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# FUZZY FRACTIONAL CONFORMABLE LAPLACE TRANSFORMS

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**Abstract.** In this paper, we define a fractional conformable fuzzy Laplace transform and prove some related theorems. Also by using this transform we solve some fuzzy fractional differential equations.

## 1. Introduction

Fractional calculus is the generalization of the standard calculus. That involves the derivative of functions to arbitrary orders and has found many applications in science, engineering, and so forth. During the last decade, there were a lots of works on discrete fractional calculus for special equations and have been discussed extensively as valuable tools in the modeling of many phenomena in various fields of science and engineering. Fractional derivatives are generalizations for a derivative of integral order. The readers can find more details in [4, 5, 7, 8, 9, 10, 12, 13, 14] and the references therein .
Fuzzy calculus is a part of mathematical analysis, widely explored in ongoing years and has risen as a viable and amazing assert for the scientific demonstrating of engineering and scientific phenomena. The uncertainty is important subject in measurement of quantities in physics. Fuzzy sets have been introduced by Lotfi Zadeh in 1965 and since then they have been used in many applications .
Fuzzy Fractional Differential Equations (FFDE) can offer a more comprehensive account of the process of phenomenon. This has recently captured much attention in FFDE. As the derivative of a function is defined in the sense of Riemann-Liouville, Grunwald-Letnikov or Caputo in fractional calculus, the used derivative is to be specified and defined in FFDE as well [3, 6, 16].
The fuzzy Laplace transform method solves FDE and corresponding fuzzy initial and boundary value problems. In this way fuzzy Laplace transforms reduce

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the problem of solving a FDE to an algebraic problem. This switching from operations of calculus to algebraic operations on transforms is called operational calculus, a very important area of applied mathematics. The fuzzy Laplace transform method solves FFDEs and corresponding fuzzy initial and boundary value problems also has the advantage that it solves problems directly without determining a general solution in the first and obtaining non homogeneous differetial equations [2, 15, 18, 19].
The derivative for fuzzy valued mappings was developed by Puri and Ralescu in 1983 [11] which generalized and extended the concept of Hukuhara differentiability for set-valued mappings to the class of fuzzy mappings.
In [14], a new well-behaved simple fractional derivative which is called "the conformable fractional derivative" depending just on the basic limit definition of the derivative was defined, namely, for a function $h:(0,\infty)\to\mathbf{R}$ the conformable fractional derivative of order $0<\alpha\leq 1$ of $h$ at $t>0$ is as follows:

$$T_{\alpha}(h)(t)=\lim_{\epsilon\to 0}\frac{h(t+\epsilon t^{1-\alpha})-h(t)}{\epsilon}.$$

This definition is very easy for calculating derivatives and solving fractional differential equations compared with other fractional definitions. Moreover it can be easily extended to generalize many integral transforms such as Laplace, Mellin, Natural and Sumudu transforms [19].
Also the conformable fractional integral has been defined of order $\alpha$ by [17]:

$$I^{\alpha}h(x)=\int_{0}^{x}h(t)t^{\alpha-1}\mathbf{dt}.$$

In fact, if $h(x)$ is an n-differentiable function at $x>0$ and $\alpha\in(0,1],n\in\mathbf{N}$, then:

$$D^{\alpha}h(x)=x^{1-\alpha}\frac{d}{dx}h(x),$$
$$D^{\alpha}I^{\alpha}h(x)=h(x).$$

## 2. Preliminaries

In this paper, we define Fuzzy Laplace Conformable transform and prove some related theorems. Then we solve some FCFDE by this transform. Finally we present the conclusions by drawing a diagram.
We denote the set of all real numbers by $\mathbf{R}$ and the set of all fuzzy numbers on $\mathbf{R}$ is indicated by $\mathbf{E}$. A fuzzy number is a mapping $u:\mathbf{R}\to[0,1]$ with the following properties:
a) $u$ is upper semi-continuous,
b) $u$ is fuzzy convex, i.e, $u(\lambda x+(1-\lambda)y)\geq\min\{u(x),u(y)\}$ for all $x,y\in\mathbf{R},\lambda\in[0,1]$,
c) $u$ is normal, i.e, there exist $x_0\in\mathbf{R}$ such that $u(x_0)=1$,
d) $suppu=\{x\in\mathbf{R},u(x)>0\}$ is the support of the $u$, and its closure is

compact.
An equivalent parametric definition is also given in as follows:

**Definition 2.1.** [18] *A fuzzy number $u$ in a parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r), 0 \le r \le 1$, which satisfy the following requirements:*
*a) $\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0,1]$ and right continuous at 0,*
*b) $\overline{u}(r)$ is a bounded non-increasing left continuous function in $(0,1]$ and right continuous at 0,*
*c) $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1$.*

According to Zadeh's extension principle, operation of addition on $\mathbf{E}$ is defined by

$$(u \oplus v)(x) = \sup_{y\in\mathbf{R}} \min\{u(y), v(x-y)\}, \quad x \in \mathbf{R},$$

and scaler multiplication of a fuzzy number is given by

$$(k \odot u)(x) = \begin{cases} u(\frac{x}{k}), & \text{k¿0}, \\ \bar{0}, & k = 0, \end{cases}$$

where $\bar{0} \in E$.
It is well known that the following properties are true for all levels

$$[u \oplus v]_r = [u]_r \oplus [v]_r, \quad [k \odot u]_r = k[u]_r.$$

The Hausdorff distance between fuzzy numbers given by $d : \mathbf{E} \times \mathbf{E} \to [0, \infty)$,

$$d(u, v) = \sup_{r\in[0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\},$$

where $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r)) \subset \mathbf{R}$ is utilized in [3]. Then it is easy to see that $d$ is a metric in $\mathbf{E}$ and has the following properties:
a) $d(u \oplus w, v \oplus w) = d(u, v), \forall u, v, w \in \mathbf{E}$,
b) $d(k \odot u, k \odot v) = |k|d(u, v), \forall k \in \mathbf{R}, u, v \in \mathbf{E}$,
c) $d(u \oplus v, w \oplus e) \le d(u, w) + d(v, e), \forall u, v, w, e \in \mathbf{E}$
d) $(d, \mathbf{E})$ is a complete metric space.
Note that a function $f : A \to \mathbf{E}, A \subseteq \mathbf{R}$ is called fuzzy valued function. The r-cut representation of fuzzy valued function $f$ can be expressed by $f(x, r) = [\underline{f}(x, r), \overline{f}(x, r)], x \in A \subseteq \mathbf{R}$ and $0 \le r \le 1$ [11].

**Definition 2.2.** [9] *Let $f : \mathbf{R} \to \mathbf{E}$ be a fuzzy-valued function. If for arbitrary fixed $t_0 \in \mathbf{R}$ and $\epsilon > 0, \delta > 0$ such that $|t - t_0| < \delta$ we have $d(f(t), f(t_0)) < \epsilon$, $f$ is said to be continuous.*

**Theorem 2.3.** [18] *Let $f(x)$ be a fuzzy value function on $[a, \infty)$ represented by $(\underline{f}(x, r), \overline{f}(x, r))$. For any fixed $r \in [0, 1]$, assume $\underline{f}(x, r)$ and $\overline{f}(x, r)$ are Riemann-integrable on $[a, b]$ for every $b \ge a$ and assume there are two positive functions $\underline{M}(r)$ and $\overline{M}(r)$ such that $\int_a^b |\underline{f}(x, r)|dx \le \underline{M}(r)$ and $\int_a^b |\overline{f}(x, r)|dx \le \overline{M}(r)$ for every $b \ge a$. Then $f(x)$ is improper fuzzy Rieman-integrable on $[a, \infty)$*

*and the improper fuzzy Rieman-integrable is a fuzzy number. Further more, we have:*

$$\int_a^\infty f(x)dx = [\int_a^\infty \underline{f}(x,r)dx, \int_a^\infty \overline{f}(x,r)dx].$$

**Definition 2.4.** [18] *Let $x, y \in \mathbf{E}$, if there exist $z \in \mathbf{E}$ such that $x = y \oplus z$, then $z$ is called the H-difference of $x$ and $y$.*

Note that $x \ominus y \neq x + (-1)y$.

**Theorem 2.5.** [18] *Let $f(x)$ and $g(x)$ are fuzzy value functions and fuzzy Riemmn integrable on $[a, \infty)$ then $f(x) \oplus g(x)$ is fuzzy Riemman-integrable on $[a, \infty)$. Moreover, we have:*

$$\int_a^\infty f(x) \oplus g(x)dx = \int_a^\infty f(x)dx \oplus \int_a^\infty g(x)dx.$$

**Theorem 2.6.** [18] *For $x_0 \in \mathbf{R}$ the fuzzy differential equation $y^{'} = f(x, y)$, $y(x_0) = y \in \mathbf{E}$ where $f : \mathbf{R} \times \mathbf{E} \to \mathbf{E}$ is supposed to be continuous, is equivalent to one of the integral equations:*

$$y(x) = y_0 \oplus \int_{x_0}^x f(t, y(t))\mathbf{d}t, \quad \forall x \in [x_0, x_1],$$

$$y^1(0) = y^1(x) \oplus (-1) \odot \int_{x_0}^x f(t, y^1(t))\mathbf{d}t, \quad \forall x \in [x_0, x_1],$$

*on some interval $(x_0, x_1)$, depending on the strong differentiability considered, $\alpha_1$ or $\alpha_2$-differentiable, respectively.*

Now, we give fuzzy conformable fractional derivatives about order $0 < \alpha < 1$ for fuzzy-value function $F$.

**Definition 2.7.** [9] *Let $F : I \to \mathbf{E}$ be a fuzzy function. $q$th order "fuzzy conformable fractional derivative " of $F$ is defined by*

$$D^\alpha(F)(t) = \lim_{\epsilon \to 0^+} \frac{F(t + \epsilon t^{1-\alpha}) \ominus F(t)}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{F(t) \ominus F(t - \epsilon t^{1-\alpha})}{\epsilon},$$

*for all $t > 0, \alpha \in (0, 1)$.*

If $F$ is $\alpha$-differentiable in some $(0, a)$ then $\lim_{t \to 0^+} F^\alpha(t) = F^\alpha(0)$ and the limit exist in metric d.

Now, we present the following definition that is more general than previous one.

**Definition 2.8.** [9] *Let $F : I \to \mathbf{E}$ be a fuzzy function and $\alpha \in (0, 1]$. One says $F$ is $\alpha_1$-differentiable at point $t > 0$ if there exists an element $D^\alpha(F)(t) \in$*

$\mathbf{R_F}$ *such that for all* $\epsilon > 0$ *sufficiently near to 0, there exist* $F(t+\epsilon t^{1-q}) \ominus F(t)$ , $F(t) \ominus F(t-\epsilon t^{1-\alpha})$ *and the limit (in the metric d):*

$$(1)\, D^{\alpha}(F)(t) = \lim_{\epsilon \to 0^+} \frac{F(t+\epsilon t^{1-\alpha}) \ominus F(t)}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{F(t) \ominus F(t-\epsilon t^{1-\alpha})}{\epsilon}.$$

*Where* $F$ *is* $\alpha_2$*-differentiable at point* $t > 0$ *if there exists an element* $D^{\alpha}(F)(t) \in \mathbf{R_F}$ *such that for all* $\epsilon < 0$ *sufficiently near to 0, there exist* $F(t+\epsilon t^{1-\alpha}) \ominus F(t)$ , $F(t) \ominus F(t-\epsilon t^{1-\alpha})$*:*

$$(2)\, D^{\alpha}(F)(t) = \lim_{\epsilon \to 0^-} \frac{F(t+\epsilon t^{1-\alpha}) \ominus F(t)}{\epsilon} = \lim_{\epsilon \to 0^-} \frac{F(t) \ominus F(t-\epsilon t^{1-\alpha})}{\epsilon}.$$

If $F$ is $\alpha_n$-differentiable at $t > 0$, we denote its $\alpha$-derivative ($\alpha \in (0,1]$) by $F_n^{\alpha}(t)$ for n=1,2.

**Theorem 2.9.** [9] *Let* $F : I \to \mathbf{E}$ *be a fuzzy function, where* $F(t) = [\underline{f}(t,r), \overline{f}(t,r)]$ , $t \in [0,1]$*:*
*i) If* $F$ *is* $\alpha_1$*-differentiable, then* $\underline{f}(t,r)$ *and* $\overline{f}(t,r)$ *are* $\alpha$*-differentiable and*

$$[F^{(\alpha_{(1)})}(t)]^{\alpha} = [\underline{f}^{(\alpha)}(t,r), \overline{f}^{(\alpha)}(t,r)].$$

*ii) If* $F$ *is* $\alpha_2$*-differentiable, then* $f_1^{\alpha}(t)$ *and* $f_2^{\alpha}(t)$ *are* $\alpha$*-differentiable and*

$$[F^{(\alpha_{(2)})}(t)]^{\alpha} = [\overline{f}^{(\alpha)}(t,r), \underline{f}^{(\alpha)}(t,r)].$$

We restrict our attention to functions which are $\alpha_1$ or $\alpha_2$-differentiable on their domain except on a finite number of points.

**Theorem 2.10.** [9] *Let* $F : I \to \mathbf{E}$ *be a fuzzy function. Then the subsequent are working:*
*i) If* $F$ *is* $\alpha_1$*-differentiable at* $t \in I$*, then* $F$ *is continuous at* $t$*.*
*ii) If* $F$ *is* $\alpha_2$*-differentiable at* $t \in I$*, then* $F$ *is continuous at* $t$*.*

More results, knowledge and properties one can refer to Abdeljawad (2015), Allahviranlo (2010) and Bede (2005).

## 3. Main Results

In this section, we discuss what is necessary to apply the fuzzy conformable Laplace transform for solving equations.

**Definition 3.1.** *Let* $f(x)$ *be a continuous fuzzy value function. Suppose that* $f(x) \odot e^{-s\frac{x^{\alpha}}{\alpha}} \odot x^{\alpha-1}$ *is improper fuzzy Riemann-integrable on* $[0,\infty)$*, then* $\int_0^{\infty} f(x) \odot e^{-s\frac{x^{\alpha}}{\alpha}} \odot x^{\alpha-1}\mathbf{dx}$ *is called fuzzy fractional Laplace transform and is denoted as*

$$L_{\alpha}[f(x)] = \int_0^{\infty} f(x) \odot e^{-s\frac{x^{\alpha}}{\alpha}} \odot x^{\alpha-1}\mathbf{dx},$$

*that which* $s > 0$ *and integer.*

From Theorem 3.2, we have:

$$\int_0^\infty f(x) \odot e^{-s\frac{x^\alpha}{\alpha}} \odot x^{\alpha-1}\mathbf{dx}$$
$$= (\int_0^\infty \underline{f}(x,r)e^{-s\frac{x^\alpha}{\alpha}} x^{\alpha-1}\mathbf{dx}, \int_0^\infty \overline{f}(x,r)e^{-s\frac{x^\alpha}{\alpha}} x^{\alpha-1}\mathbf{dx}),$$

also by using the definition of classical Laplace transform, we get $l_\alpha[\underline{f}(x,r)] = \int_0^\infty \underline{f}(x,r)e^{-s\frac{x^\alpha}{\alpha}} x^{\alpha-1}\mathbf{dx}$ and $l_\alpha[\overline{f}(x,r)] = \int_0^\infty \overline{f}(x,r)e^{-s\frac{x^\alpha}{\alpha}} x^{\alpha-1}\mathbf{dx}$, so we have

$$L_\alpha[f(x)] = (l_\alpha[\underline{f}(x,r)], l_\alpha[\overline{f}(x,r)]),$$

**Theorem 3.2.** *Let $f(x), g(x)$ be continuous fuzzy-value functions, suppose that $c_1, c_2$ are constants, then*

$$L_\alpha[(c_1 \odot f(x)) \oplus (c_2 \odot g(x)] = (c_1 \odot L_\alpha[f(x)]) \oplus (c_2 \odot L_\alpha[g(x)]).$$

*Proof.* By Definition 3 and Theorem 2.5 we have,

$$\begin{aligned}
&L_\alpha[(c_1 \odot f(x)) \oplus (c_2 \odot g(x)] \\
&= \int_0^\infty (c_1 \odot f(x) \oplus c_2 \odot g(x)) \odot e^{-s\frac{x^\alpha}{\alpha}} \odot x^{\alpha-1}\mathbf{dx} \\
&= \int_0^\infty c_1 \odot f(x) \odot e^{-s\frac{x^\alpha}{\alpha}} \odot x^{\alpha-1}\mathbf{dx} \oplus \int_0^\infty c_2 \odot g(x) \odot e^{-s\frac{x^\alpha}{\alpha}} \odot x^{\alpha-1}\mathbf{dx} \\
&= (c_1 \odot \int_0^\infty f(x) \odot e^{-s\frac{x^\alpha}{\alpha}} \odot x^{\alpha-1}\mathbf{dx}) \oplus (c_2 \odot \int_0^\infty g(x) \odot e^{-s\frac{x^\alpha}{\alpha}} \odot x^{\alpha-1}\mathbf{dx}) \\
&= c_1 \odot L_\alpha[f(x)] \oplus c_2 \odot L_\alpha[g(x)].
\end{aligned}$$

□

**Lemma 3.3.** *Let $f(x)$ be continuous fuzzy-value function on $[0, \infty)$ and $\lambda \geq 0$, then*

$$L_\alpha[\lambda \odot f(x)] = \lambda \odot L_\alpha[f(x)].$$

*Proof.* By Definition 3 and Theorem 2.5 we have,

$$\begin{aligned}
L_\alpha[\lambda \odot f(x)] &= \int_0^\infty \lambda \odot f(x) \odot e^{-s\frac{x^\alpha}{\alpha}} \odot x^{\alpha-1}\mathbf{dx} \\
&= \lambda \odot \int_0^\infty f(x) \odot e^{-s\frac{x^\alpha}{\alpha}} \odot x^{\alpha-1}\mathbf{dx}
\end{aligned}$$

then

$$L_\alpha[\lambda \odot f(x)] = \lambda \odot L_\alpha[f(x)].$$

□

**Remark 3.4.** *Suppose that $f(x)$ is a continuous fuzzy-value function and $g(x) \geq 0$ and $(f(x) \odot g(x)) \odot e^{-s\frac{x^\alpha}{\alpha}} \odot x^{\alpha-1}$ is a improper fuzzy Riemann-integrable on $[0, \infty)$, then*

$$\int_0^\infty (f(x) \odot g(x)) \odot e^{-s\frac{x^\alpha}{\alpha}} \odot x^{\alpha-1}\mathbf{dx}$$
$$= (\int_0^\infty g(x)\underline{f}(x,r)e^{-s\frac{x^\alpha}{\alpha}}x^{\alpha-1}\mathbf{dx}, \int_0^\infty g(x)\overline{f}(x,r)e^{-s\frac{x^\alpha}{\alpha}}x^{\alpha-1}\mathbf{dx}).$$

**Theorem 3.5.** *Suppose that $f(x)$ is a continuous fuzzy-value function and $L_\alpha[f(x)] = F(s)$, then*

$$L_\alpha[e^{a\frac{x^\alpha}{\alpha}} \odot f(x)] = F(s-a),$$

*where $e^{a\frac{x^\alpha}{\alpha}}$ is a real value function and $s - a > 0$.*

*Proof.*

$$L_\alpha[e^{a\frac{x^\alpha}{\alpha}} \odot f(x)]$$
$$= \int_0^\infty e^{a\frac{x^\alpha}{\alpha}} \odot f(x) \odot e^{-s\frac{x^\alpha}{\alpha}} \odot x^{\alpha-1}\mathbf{dx}$$
$$= (\int_0^\infty \underline{f}(x,r)e^{-(s-a)\frac{x^\alpha}{\alpha}}x^{\alpha-1}\mathbf{dx}, \int_0^\infty \overline{f}(x,r)e^{-(s-a)\frac{x^\alpha}{\alpha}}x^{\alpha-1}\mathbf{dx})$$
$$= \int_0^\infty e^{-(s-a)\frac{x^\alpha}{\alpha}} \odot x^{\alpha-1} \odot f(x)\mathbf{dx} = F(s-a). \tag{3}$$

□

In order to solve fuzzy conformable fractional differential equations, it is necessary to know the fuzzy fractional conformable Laplace transform of the conformable fractional derivative of $f$.

**Theorem 3.6.** *suppose that $f \in C^F[0, \infty) \cap L^F[0, \infty)$, then*

$$L_\alpha[D^\alpha f(x)] = s \odot L_\alpha[f(x)] \ominus f(0),$$

*if $f$ is $\alpha_1$-differentiable and*

$$L_\alpha[D^\alpha f(x)] = -f(0) \ominus s \odot L_\alpha[f(x)],$$

*if $f$ is $\alpha_2$-differentiable.*

*Proof.* For arbitrary fixed $r \in [0, 1]$, we have:

$$sL_\alpha[f(x,r)] \ominus f(0,r) = [sl_\alpha[\underline{f}(x,r)] - \underline{f}(0,r), l_\alpha[\overline{f}(x,r)] - \overline{f}(0,r)],$$

since $f$ is $\alpha_1$-differentiable, we get:

$$[D^\alpha f(x,r)] = [\underline{D^\alpha f}(x,r), \overline{D^\alpha f}(x,r)] = [D^\alpha \underline{f}(x,r), D^\alpha \overline{f}(x,r)],$$

hence by integration by parts we get:

$$\begin{aligned} l_\alpha[\underline{D^\alpha f}(x,r)] &= \int_0^\infty x^{\alpha-1}\underline{D^\alpha f}(x,r)e^{-s\frac{x^\alpha}{\alpha}}\mathbf{dx} = \int_0^\infty x^{\alpha-1}\underline{f}'(x,r)x^{1-\alpha}e^{-s\frac{x^\alpha}{\alpha}}\mathbf{dx} \\ &= \lim_{c\to\infty}[\underline{f}(x,r)e^{-s\frac{x^\alpha}{\alpha}}]_0^c + s\int_0^\infty x^{\alpha-1}\underline{f}(x,r)e^{-s\frac{x^\alpha}{\alpha}}\mathbf{dx}, \\ (4) \qquad &= sl_\alpha[\underline{f}(x,r)] - \underline{f}(0,r), \end{aligned}$$

and similarly:

$$l_\alpha[\overline{D^\alpha f}(x,r)] = l_\alpha[D^\alpha\overline{f}(x,r)] = sl_\alpha[\overline{f}(x,r)] - \overline{f}(0,r),$$

Then we conclude that:

$$s\odot L_\alpha[f(x,r)]\ominus f(0,r) = L_\alpha[D^\alpha\underline{f}(x,r), D^\alpha\overline{f}(x,r)] = L_\alpha[D^\alpha f(x,r)].$$

Now, we assume that $f$ is $\alpha_2$-differentiable, for fixed $r\in[0,1]$, we have

$$\begin{aligned} -f(0,r)+s\odot L_\alpha[f(x,r)] &= (-\overline{f}(0,r)+sl_\alpha[\overline{f}(x,r)], -\underline{f}(0,r)+sl_\alpha[\underline{f}(x,r)]] \\ (5) \qquad &= (sl_\alpha[\overline{f}(x,r)]-\overline{f}(0,r), sl_\alpha[\underline{f}(x,r)]-\underline{f}(0,r)), \end{aligned}$$

since

$$l_\alpha[D^\alpha\overline{f}(x,r)] = sl_\alpha[\overline{f}(x,r)] - \overline{f}(0,r),$$

and

$$l_\alpha[D^\alpha\underline{f}(x,r)] = sl_\alpha[\underline{f}(x,r)] - \underline{f}(0,r),$$

then

$$-f(0,r) - (-sl_\alpha[f(x,r)]) = L_\alpha[D^\alpha\overline{f}(x,r), D^\alpha\underline{f}(x,r)],$$

so

$$-f(0,r) - (-sL_\alpha[f(x,r)]) = L_\alpha[D^\alpha f(x,r)].$$

Since $f$ is $\alpha_2$-differentiable, we have:

$$D^\alpha f(x,r) = [D^\alpha\overline{f}(x,r), D^\alpha\underline{f}(x,r)],$$

hence

$$\begin{aligned} -f(0,r)\ominus(-s\odot L_\alpha[f(x,r)] &= [sl_\alpha[\overline{f}(x,r)]-f(0,r), sl_\alpha[\underline{f}(x,r)]-f(0,r)] \\ &= L_\alpha[D^\alpha\overline{f}(x,r), D^\alpha\underline{f}(x,r)] \end{aligned}$$

□

Now, we present some examples, which indicate how our theorem can be applied to concrete problems.

**Example 3.7.** *Consider the initial value problem*

$$(6) \qquad \begin{cases} D^\alpha y(x) = e^{\frac{x^\alpha}{\alpha}} - y(x) + 1, & 0\le t\le T, \\ y(0) = (\underline{y}(0,r), \overline{y}(0,r)). \end{cases}$$

*By using fuzzy fractional Laplace transform method and applying $\alpha_2$-differentiability, we have:*

$$L_\alpha[D^\alpha y(x)] = L_\alpha[e^{\frac{x^\alpha}{\alpha}} - y(x) + 1],$$

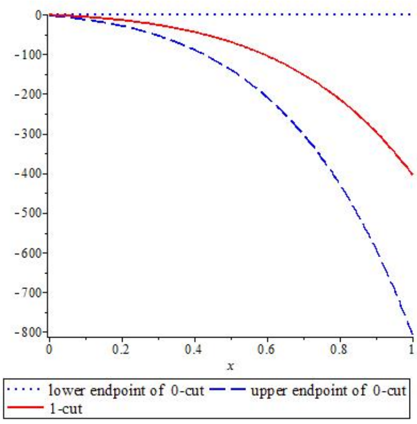


FIGURE 1. Solid, dash and dot denote upper endpoint of 0-cuts, 1-cut and upper lowerpoints of 0-cuts in which $\alpha = \frac{1}{2}$, respectively.

*so*

$$sl_\alpha[\underline{y}(x,r)] - D^\alpha \underline{y}(0,r) = l_\alpha[e^{\frac{x^\alpha}{\alpha}}] - l_\alpha[\underline{y}(x,r)] + l_\alpha[1],$$
$$sl_\alpha[\overline{y}(x,r)] - D^\alpha \overline{y}(0,r) = l_\alpha[e^{\frac{x^\alpha}{\alpha}}] - l_\alpha[\overline{y}(x,r)] + l_\alpha[1],$$

*then*

$$(s+1)l_\alpha[\underline{y}(x,r)] = l_\alpha[e^{\frac{x^\alpha}{\alpha}}] + l[1] + D^\alpha \underline{y}(0,r),$$
$$(s+1)l_\alpha[\overline{y}(x,r)] = l_\alpha[e^{\frac{x^\alpha}{\alpha}}] + l[1] + D^\alpha \overline{y}(0,r),$$

*hence we get*

$$l_\alpha[\underline{y}(x,r)] = \frac{1}{(s-1)(s+1)} + \frac{1}{s(s+1)} + \frac{D^\alpha \underline{y}(0,r)}{(s+1)},$$
$$l_\alpha[\overline{y}(x,r)] = \frac{1}{(s-1)(s+1)} + \frac{1}{s(s+1)} + \frac{D^\alpha \overline{y}(0,r)}{(s+1)}, \tag{7}$$

*consequently, applying inverse of Laplace on the both sides of* 7, *we have:*

$$\underline{y}(x,r) = \frac{1}{2}(e^{\frac{x^\alpha}{\alpha}} - e^{\frac{-x^\alpha}{\alpha}}) + (1 - e^{\frac{x^\alpha}{\alpha}}) + e^{\frac{x^\alpha}{\alpha}} D^\alpha \underline{y}(0,r),$$

*and*

$$\overline{y}(x,r) = \frac{1}{2}(e^{\frac{x^\alpha}{\alpha}} - e^{\frac{-x^\alpha}{\alpha}}) + (1 - e^{\frac{x^\alpha}{\alpha}}) + e^{\frac{x^\alpha}{\alpha}} D^\alpha \overline{y}(0,r),$$

Figure (1) denot upper endpoint of 0-cuts, 1-cut and upper lowerpoints of 0-cuts of the solution of (6) in which $\alpha = \frac{1}{2}$, respectively.

**Example 3.8.** *Consider the following FCFDE*

$$\begin{cases} D^{\alpha}y(x) = \lambda \odot y(x), \quad 0 \leq t \leq T, \\ y(0) = (\underline{y}(0,r), \overline{y}(0,r)). \end{cases} \tag{8}$$

*Applying fuzzy fractional Laplace transform on both sides of above equation, we obtain:*

$$L_{\alpha}[D^{\alpha}y(x)] = L_{\alpha}[\lambda \odot y(x)].$$

*Using $\alpha_1$-differentiability and Theorem 7, We have the following:*

$$\lambda l_{\alpha}\underline{y}(x,r) = sl_{\alpha}[\underline{y}(x,r)] - D^{\alpha}\underline{y}(0,r),$$
$$\lambda l_{\alpha}\overline{y}(x,r) = sl_{\alpha}[\overline{y}(x,r)] - D^{\alpha}\overline{y}(0,r),$$

*so*

$$(s-\lambda)l_{\alpha}[\underline{y}(x,r)] = D^{\alpha}\underline{y}(0,r),$$
$$(s-\lambda)l_{\alpha}[\overline{y}(x,r)] = D^{\alpha}\overline{y}(0,r).$$

*After some manipulations, we get:*

$$l_{\alpha}[\underline{y}(x,r)] = \frac{1}{s-\lambda}D^{\alpha}\underline{y}(0,r)$$
$$l_{\alpha}[\overline{y}(x,r)] = \frac{1}{s-\lambda}D^{\alpha}\overline{y}(0,r). \tag{9}$$

*Applying inverse of Laplace Fractional transform on the both sides of Equation 9, we get the following:*

$$\underline{y}(x,r) = e^{\lambda\frac{x^{\alpha}}{\alpha}}D^{\alpha}\underline{y}(0,r),$$
$$\overline{y}(x,r) = e^{\lambda\frac{x^{\alpha}}{\alpha}}D^{\alpha}\overline{y}(0,r).$$

*Suppose $\lambda \in (-\infty, 0)$, then using $\alpha_2$-differentiability, the solution will obtain similarly.*

Figure (2) denot upper endpoint of 0-cuts, 1-cut and upper lowerpoints of 0-cuts of the solution of (8) in which $\alpha = \frac{1}{2}$ and $\lambda = 3$, respectively. Now, we solve one nonlinear fuzzy fractional differential equation.

**Example 3.9.** *Consider the following FCFDE*

$$\begin{cases} D^{\alpha}y(x) = [y(x)]^{q}, \quad q \in \mathbf{R}, q \neq 0, 1, \\ y(0) = (\underline{y}(0,r), \overline{y}(0,r)). \end{cases} \tag{10}$$

*We make the change of variables;*

$$\underline{z}(x,r) = [\underline{y}(x,r)]^{1-q},$$

*consequently, by using chain rule, we have*

$$D^{\alpha}\underline{y}(x,r) = \frac{1}{1-q}[\underline{z}(x,r)]^{\frac{q}{1-q}}D^{\alpha}\underline{z}(x,r),$$

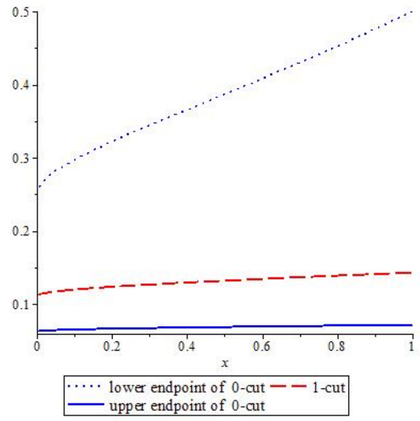


FIGURE 2. Solid ,dash and dot denote upper endpoint of 0-cuts, 1-cut and upper lowerpoints of 0-cuts in which $\alpha = \frac{1}{2}$ and $\lambda = 3$, respectively.

*After an algebraic manipulation, we have:*

$$D^{\alpha}\underline{z}(x,r) = 1 - q,$$

*then using $\alpha_1$-differentiability and applying the conformable Laplace transform, we get*

$$l_{\alpha}(D^{\alpha}\underline{z}(x,r)) = l_{\alpha}(1-q),$$

*so*

$$sl_{\alpha}(\underline{z}(x,r)) - \underline{z}(0,r) = \frac{1-q}{s},$$

*hence*

$$l_{\alpha}(\underline{z}(x,r)) = \frac{(1-q)}{s^2} + \frac{\underline{z}(0,r)}{s}.$$

*Applying the inverse conformable Laplace transform we get:*

$$\underline{z}(x,r) = \frac{(1-q)x^{\alpha}}{\alpha\Gamma(2)} + \underline{z}(0,r),$$

*so*

$$\underline{y}(x,r) = (\frac{(1-q)x^{\alpha}}{\alpha\Gamma(2)} + \underline{y}^{1-q}(0,r))^{\frac{1}{1-q}}.$$

*Similarly,if we make the change of variables $\overline{z}(x,r) = [\overline{y}(x,r)]^{1-q}$, then we get*

$$\overline{y}(x,r) = (\frac{(1-q)x^{\alpha}}{\alpha\Gamma(2)} + \overline{y}^{1-q}(0,r))^{\frac{1}{1-q}}.$$

Figure (3) denot upper endpoint of 0-cuts, 1-cut and upper lowerpoints of 0-cuts of the solution of (10) in which $\alpha = \frac{1}{2}$ and $q = 3$, respectively.

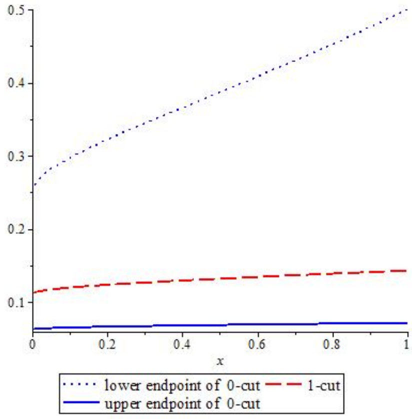


FIGURE 3. Solid ,dash and dot denote upper endpoint of 0-cuts, 1-cut and upper lowerpoints of 0-cuts in which $\alpha = \frac{1}{2}$ and$q = 3$, respectively.

## 4. Conclusion

We have investigated Fuzzy Fractional Conformable Laplace transforms. The conformable Laplace Transform introduced here is a general concept, being also practically applicable.
For further research we propose to extend the results of present paper and also to solve equations involving special functions for example Hyperbolic functions. Moreover one can find a new applications for Fuzzy conformable fractional Laplace Transform like as solving fuzzy output equation, fuzzy input equation, transfer function, etc.

## References


[1] A. Harir, S. Melliani and L.Chadli, *Fuzzy gneralized conformable fractional derivative* , Advance in Fuzzy Systems, **17**(2020).

[2] A. Jafarian, A. K. Golmankhaneh and D. Baleanu, *On Fuzzy fractional laplace transformation*, Advances in Mathematical Physic, **385** (2014).

[3] B. Bede, SG. Gal, *Generalizations of the differetiability of fuzzy-number-valued functions with applications to fuzzy differential equations*,Fuzzy set syst.,(2005), 151:581-599.

[4] Ch. S. Goodrich, *Some new existence results for fractional difference equations*, Int. J. Dynamical Syst. Diff. Eq., **3** (2011), 145-162.

[5] Ch. S. Goodrich, *On discrete sequential fractional boundary value problems*, J.Math.Anal.Appl., **385** (2012), 111-124.

[6] D. Dubios, H. Prade, *Towards fuzzy differential calculus*,Fuzzy set syst.,(1982), 8:1-7.

[7] E. E. Olivera, J. A. T. Machado, *A review of definitions for fractional derivatives and integral* , Mathematics Problems in Engineering, (2014).

[8] F. M. Atici, P. W. Eloe, *Discrete fractional calculus with the nabla operator*, Electron. J. Qual. Theory Differ. Equ., (2009), 1-12.

[9] G. Al Nemer, M. Kenawy, M. Zakarya, C. Cesarano, H. M Rezk, *Generalizations of Hardy's Type Inequalities Via Conformable Calculus*. Symmetry, **13**(2021),1-13.

[10] K. B. Oldham, J. Spanier, *The fractional calculus*, Academic Press, New York, NY, USA,1974.

[11] ML. Puri, D. Ralescu, *Differential for fuzzy function* , J. Math. Anal. Appl, **91** (1983), 522-558.

[12] M. Zakarya, F. Altanji, G. AlNemer, HA. Abd El-Hamid, C. Cesarano, HM. Rezk, *Fractional Reverse Coposn's Inequalities via Conformable Calculus on Time Scales*, Symmetry, **13** (2021), 1-16.

[13] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific,2000.

[14] R. Khalil, M. Horani, A.Yousf and M. Sababheh, *A new definitionof fractional derivative* , J.Comput. Appl. Math., **264** (2009), 65-70.

[15] S. Liang, R. Wu and L. Chen, *Laplace transform of fractional order diffrential equations*, Electronic Journal of Differential Equations, **139** (2015).

[16] SSL. Chang, L. Zade, *on fuzzy mapping and control*, IEEE. trans syst. Cybem.(1972), **2**30-34.

[17] T. Abdeljawad, *On conformable fractional calculus* , Journal of Computional and Applied Mathematics, **279** (2015), 57–66.

[18] T. Allahviranloo, M. Barkhordari, *Fuzzy laplace transforms*, Soft comput., **14** (2010), 235-243.

[19] Z. AL-zhour, F. Alrawajeh, N. AL-mutairi and R. Alkhasawnef, *New results on the conformable fractional sumudu transform* , International Journal of Analysis and Applications, 17(2019),**6**, 1019-1033.

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