$\left|\left\{0\leq j\leq p_0-1,0\leq k\leq q_0-1,0\leq l\leq m_0-1:\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}}-L\right)\geq\varepsilon\right\}\right|=P.$$

Now, for $r > p_0$, $s > q_0$ and $t > m_0$, we have

$$\frac{1}{rst}\left|\left\{0 \leq j \leq r-1, 0 \leq k \leq s-1, 0 \leq l \leq t-1 : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\}\right|$$
$$\leq \frac{1}{rst}\left|\left\{0 \leq j \leq p_0-1, 0 \leq k \leq q_0-1, 0 \leq l \leq m_0-1 : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\}\right|$$
$$+ \frac{1}{rst}\left|\left\{p_0 \leq j \leq p-1, q_0 \leq k \leq q-1, m_0 \leq l \leq r-1 : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\}\right|$$
$$\leq \frac{P}{rst} + \frac{1}{rst}\left|\{p_0 \leq j \leq p-1, q_0 \leq k \leq q-1, m_0 \leq l \leq r-1 : \right.$$
$$\left.\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right|\Bigg\}$$
$$\leq \frac{P}{rst} + \frac{\delta}{2} < \delta.$$

for sufficiently large r, s, t. This established the result. □

Theorem 3.11. *Let $\mathcal{I}_3 \subset \mathcal{P}(\mathbb{N}\times\mathbb{N}\times\mathbb{N})$ be a non-trivial ideal. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \{(j_r, k_s, l_t)\}$ be a lacunary triple sequence with* $\liminf q_{r,s,t} > 1$*. Then $x \overset{S^L(\mathcal{I}_3-\widetilde{\phi})}{\sim} y$ implies $x \overset{S^L_{\theta_3}(\mathcal{I}_3-\widetilde{\phi})}{\sim} y$.*

Proof. Suppose that $\liminf q_{r,s,t} > 1$, then there exists a $\gamma > 0$ such that $q_{r,s,t} \geq 1+\gamma$ for sufficiently large r, s, t, which implies

$$\frac{h_{r,s,t}}{k_{r,s,t}} \geq \frac{\gamma}{1+\gamma}.$$

If $x \overset{S^L(\mathcal{I}_3-\widetilde{\phi})}{\sim} y$, then for every $\varepsilon > 0$ and for sufficiently large r, s, t, we have

$$\frac{1}{k_{r,s,t}}\left|\left\{j \leq j_r, k \leq k_s, l \leq l_t : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\}\right|$$
$$\geq \frac{1}{k_{r,s,t}}\left|\left\{(j,k,l) \in I_{r,s,t} : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\}\right|$$
$$\geq \frac{\gamma}{1+\gamma}\frac{1}{h_{r,s,t}}\left|\left\{(j,k,l) \in I_{r,s,t} : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\}\right|.$$

Then, for any $\delta > 0$, we get

$$\left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s,t}}\left|\left\{(j,k,l) \in I_{r,s,t} : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\}\right| \geq \delta\right\}$$
$$\subseteq \left\{(r,s,t) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{k_{r,s,t}}\left|\{j \leq j_r, k \leq k_s, l \leq l_t : \right.\right.$$
$$\left.\left.\widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L\right) \geq \varepsilon\right\}\right| \geq \frac{\gamma\delta}{1+\gamma}\right\} \in \mathcal{I}_3.$$

This completes the proof. □

Theorem 3.12. *Let $\mathcal{I}_3 = \mathcal{I}_3^{\text{fin}} = \{A : A \text{ is a finite set}\}$ be a non-trivial ideal, and let $\theta_3 = \{(j_r, k_s, l_t)\}$ be a lacunary triple sequence with* $\limsup q_{r,s,t} < \infty$. *Then*

$$x \overset{S_{\theta_3}^L(\mathcal{I}_3-\widetilde{\phi})}{\sim} y \quad \textit{implies } x \overset{S^L(\mathcal{I}_3-\widetilde{\phi})}{\sim} y .$$

Proof. If $\limsup_{r,s,t} q_{r,s,t} < \infty$, then without loss of generality, we can assume that there exists a $D > 0$ such that $q_{r,s,t} < D$ for all $r, s, t \geq 1$. Suppose that $x \overset{S_{\theta_3}^L(\mathcal{I}_3-\widetilde{\phi})}{\sim} y$ and for $\varepsilon > 0, \delta > 0$ define the sets

$$F_{r,s,t} = \left| \left\{ (j,k,l) \in I_{r,s,t} : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L \right) \geq \varepsilon \right\} \right|$$

and

$$\left\{ (r,s,t) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s,t}} \left| \left\{ (j,k,l) \in I_{r,s,t} : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L \right) \geq \varepsilon \right\} \right| \geq \delta \right\}$$
$$= \left\{ (r,s,t) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{F_{r,s,t}}{h_{r,s,t}} \geq \delta \right\} \in \mathcal{I}_3,$$

and, therefore, it is a finite set. We choose integers $r_0, s_0, t_0 \in \mathbb{N}$ such that

$$\frac{F_{r,s,t}}{h_{r,s,t}} < \delta \text{ for all } r > r_0, s > s_0, t > t_0.$$

Let $F = \max\{F_{r,s,t} : 1 \leq r \leq r_0, 1 \leq s \leq s_0, 1 \leq t \leq t_0\}$ and p, q, r be three integers with $j_{r-1} < p \leq j_r, k_{s-1} < q \leq k_s$ and $l_{t-1} < r \leq l_t$, then we have

$$\frac{1}{pqr} \left| \left\{ j \leq p, k \leq q, l \leq r : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L \right) \geq \varepsilon \right\} \right|$$
$$\leq \frac{1}{j_{r-1}k_{s-1}l_{t-1}} \left| \left\{ j \leq j_r, k \leq k_s, l \leq l_t : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L \right) \geq \varepsilon \right\} \right|$$
$$= \frac{1}{j_{r-1}k_{s-1}l_{t-1}} \left\{ \left| \left\{ (j,k,l) \in I_{1,1,1} : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L \right) \geq \varepsilon \right\} \right| + ... \right.$$
$$\left. + \frac{1}{j_{r-1}k_{s-1}l_{t-1}} \left| \left\{ (j,k,l) \in I_{r,s,t} : \widetilde{\phi}\left(\frac{x_{jkl}}{y_{jkl}} - L \right) \geq \varepsilon \right\} \right| \right\}$$
$$= \frac{1}{j_{r-1}k_{s-1}l_{t-1}} \{F_{1,1,1} + F_{2,2,2} + ... + F_{r_0,s_0,t_0} + F_{r_0+1,s_0+1,t_0+1} + ... + F_{r,s,t}\}$$
$$= \frac{F}{j_{r-1}k_{s-1}l_{t-1}} r_0 s_0 t_0$$
$$+ \frac{1}{j_{r-1}k_{s-1}l_{t-1}} \left\{ h_{r_0+1,s_0+1,t_0+1} \left(\frac{F_{r_0+1,s_0+1,t_0+1}}{h_{r_0+1,s_0+1,t_0+1}} \right) + ... + h_{r,s,t} \frac{F_{r,s,t}}{h_{r,s,t}} \right\}$$

$$= \frac{F}{j_{r-1}k_{s-1}l_{t-1}} r_0 s_0 t_0$$
$$+ \frac{1}{j_{r-1}k_{s-1}l_{t-1}} \left(\sup_{r>r_0, s>s_0, t>t_0} \frac{F_{r,s,t}}{h_{r,s,t}} \right) \{h_{r_0+1,s_0+1,t_0+1} + ... + h_{r,s,t}\}$$
$$\leq \frac{F}{j_{r-1}k_{s-1}l_{t-1}} r_0 s_0 t_0 + \delta \left(\frac{j_r k_s l_t - j_{r_0} k_{s_0} l_{t_0}}{j_{r-1}k_{s-1}l_{t-1}} \right)$$
$$\leq \frac{F}{j_{r-1}k_{s-1}l_{t-1}} r_0 s_0 t_0 + \delta q_{r,s,t}$$
$$\leq \frac{F}{j_{r-1}k_{s-1}l_{t-1}} r_0 s_0 t_0 + \delta D.$$

This completes the proof of the theorem. □

Definition 3.13. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $p \in (0, \infty)$. Two number triple sequences $x = (x_{jkl})$ and $y = (y_{jkl})$ are said to be strongly $\mathcal{I}$-asymptotically lacunary $\widetilde{\phi}(p)$-equivalent of multiple L if*

$$\left\{ (r,s,t) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s,t}} \sum_{(j,k,l) \in I_{r,s,t}} \left(\widetilde{\phi} \left(\frac{x_{jkl}}{y_{jkl}} - L \right) \right)^p \geq \varepsilon \right\} \in \mathcal{I}_3$$

for every $\varepsilon > 0$ (denoted by $x \overset{N^L_{\theta_3}(\mathcal{I}_3 - \widetilde{\phi}(p))}{\sim} y$) and strongly simply $\mathcal{I}$-asymptotically lacunary $\widetilde{\phi}(p)$-equivalent if $L = 1$.

Definition 3.14. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $p \in (0, \infty)$. Two number triple sequences $x = (x_{jkl})$ and $y = (y_{jkl})$ are said to be $\mathcal{I}$-asymptotically lacunary statistical $\widetilde{\phi}(p)$-equivalent of multiple L if*

$$\left\{ (r,s,t) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s,t}} \left| \left\{ (j,k,l) \in I_{r,s,t} : \left(\widetilde{\phi} \left(\frac{x_{jkl}}{y_{jkl}} - L \right) \right)^p \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}_3$$

for every $\varepsilon > 0, \delta > 0$ (denoted by $x \overset{S^L_{\theta_3}(\mathcal{I}_3 - \widetilde{\phi}(p))}{\sim} y$) and simply $\mathcal{I}$-asymptotically lacunary statistical $\widetilde{\phi}(p)$-equivalent if $L = 1$.

Theorem 3.15. *Let $\mathcal{I}_3 \subset \mathcal{P}(\mathbb{N} \times \mathbb{N} \times \mathbb{N})$ be a non-trivial ideal, and $\theta_3 = \{(j_r, k_s, l_t)\}$ be a lacunary triple sequence. Then, the following statements hold:*

(i) If $x \overset{N^L_{\theta_3}(\mathcal{I}_3 - \widetilde{\phi}(p))}{\sim} y$ then $x \overset{S^L_{\theta_3}(\mathcal{I}_3 - \widetilde{\phi}(p))}{\sim} y$;

(ii) $x \overset{N^L_{\theta_3}(\mathcal{I}_3 - \widetilde{\phi}(p))}{\sim} y$ is a proper subset of $x \overset{S^L_{\theta_3}(\mathcal{I}_3 - \widetilde{\phi}(p))}{\sim} y$;

(iii) If $x, y \in \ell^3_\infty$ and $x \overset{S^L_{\theta_3}(\mathcal{I}_3 - \widetilde{\phi}(p))}{\sim} y$ then $x \overset{N^L_{\theta_3}(\mathcal{I}_3 - \widetilde{\phi}(p))}{\sim} y$;

(iv) $x \overset{S^L_{\theta_3}(\mathcal{I}_3 - \widetilde{\phi}(p))}{\sim} y \cap \ell^3_\infty = x \overset{N^L_{\theta_3}(\mathcal{I}_3 - \widetilde{\phi}(p))}{\sim} y \cap \ell^3_\infty$.

Proof. The proof of the theorem follows from the proof of Theorem 3.9. □

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