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COINCIDENCE POINT RESULTS FOR (ϕ, ψ)-WEAK CONTRACTIVE MAPPINGS IN CONE 2-METRIC SPACES

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Abstract. In the present paper, utilizing (ϕ, ψ)-weak contractive conditions, unique fixed point and some coincidence point results have been studied in the context of cone 2- metric spaces. Also, our obtained results generalize some results from cone metric space to cone 2-metric space. For the authenticity of the presented work, a non trivial example is also provided.

1. Introduction and Preliminaries

A contraction mapping is $\mathcal{D}$ from a space (S, d) to itself, if there exists η with $0 \leq \eta < 1$ such that

$$d(\mathcal{D}s, \mathcal{D}t) \leq \eta\, d(s,t) \text{ for all } s, t \in S. \tag{1}$$

Theorem 1.1. [1] *A mapping $\mathcal{D}$ has unique fixed point in S if $\mathcal{D}$ satisfy (1) and (S, d) is complete metric space.*

The Banach contraction principle (BCP) [1] is usually expanded and improved in two ways, either by generalizing the contraction condition or by replacing complete metric space with some certain generalized spaces. In this way, a lot of works has been reported in the literature on this line by using different classes of contraction type conditions. In Hilbert spaces, the study of single-valued maps satisfying weak contractions was first initiated in 1997 by Alber and Guarre-Delabriere [2] which generalized the contraction principle of Banach. While Rhoades [3] proved that Alber [2] most results are true for any Banach space.

Definition 1.2. The mapping $\mathcal{D} : (S, d) \to (S, d)$ is a weak contraction for all s, t belongs to the complete metric space (S, d) such that

$$d(\mathcal{D}s, \mathcal{D}t) \leq d(s,t) - \phi(d(s,t)), \tag{2}$$

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where $\phi : [0,\infty) \to [0,\infty)$ is continuous and monotone non-decreasing with $\phi(\xi) = 0$ iff $\xi = 0$.

If $0 \leq \eta < 1$ and $\phi(\xi) = (1-\eta)\xi$, then (1) is a special case of (2). In this linkage, Rhoades [3] proved the preceding result.

Theorem 1.3. *([3]) Let metric space (S,d) be complete and mapping $\mathcal{D} : (S,d) \to (S,d)$ satisfying (2), then $\mathcal{D}$ has a unique fixed point.*

The weak contractive condition (ϕ-weak contraction [3]) was extended for two mappings by Song [4] and Zhang and Song [5].

In 2008, Dutta and Chaudhary [6] generalized (1) and (2) by using the concept of ψ functions and demonstrated a fixed point theorem involving the existence of a unique fixed point.

Theorem 1.4. *([6]) Let metric space (S,d) be complete and mapping $\mathcal{D} : (S,d) \to (S,d)$ satisfy*

$$\psi\big(d(\mathcal{D}s,\mathcal{D}t)\big) \leq \psi\big(d(s,t)\big) - \phi\big(d(s,t)\big) \quad \text{for all} \quad s,t \in S, \tag{3}$$

where $\psi,\phi : [0,\infty) \to [0,\infty)$ are continuous, monotone non-decreasing with $\psi(\xi) = 0 = \phi(\xi)$ iff $\xi = 0$. Then $\mathcal{D}$ possesses a unique fixed point.

To put $\psi(\xi) = \xi$ in (3), one can get (2) for all $\xi \geq 0$.
After Dutta and Chaudhary [6], Dorić [7] advanced the abstraction of $(\phi,\psi)-$weak contractive mappings and studied common fixed point results under this contraction. In the recent years, many authors used varieties of weak contractive conditions to illustrate the existence of fixed point theorems in different settings. For the readers, we refer to ([8], [9], [10], [13], [14], [11], [15], [16], [17]) and the references therein.

In 2007, a new generalized space was reintroduced by Huang and Zhang [18] called cone metric space which extends the idea of a metric to cone metric and also presented some fixed point results for contractive type mappings in the underlying spaces.

Let 0_{E_R} be the zero element of real Banach space E_R, then $\mathcal{P} \subseteq E_R$ is called cone if:

(p_1). $\mathcal{P}$ is non-empty, closed and $\mathcal{P} \neq \{0_{E_R}\}$,
(p_2). for all $s,t \in \mathcal{P}$ and for all $a,b \in \mathbb{R}^+$, $as + bt \in \mathcal{P}$,
(p_3). if $w \in \mathcal{P} \cap (-\mathcal{P})$, then $w = 0_{E_R}$.

A partial order $\preccurlyeq$ is define in E_R with respect to $\mathcal{P}$ by $s \preccurlyeq t$ if and only if $t - s \in \mathcal{P}$. Further, $s \prec t$ implies $s \preccurlyeq t$ but $s \neq t$, while $s \lll t$ stands for $t - s \in \text{int}\mathcal{P}$, where $\text{int}\mathcal{P}$ is the interior of $\mathcal{P}$. We called $\mathcal{P}$ normal, if for a real number $k > 0$ and for all $s,t \in E_R$

$$0_{E_R} \preccurlyeq s \preccurlyeq t \text{ implies } \|s\| \leqslant k\,\|t\|\,.$$

If for least k, the above is true then k is known the normal constant of $\mathcal{P}$. The cone $\mathcal{P}$ is regular if for every increasing sequence which is bounded from above is convergent, that is, if $\{v_n\}$ is a sequence in E_R and

$$v_1 \preccurlyeq v_2 \preccurlyeq \cdots \preccurlyeq v_n \preccurlyeq \cdots \preccurlyeq u$$

for some $u \in E_R$, then there exists $v_0 \in E_R$ such that $\|v_n - v_0\| \to 0_{E_R}$ $(n \to \infty)$. The cone $\mathcal{P}$ is also regular if it is convergent for every decreasing sequence which is bounded from below. The cone $\mathcal{P}$ is solid if int$\mathcal{P} \neq \emptyset$.
In the rest of this paper E_R stands for a real Banach space, $\mathcal{P}$ a solid cone in E_R and $\preccurlyeq$ a partial order in E_R w.r.t the cone $\mathcal{P}$.

Definition 1.5. ([18]) Let $S \neq \emptyset$ and $d : S \times S \to E_R$ a function satisfying:
(c_1). for all $s, t \in S$, $0_{E_R} \preccurlyeq d(s,t)$ and $d(s,t) = 0_{E_R}$ iff $s = t$;
(c_2). $d(s,t) = d(t,s)$ for all $s, t \in S$;
(c_3). $d(s,t) \preccurlyeq d(s,w) + d(w,t)$ for all $s, w, t \in S$.
Then the function d is cone metric and (S,d) is cone metric space.

Example 1.6. ([18]) Let $E_R = \mathbb{R}^2$ and $\mathcal{P} = \{(s,t) \in E_R : s,t \geq 0\}$. Take $d : S \times S \to E_R$ as $d(s,t) = (|s-t|, \lambda|s-t|)$, where $\lambda \geq 0$ and $S = \mathbb{R}$. Then (S,d) is a cone metric space.

In [19], Gähler investigated the notion of 2-metric spaces. Let $S \neq \emptyset$, $d : S \times S \times S \to \mathbb{R}^+$ satisfy the following:
(t_1). for $s, t \in S$, there is a point $w \in S$ with at least two of s, t, w are not equal, then $d(s,t,w) \neq 0$;
(t_2). $d(s,t,w) = 0$ iff at least two of s, t, w are equal;
(t_3). for all $s, t, w \in S$, $d(s,t,w) = d(p(s,t,w))$ where $p(s,t,w)$ stands for all permutations of s, t, w;
(t_4). for all $s, t, w, j \in S$, $d(s,t,w) \leq d(s,t,j) + d(s,j,w) + d(j,t,w)$.
Then the function d is 2-metric and (S,d) is a 2-metric space.

Recently, a new generalization is done in 2012 by Singh *et al.* [20] called cone 2-metric space by merging the concepts of cone and 2-metric. For other related works, see [21], [22], [23] and references therein.

Definition 1.7. ([20]) Let $S \neq \emptyset$ and assume that the mapping $d : S \times S \times S \to \mathcal{P}$ satisfying:
(s_1). for all $s, t, w \in S$, $0_{E_R} \preccurlyeq d(s,t,w)$ and $d(s,t,w) = 0_{E_R}$ iff at least two of s, t, w are equal;
(s_2). for all $s, t, w \in S$, $d(s,t,w) = d(p(s,t,w))$ where $p(s,t,w)$ stands for all permutations of s, t, w;
(s_3). $d(s,t,w) \preccurlyeq d(s,t,j) + d(s,j,w) + d(j,t,w)$ for all $s, t, w, j \in S$.
Then the function d is cone 2-metric and (S,d) is cone 2-metric space.

Example 1.8. ([20]) Let $E_R = \mathbb{R}^2$, $\mathcal{P} = \{(s,t) \in E_R : s,t \geq 0_{E_R}\}$ and $d : S \times S \times S \to E_R$ be defined by $d(s,t,w) = (\varrho^\lambda, n\varrho)$, where $\varrho = \min(|s-t|, |t-w|, |s-w|)$, n and λ are fixed positive integers. Then (S,d) is cone 2-metric space.

Definition 1.9. ([20]) Let cone 2-metric space be (S, d) and $\mathcal{P} \subset E_R$. Let $\{u_n\}$ be a sequence in (S, d), Then

(i) $\{u_n\}$ is convergent to $u_0 \in \mathcal{P}$ (i.e $\lim_{n\to\infty} u_n = u_0$ *or* $u_n \to u_0$ *as* $n \to \infty$) if for each $c \in E_R$ with $0_{E_R} \lll c$, there is $n_0 \in \mathbb{N}$ such that $d(u_n, u_0, w) \lll c$ for all $w \in S$ and for all $n > n_0$.

(ii) $\{u_n\}$ is Cauchy sequence if for every $c \in \text{int}\mathcal{P}$, there is $n_0 \in \mathbb{N}$ such that $d(u_n, u_m, w) \lll c$ for all $w \in S$ and for all $n, m \geq n_0$.

(iii) (S, d) is complete if every Cauchy sequence is convergent.

Definition 1.10. ([24]) Let $S \neq \emptyset$ and $f_1, f_2 : S \to S$. If $s = f_1(t) = f_2(t)$ for some $t \in S$, then t is coincidence point and s is point of coincidence of f_1 and f_2. If f_1, f_2 commute at t then f_1, f_2 are weakly compatible.

Lemma 1.11. *([25]) Let $S \neq \emptyset$ and $f_1, f_2 : S \to S$ be weakly compatible. If $s = f_1(t) = f_2(t)$ for some $t \in S$ and s is unique, then s is unique common fixed point for f_1 and f_2.*

Lemma 1.12. *Let $\mathcal{P} \subset E_R$ a cone and $s, t, w \in E_R$.*

(i) If $s \preccurlyeq t$ and $t \lll w$, then $s \lll w$ ([26]).

(ii) If $s \lll t$ and $t \lll w$, then $s \lll w$ ([26]).

(iii) If $0_{E_R} \preccurlyeq s \preccurlyeq t$ and $a \in \mathbb{R}^+$, then $0_{E_R} \preccurlyeq as \preccurlyeq at$ ([26]).

(iv) If $0_{E_R} \preccurlyeq s_n \preccurlyeq t_n$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = s_0$, $\lim_{n\to\infty} t_n = t_0$, then $0_{E_R} \preccurlyeq s_0 \preccurlyeq t_0$ ([26]).

(v) If $s \preccurlyeq t + w$ and $0_{E_R} \lll w$, then $s \preccurlyeq t$ ([26]).

(vi) $\mathcal{P}$ is normal iff $s_n \preccurlyeq t_n \preccurlyeq w_n$ and if $\lim_{n\to\infty} s_n = \lim_{n\to\infty} w_n = u_0$, then $\lim_{n\to\infty} t_n = u_0$ ([28]).

Definition 1.13. ([27]) Let S be a partially order set w.r.t the relation $\preccurlyeq$. Then a function $\mathcal{D}$ from S to itself is monotone increasing if for given $s, t \in S$ with $s \preccurlyeq t$, we have $\mathcal{D}(s) \preccurlyeq \mathcal{D}(t)$.

2. Main results

In the present section, we firstly prove a lemma which will help us in proving our results. Further, in our first Theorem we have studied fixed point for a mapping satisfying (ϕ, ψ)-weak contraction. Secondly, the remaining theorems are concerns with common and coincidence points for mappings satisfy certain contractive type conditions. In at last, a non-trivial example is given to validate our main result.

Lemma 2.1. *Let cone 2-metric space be (S, d) and for a regular cone $\mathcal{P}$, $d(s, t, w) \in int\mathcal{P}$ with at least two of s, t, w are not equal for all $s, t, w \in S$. Let ϕ be a function from $int\mathcal{P} \cup \{0_{E_R}\}$ to itself satisfy*

(a). $\phi(\xi) = 0_{E_R}$ iff $\xi = 0_{E_R}$;

(b). $\phi(\xi) \lll \xi$, for $\xi \in int\mathcal{P}$;

(c). either $\phi(\xi) \preccurlyeq d(s,t,w)$ or $d(s,t,w) \preccurlyeq \phi(\xi)$, for $\xi \in int\mathcal{P} \cup \{0_{E_R}\}$ and for all $s,t,w \in S$.

Let $\{O_n\}$ be a sequence for which $\{d(O_n, O_{n+1}, w)\}$ is a monotonic decreasing sequence in S for all $w \in S$. Then there is $p_0 \in \mathcal{P}$ such that $\{d(O_n, O_{n+1}, w)\}$ is convergent to either $p_0 = 0_{E_R}$ or $p_0 \in int\mathcal{P}$.

Proof. Let $\{O_n\}$ be a sequence in S and $\mathcal{P}$ be a regular cone. Assume that $\{d(O_n, O_{n+1}, w)\}$ is monotonic decreasing for a sequence $\{O_n\}$ in S and $0_{E_R} \preccurlyeq d(O_n, O_{n+1}, w)$ for all $n \in \mathbb{N}$ and $w \in S$, then there exists $p_0 \in \mathcal{P}$ such that

$$d(O_n, O_{n+1}, w) \to p_o \text{ as } n \to \infty. \tag{4}$$

If $d(O_n, O_{n+1}, w) = 0_{E_R}$ for some $n \in \mathbb{N}$, then surely $p_0 = 0_{E_R}$. Hence we assume that $d(O_n, O_{n+1}, w) \neq 0_{E_R}$ for all $n \in \mathbb{N}$ and $w \in S$. Also, according to the condition of the lemma, $d(O_n, O_{n+1}, w) \in \text{int}\mathcal{P}$. Let $p_0 \neq 0_{E_R}$. The cone $\mathcal{P}$ is regular, so it is normal. Let $\mathcal{C} = \{\xi \in \text{int}\mathcal{P} : \|\xi\| < \frac{\|p_0\|}{k}\}$, where k is the normal constant of $\mathcal{P}$. For all $\alpha \in \mathbb{R}^+ - \{0\}$ with $\alpha < \frac{\|p_0\|}{k}$ and $\xi \in \text{int}\mathcal{P}$, we have $\left\| \frac{\alpha\xi}{\|\xi\|} \right\| = \alpha < \frac{\|p_0\|}{k}$. This implies that $\frac{\alpha\xi}{\|\xi\|} \in \mathcal{C}$, therefore $\mathcal{C}$ is non-empty. Next, we affirm that for every $\xi \in \mathcal{C}$, $\phi(\xi) \preccurlyeq d(O_n, O_{n+1}, w)$ for all $n \in \mathbb{N}$ and for all $w \in S$. Otherwise, there exists $\xi_0 \in \mathcal{C}$ and a positive integer m such that

$$d(O_m, O_{m+1}, w) \preccurlyeq \phi(\xi_0) \quad \text{(using (c) property of } \phi \text{ in the lemma).} \tag{5}$$

Now, for all $n \geq m$, $\{d(O_n, O_{n+1}, w)\}$ being monotonic decreasing, therefore, we have

$$d(O_n, O_{n+1}, w) \preccurlyeq d(O_m, O_{m+1}, w) \preccurlyeq \phi(\xi_0), \tag{6}$$

which implies that $d(O_n, O_{n+1}, w) \preccurlyeq \phi(\xi_0)$ for all $n \geq m$ and $w \in S$. As $n \to \infty$ in $d(O_n, O_{n+1}, w) \preccurlyeq \phi(\xi_0)$, using (b) property of ϕ and part (i) of lemma 1.12, we have $p_0 \lll \xi_0$. Hence $\|p_0\| \leq k\,\|\xi_0\|$, for normal constant k of $\mathcal{P}$. Which is a contradiction to the fact that $\xi_0 \in \mathcal{C}$. Thus, for all $\xi \in \mathcal{C}$ and $n \in \mathbb{N}$

$$\phi(\xi) \preccurlyeq d(O_n, O_{n+1}, w) \quad \text{and for all } w \in S. \tag{7}$$

As $n \to \infty$ in above we have $\phi(\xi) \preccurlyeq p_0$. Therefore, for some $p\prime \in \mathcal{P}$, $p_0 = \phi(\xi) + p\prime$ for all $\xi \in \mathcal{C}$. Now, $0_{E_R} \preccurlyeq p\prime \lll \phi(\xi) + p\prime$ (because for every $\xi \in \mathcal{C}$, $\phi(\xi) \in \text{int}\mathcal{P}$). Then by (i) of lemma 1.12, we have

$$0_{E_R} \lll \phi(\xi) + p\prime = p_0 \text{ (because } p_0 = \phi(\xi) + p\prime). \tag{8}$$

Therefore, $p_0 \in \text{int}\mathcal{P}$. □

Theorem 2.2. *Let cone 2-metric space (S,d) be complete and for regular cone $\mathcal{P}$, $d(s,t,w) \in int\mathcal{P}$ with at least two of s,t,w are not equal for all $s,t,w \in S$. Further, if a mapping $\mathcal{D} : S \to S$ is such that*

$$\psi\big(d(\mathcal{D}s, \mathcal{D}t, w)\big) \preccurlyeq \psi\big(d(s,t,w)\big) - \phi\big(d(s,t,w)\big), \tag{9}$$

where ψ from $\mathcal{P}$ to itself and ϕ from int$\mathcal{P} \cup \{0_{E_R}\}$ to itself are two continuous functions such that

(a). ψ is monotone increasing;
(b). $\psi(\xi) = \phi(\xi) = 0_{E_R}$ iff $\xi = 0_{E_R}$;
(c). $\phi(\xi) \lll \xi$, for $\xi \in$ int$\mathcal{P}$;
(d). $\phi(\xi) \preccurlyeq \phi\big(d(s,t,w)\big)$ or $\phi\big(d(s,t,w)\big) \lll \phi(\xi)$ for $\xi \in$ int$\mathcal{P} \cup \{0_{E_R}\}$ and $s, t, w \in S$.

Then $\mathcal{D}$ has a unique fixed point in S.

Proof. Let $O_0 \in S$. The sequence $\{O_n\}$ is constructed by $O_{n+1} = \mathcal{D}O_n$. If $O_n = O_{n+1}$ for some $n \in \mathbb{N}$, then O_n is fixed point of $\mathcal{D}$.
We assume for all $n \in \mathbb{N}$, $O_n \neq O_{n+1}$. Then by (9) we have

$$\begin{aligned} \psi\big(d(O_{n+1}, O_n, w)\big) &= \psi\big(d(\mathcal{D}O_n, \mathcal{D}O_{n-1}, w)\big) \\ &\preccurlyeq \psi\big(d(O_n, O_{n-1}, w)\big) - \phi\big(d(O_n, O_{n-1}, w)\big). \end{aligned} \tag{10}$$

Set $\Delta_n = d(O_n, O_{n-1}, w)$. Then by (10) we have $\psi\big(\Delta_{n+1}\big) \preccurlyeq \psi(\Delta_n) - \phi(\Delta_n) \preccurlyeq \psi(\Delta_n)$. But since ψ is monotone increasing, therefore $\Delta_{n+1} \preccurlyeq \Delta_n$ for $n \in \mathbb{N}$. It indicates that the sequence $\{\Delta_n\}$ is decreasing monotonically and so by lemma 2.1, there exists $p_0 \in \mathcal{P}$ with either $p_0 = 0_{E_R}$ or $p_0 \in$ int$\mathcal{P}$ such that

$$\lim_{n\to\infty} \Delta_n = p_0. \tag{11}$$

Letting $n \to \infty$ in (10) and using the continuities of ϕ and ψ, we have $\psi(p_0) \preccurlyeq \psi(p_0) - \phi(p_0)$, which implies that $-\phi(p_0) \in \mathcal{P}$ and hence $p_0 = 0_{E_R}$. Therefore (11) becomes

$$\lim_{n\to\infty} \Delta_n = 0_{E_R}. \tag{12}$$

Next we affirm that $\{O_n\}$ is a Cauchy sequence. If not, there exists $c \in E_R$ with $0_{E_R} \lll c$ and for all $n_0 \in \mathbb{N}$, there exist $n, m \in \mathbb{N}$ with $n > m \geq n_0$ such that $d(O_m, O_n, w) \not\lll \phi(c)$. Using (d), we have $d(O_m, O_n, w) \succcurlyeq \phi(c)$ and so there exists $\{O_{m(l)}\}$ and $\{O_{n(l)}\}$ of $\{O_n\}$ such that $d(O_{m(l)}, O_{n(l)}, w) \succcurlyeq \phi(c)$ for $n(l) > m(l) > l$ and for all $w \in S$. Assume for smallest such positive integer $n(l)$ with $n(l) > m(l) > l$ such that

$$d(O_{m(l)}, O_{n(l)}, w) \succcurlyeq \phi(c) \quad \text{and} \quad d(O_{m(l)}, O_{n(l)-1}, w) \lll \phi(c). \tag{13}$$

Now consider above we have

$$\begin{aligned} \phi(c) &\preccurlyeq d(O_{m(l)}, O_{n(l)}, w) \\ &\preccurlyeq d(O_{m(l)}, O_{n(l)}, O_{n(l)-1}) + d(O_{m(l)}, O_{n(l)-1}, w) + d(O_{n(l)-1}, O_{n(l)}, w) \\ &\preccurlyeq d(O_{m(l)}, O_{n(l)}, O_{n(l)-1}) + \phi(c) + d(O_{n(l)-1}, O_{n(l)}, w). \end{aligned}$$

By taking $l \to \infty$ in above inequality, using (12) and Lemma 1.12, we have

$$\lim_{l\to\infty} d(O_{m(l)}, O_{n(l)}, w) = \phi(c). \tag{14}$$

Next consider the following four inequalities, that is

$$d(O_{m(l)},O_{n(l)},w) \preccurlyeq d(O_{m(l)},O_{n(l)},O_{m(l)-1}) + d(O_{m(l)},O_{m(l)-1},w) + d(O_{m(l)-1},O_{n(l)},w), \tag{15}$$

$$d(O_{m(l)-1},O_{n(l)},w) \preccurlyeq d(O_{m(l)-1},O_{n(l)},O_{m(l)}) + d(O_{m(l)-1},O_{m(l)},w) + d(O_{m(l)},O_{n(l)},w), \tag{16}$$

$$d(O_{m(l)-1},O_{n(l)-1},w) \preccurlyeq d(O_{m(l)-1},O_{n(l)-1},O_{n(l)}) + d(O_{m(l)-1},O_{n(l)},w) + d(O_{n(l)},O_{n(l)-1},w), \tag{17}$$

$$d(O_{m(l)-1},O_{n(l)},w) \preccurlyeq d(O_{m(l)-1},O_{n(l)},O_{n(l)-1}) + d(O_{m(l)-1},O_{n(l)-1},w) + d(O_{n(l)-1},O_{n(l)},w). \tag{18}$$

Letting $l \to \infty$ in (15-18), using (12) and (14), we get

$$\lim_{l\to\infty} d(O_{m(l)-1},O_{n(l)-1},w) = \phi(c). \tag{19}$$

As $\phi(c) \preccurlyeq d(O_{m(l)},O_{n(l)},w)$ and ψ is monotone increasing, therefore

$$\begin{aligned}\psi\big(\phi(c)\big) &\preccurlyeq \psi\big(d(O_{m(l)},O_{n(l)},w)\big)\\ &= \psi\big(d(\mathcal{D}O_{m(l)-1},\mathcal{D}O_{n(l)-1},w)\big)\\ &\preccurlyeq \psi\big(d(O_{m(l)-1},O_{n(l)-1},w)\big) - \phi\big(d(O_{m(l)-1},O_{n(l)-1},w)\big).\end{aligned} \tag{20}$$

By taking $l \to \infty$ in above inequality, using (19) and continuity property of ϕ, ψ, we get

$$\psi\big(\phi(c)\big) \preceq \psi\big(\phi(c)\big) - \phi\big(\phi(c)\big). \tag{21}$$

Which is not true by definition of ϕ. Therefore, $\{O_n\}$ is a Cauchy sequence in S. By completeness of S, there exists $O_0 \in S$ such that $\lim_{n\to\infty} O_n = O_0$. Now consider

$$\begin{aligned}\psi\big(d(O_n,\mathcal{D}O_0,w)\big) &= \psi\big(d(\mathcal{D}O_{n-1},\mathcal{D}O_0,w)\big)\\ &\preccurlyeq \psi\big(d(O_{n-1},O_0,w)\big) - \phi\big(d(O_{n-1},O_0,w)\big).\end{aligned} \tag{22}$$

As ϕ,ψ are continuous and by taking $n \to \infty$ in above, we have

$$\psi\big(d(O_0,\mathcal{D}O_0,w)\big) \preccurlyeq 0_{E_R}. \tag{23}$$

But $\psi\big(d(O_0,\mathcal{D}O_0,w)\big) \succcurlyeq 0_{E_R}$, this implies that $\psi\big(d(O_0,\mathcal{D}O_0,w)\big) = 0_{E_R}$, and so $d(O_0,\mathcal{D}O_0,w) = 0_{E_R}$. Hence $\mathcal{D}O_0 = O_0$, that is O_0 is a fixed point of $\mathcal{D}$ in S.

For uniqueness of O_0 we let $t_0 \in S$ be another fixed point of $\mathcal{D}$ with $(O_0 \neq t_0)$ in S then

$$\begin{aligned}\psi\big(d(O_0,t_0,w)\big) &= \psi\big(d(\mathcal{D}O_0,\mathcal{D}t_0,w)\big)\\ &\preccurlyeq \psi\big(d(O_0,t_0,w)\big) - \phi\big(d(O_0,t_0,w)\big).\end{aligned} \tag{24}$$

As $(O_0 \neq t_0)$, therefore $\phi\big(d(O_0, t_0, w)\big) \in \text{int}\mathcal{P}$, which from above implies that $\phi\big(d(O_0, t_0, w)\big) = 0_{E_R}$, and hence $O_0 = t_0$. It completes the proof. $\square$

On taking $\psi(\xi) = \xi$ for $\xi \geq 0_{E_R}$ in Theorem 2.2, then we have the following corollary.

Corollary 2.3. *Let cone 2-metric space (S, d) be complete and for regular cone $\mathcal{P}$, $d(s, t, w) \in int\mathcal{P}$ with at least two of s, t, w are not equal for all $s, t, w \in S$. If a mapping $\mathcal{D} : S \to S$ is such that*

$$d(\mathcal{D}s, \mathcal{D}t, w) \preccurlyeq d(s, t, w) - \phi\big(d(s, t, w)\big), \tag{25}$$

for $s, t, w \in S$, where ϕ is defined with given properties in Theorem 2.2. Then $\mathcal{D}$ has a unique fixed point in S.

Remark 2.4. In fact, by taking $\phi(\xi) = K\xi$ where $0 < K < 1$ then corollary 2.3 generalize and improve corollary 2.2 of [20].

Theorem 2.5. *Let cone 2-metric space be (S, d) and for regular cone $\mathcal{P}$, $d(s, t, w) \in int\mathcal{P}$ with at least two of s, t, w are not equal for all $s, t, w \in S$. Let $\mathcal{R}, \mathcal{K} : S \to S$ are such that*

$$\psi\big(d(\mathcal{R}s, \mathcal{R}t, w)\big) \preccurlyeq \psi\big(d(\mathcal{K}s, \mathcal{K}t, w)\big) - \phi\big(d(\mathcal{K}s, \mathcal{K}t, w)\big) \tag{26}$$

for all $s, t, w \in S$, where ϕ and ψ are defined in Theorem 2.2 with given properties.
If $\mathcal{K}(S) \subseteq S$ is complete and $\mathcal{R}(S) \subseteq \mathcal{K}(S)$, then $\mathcal{R}$, $\mathcal{K}$ has unique point of coincidence.

Proof. Let $O_0 \in S$. Since $\mathcal{R}(S) \subseteq \mathcal{K}(S)$, $\{O_n\}$ is constructed in S as $\mathcal{R}O_n = \mathcal{K}O_{n+1}$ for $n \in \mathbb{N} \cup \{0\}$. If for some $n_0 \in \mathbb{N} \cup \{0\}$, we have $\mathcal{R}O_{n_0} = \mathcal{R}O_{n_0+1}$, then O_{n_0+1} is a point of coincidence of $\mathcal{R}$ and $\mathcal{K}$.
Hence, we shall assume that $\mathcal{R}O_n \neq \mathcal{R}O_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. By (26) we have

$$\begin{aligned}\psi\big(d(\mathcal{R}O_{n+2}, \mathcal{R}O_{n+1}, w)\big) \preccurlyeq\ & \psi\big(d(\mathcal{K}O_{n+2}, \mathcal{K}O_{n+1}, w)\big) \\ & - \phi\big(d(\mathcal{K}O_{n+2}, \mathcal{K}O_{n+1}, w)\big)\end{aligned}$$

for all $w \in S$, and so

$$\begin{aligned}\psi\big(d(\mathcal{R}O_{n+2}, \mathcal{R}O_{n+1}, w)\big) \preccurlyeq\ & \psi\big(d(\mathcal{R}O_{n+1}, \mathcal{R}O_n, w)\big) \\ & - \phi\big(d(\mathcal{R}O_{n+1}, \mathcal{R}O_n, w)\big).\end{aligned} \tag{27}$$

Using definition of ϕ and monotone increasing property of ψ, we have

$$d(\mathcal{R}O_{n+2}, \mathcal{R}O_{n+1}, w) \preccurlyeq d(\mathcal{R}O_{n+1}, \mathcal{R}O_n, w). \tag{28}$$

Therefore, the sequence $\{d(\mathcal{R}O_{n+1}, \mathcal{R}O_n, w)\}$ is decreasing monotonically. Hence, by Lemma 2.1 there exists $p_0 \in \mathcal{P}$ with either $p_0 = 0_{E_R}$ or $p_0 \in \text{int}\mathcal{P}$ and

$$d(\mathcal{R}O_{n+1}, \mathcal{R}O_n, w) \to p_0 \ as\ n \to \infty. \tag{29}$$

Letting $n \to \infty$ in (27) and using continuities of ϕ and ψ, we have

$$\psi(p_0) \preccurlyeq \psi(p_0) - \phi(p_0). \tag{30}$$

The above is true only when $p_0 = 0_{E_R}$. Therefore, for all $w \in S$ (29) becomes

$$d(\mathcal{R}O_{n+1}, \mathcal{R}O_n, w) \to 0_{E_R} \ \textit{as} \ n \to \infty. \tag{31}$$

Next, we affirm that $\{\mathcal{R}O_n\}$ is a Cauchy sequence. In case if it is not, then there exist $c \in E_R$ with $0_{E_R} \lll c$ and for all $n_0 \in \mathbb{N}$ there exists $n, m \in \mathbb{N}$ with $n > m \geq n_0$ such that

$$d(\mathcal{R}O_m, \mathcal{R}O_n, w) \not\lll \phi(c). \tag{32}$$

Using (d) we have $d(\mathcal{R}O_m, \mathcal{R}O_n, w) \succcurlyeq \phi(c)$. Hence, there exist $\{\mathcal{R}O_{m(l)}\}$ and $\{\mathcal{R}O_{n(l)}\}$ of $\{\mathcal{R}O_n\}$ such that $d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{n(l)}, w) \succcurlyeq \phi(c)$ and $n(l) > m(l) > l$ for all $w \in S$.
Assume for smallest such positive integer $n(l)$ with $n(l) > m(l) > l$ such that

$$d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{n(l)}, w) \succcurlyeq \phi(c) \quad \text{and} \quad d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{n(l)-1}, w) \lll \phi(c). \tag{33}$$

Now by considering above we have

$$\begin{aligned}\phi(c) &\preccurlyeq d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{n(l)}, w)\\ &\preccurlyeq d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{n(l)}, \mathcal{R}O_{n(l)-1}) + d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{n(l)-1}, w)\\ &\quad + d(\mathcal{R}O_{n(l)-1}, \mathcal{R}O_{n(l)}, w).\end{aligned}$$

By using (33) and part (v) of Lemma 1.12, we have

$$\begin{aligned}\phi(c) &\preccurlyeq d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{n(l)}, w)\\ &\preccurlyeq d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{n(l)}, \mathcal{R}O_{n(l)-1}) + \phi(c) + d(\mathcal{R}O_{n(l)-1}, \mathcal{R}O_{n(l)}, w).\end{aligned}$$

As $l \to \infty$ in above and using (31), we get for all $w \in S$

$$\lim_{l \to \infty} d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{n(l)}, w) = \phi(c). \tag{34}$$

Next consider the following four inequalities, that is

$$\begin{aligned}&d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{n(l)}, w)\\ &\quad \preccurlyeq d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{n(l)}, \mathcal{R}O_{m(l)+1}) + d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{m(l)+1}, w)\\ &\qquad + d(\mathcal{R}O_{m(l)+1}, \mathcal{R}O_{n(l)}, w),\end{aligned}$$

$$\begin{aligned}&d(\mathcal{R}O_{m(l)+1}, \mathcal{R}O_{n(l)}, w)\\ &\quad \preccurlyeq d(\mathcal{R}O_{m(l)+1}, \mathcal{R}O_{n(l)}, \mathcal{R}O_{m(l)}) + d(\mathcal{R}O_{m(l)+1}, \mathcal{R}O_{m(l)}, w)\\ &\qquad + d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{n(l)}, w),\end{aligned}$$

$$\begin{aligned}&d(\mathcal{R}O_{m(l)+1}, \mathcal{R}O_{n(l)+1}, w)\\ &\quad \preccurlyeq d(\mathcal{R}O_{m(l)+1}, \mathcal{R}O_{n(l)+1}, \mathcal{R}O_{n(l)}) + d(\mathcal{R}O_{m(l)+1}, \mathcal{R}O_{n(l)}, w)\\ &\qquad + d(\mathcal{R}O_{n(l)}, \mathcal{R}O_{n(l)+1}, w),\end{aligned}$$

$$\begin{aligned} &d(\mathcal{R}O_{m(l)+1}, \mathcal{R}O_{n(l)}, w) \\ &\quad \preccurlyeq d(\mathcal{R}O_{m(l)+1}, \mathcal{R}O_{n(l)}, \mathcal{R}O_{n(l)+1}) + d(\mathcal{R}O_{m(l)+1}, \mathcal{R}O_{n(l)+1}, w) \\ &\qquad + d(\mathcal{R}O_{n(l)+1}, \mathcal{R}O_{n(l)}, w). \end{aligned}$$

As $l \to \infty$ in above four listed inequalities, using (31) and (34), we get

$$\lim_{l\to\infty} d(\mathcal{R}O_{m(l)+1}, \mathcal{R}O_{n(l)+1}, w) = \phi(c). \tag{35}$$

Substituting $s = O_{m(l)+1}$ and $t = O_{n(l)+1}$ in (26)

$$\begin{aligned} &\psi\big(d(\mathcal{R}O_{m(l)+1}, \mathcal{R}O_{n(l)+1}, w)\big) \preccurlyeq \\ &\qquad \psi\big(d(\mathcal{K}O_{m(l)+1}, \mathcal{K}O_{n(l)+1}, w)\big) - \phi\big(d(\mathcal{K}O_{m(l)+1}, \mathcal{K}O_{n(l)+1}, w)\big). \end{aligned}$$

That is

$$\begin{aligned} \psi\big(d(\mathcal{R}O_{m(l)+1}, \mathcal{R}O_{n(l)+1}, w)\big) \preccurlyeq\ & \psi\big(d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{n(l)}, w)\big) \\ & - \phi\big(d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{n(l)}, w)\big). \end{aligned}$$

Thus we have, as $l \to \infty$ in above and using (34), (35) and continuities of ϕ and ψ

$$\psi\big(\phi(c)\big) \preccurlyeq \psi\big(\phi(c)\big) - \phi\big(\phi(c)\big),$$

which contradicts the fact that $0_{E_R} \lll c$.
Therefore, $\{\mathcal{R}O_n\}$ is a Cauchy sequence in $\mathcal{K}(S)$. As $\mathcal{K}(S)$ is complete, there is $O \in \mathcal{K}(S)$ such that

$$\mathcal{R}O_n \to O \text{ as } n \to \infty. \tag{36}$$

Since $O \in \mathcal{K}(S)$, there is $e \in S$ such that $\mathcal{K}e = O$. Now by putting $s = O_{n+1}$ and $t = e$ in (26), we have

$$\psi\big(d(\mathcal{R}O_{n+1}, \mathcal{R}e, w)\big) \preccurlyeq \psi\big(d(\mathcal{K}O_{n+1}, \mathcal{K}e, w)\big) - \phi\big(d(\mathcal{K}O_{n+1}, \mathcal{K}e, w)\big). \tag{37}$$

That is

$$\psi\big(d(\mathcal{R}O_{n+1}, \mathcal{R}e, w)\big) \preccurlyeq \psi\big(d(\mathcal{R}O_n, O, w)\big) - \phi\big(d(\mathcal{R}O_n, O, w)\big). \tag{38}$$

By letting $n \to \infty$ in above, using (36) and properties of ϕ and ψ we have $\psi\big(d(O, \mathcal{R}e, w)\big) \preccurlyeq 0_{E_R}$, which is true unless $\psi\big(d(O, \mathcal{R}e, w)\big) = 0_{E_R}$, and so that $\mathcal{R}e = O$. Hence, $\mathcal{R}e = \mathcal{K}e = O$. That is, O is a point of coincidence and e is a coincidence point of $\mathcal{R}$ and $\mathcal{K}$.

Further, we reveal that the point of coincidence O is unique. For this, let $e_1 \in S$ be another point such that $\mathcal{R}e_1 = \mathcal{K}e_1 = O_1$ and assume that $O \neq O_1$. Putting $s = e$ and $t = e_1$ in (26)

$$\psi\big(d(\mathcal{R}e, \mathcal{R}e_1, w)\big) \preccurlyeq \psi\big(d(\mathcal{K}e, \mathcal{K}e_1, w)\big) - \phi\big(d(\mathcal{K}e, \mathcal{K}e_1, w)\big). \tag{39}$$

That is

$$\psi\big(d(O, O_1, w)\big) \preccurlyeq \psi\big(d(O, O_1, w)\big) - \phi\big(d(O, O_1, w)\big), \tag{40}$$

which is true unless $O = O_1$.
Hence, the point of coincidence O is unique for $\mathcal{R}$ and $\mathcal{K}$. □

Theorem 2.6. *Let cone 2-metric space be (S,d) and for regular cone $\mathcal{P}$, $d(s,t,w)\in int\mathcal{P}$ with at least two of s,t,w are not equal for all $s,t,w\in S$. Let $\mathcal{R},\mathcal{K}:S\to S$ are such that*

$$\psi\big(d(\mathcal{R}s,\mathcal{R}t,w)\big)\preccurlyeq\psi\Big(\frac{1}{2}\big(d(\mathcal{R}s,\mathcal{K}s,w)+d(\mathcal{R}t,\mathcal{K}t,w)\big)\Big)-\phi\big(d(\mathcal{K}s,\mathcal{K}t,w)\big) \tag{41}$$

for all $s,t,w\in S$, where ϕ and ψ are defined in Theorem 2.2 with given properties.
If $\mathcal{K}(S)\subseteq S$ is complete and $\mathcal{R}(S)\subseteq\mathcal{K}(S)$, then $\mathcal{R}$, $\mathcal{K}$ has unique point of coincidence.

Proof. The sequence $\{O_n\}$ is constructed in the same manner as in Theorem 2.5. Also, we prove with the help of (41) that for all $w\in S$, the sequence $\{d(\mathcal{R}O_{n+1},\mathcal{R}O_n,w)\}$ is monotonically decreasing and

$$d(\mathcal{R}O_{n+1},\mathcal{R}O_n,w)\to 0_{E_R}\ as\ n\to\infty. \tag{42}$$

Next, we claim that $\{\mathcal{R}O_n\}$ is a Cauchy sequence. If it is not the case, then using the same method as given in Theorem 2.5, there exist two subsequences $\{\mathcal{R}O_{m(l)}\}$ and $\{\mathcal{R}O_{n(l)}\}$ of $\{\mathcal{R}O_n\}$ with

$$\lim_{l\to\infty}d(\mathcal{R}O_{m(l)},\mathcal{R}O_{n(l)},w)=\phi(c) \tag{43}$$

and

$$\lim_{l\to\infty}d(\mathcal{R}O_{m(l)+1},\mathcal{R}O_{n(l)+1},w)=\phi(c). \tag{44}$$

By putting $s=O_{m(l)+1}$ and $t=O_{n(l)+1}$ in (41), we have

$$\begin{aligned}&\psi\big(d(\mathcal{R}O_{m(l)+1},\mathcal{R}O_{n(l)+1},w)\big)\\&\quad\preccurlyeq\psi\Big(\frac{1}{2}\big(d(\mathcal{R}O_{m(l)+1},\mathcal{K}O_{m(l)+1},w)+d(\mathcal{R}O_{n(l)+1},\mathcal{K}O_{n(l)+1},w)\big)\Big)\\&\qquad-\phi\big(d(\mathcal{K}O_{m(l)+1},\mathcal{K}O_{n(l)+1},w)\big).\end{aligned}$$

That is

$$\begin{aligned}&\psi\big(d(\mathcal{R}O_{m(l)+1},\mathcal{R}O_{n(l)+1},w)\big)\\&\quad\preccurlyeq\psi\Big(\frac{1}{2}\big(d(\mathcal{R}O_{m(l)+1},\mathcal{R}O_{m(l)},w)+d(\mathcal{R}O_{n(l)+1},\mathcal{R}O_{n(l)},w)\big)\Big)\\&\qquad-\phi\big(d(\mathcal{R}O_{m(l)},\mathcal{R}O_{n(l)},w)\big).\end{aligned}$$

By letting $l\to\infty$ in above, using (42), (43), (44) and properties of ϕ and ψ, we get

$$\psi\big(\phi(c)\big)\preccurlyeq-\phi\big(\phi(c)\big),$$

which contradicts the fact that $0_{E_R}\lll c$, and hence $\{\mathcal{R}O_n\}$ is a Cauchy sequence in $\mathcal{K}(S)$. As $\mathcal{K}(S)$ is complete, there exists $O\in\mathcal{K}(S)$ such that

$$\mathcal{R}O_n\to O\ as\ n\to\infty. \tag{45}$$

Since $O \in \mathcal{K}(S)$, there is $e \in S$ such that $\mathcal{K}e = O$. By putting $s = O_{n+1}$ and $t = e$ in (41)

$$\begin{aligned}\psi\big(d(\mathcal{R}O_{n+1}, \mathcal{R}e, w)\big) \preccurlyeq &\psi\Big(\frac{1}{2}\big(d(\mathcal{R}O_{n+1}, \mathcal{K}O_{n+1}, w) + d(\mathcal{R}e, \mathcal{K}e, w)\big)\Big)\\ &- \phi\big(d(\mathcal{K}O_{n+1}, \mathcal{K}e, w)\big).\end{aligned}$$

Thus

$$\psi\big(d(\mathcal{R}O_{n+1}, \mathcal{R}e, w)\big) \preccurlyeq \psi\Big(\frac{1}{2}\big(d(\mathcal{R}O_{n+1}, \mathcal{R}O_n, w)+d(\mathcal{R}e, O, w)\big)\Big)-\phi\big(d(\mathcal{R}O_n, O, w)\big).$$

By letting $n \to \infty$, using (42), (45) and properties of ϕ and ψ, we get

$$\psi\big(d(O, \mathcal{R}e, w)\big) \preccurlyeq \psi\big(\frac{1}{2}d(\mathcal{R}e, O, w)\big),$$

that is

$$d(O, \mathcal{R}e, w) \preccurlyeq \frac{1}{2}d(\mathcal{R}e, O, w), \quad \text{(by monotonically increasing property of } \psi)$$

which is contradiction unless $\mathcal{R}e = O$ and therefore $\mathcal{R}e = \mathcal{K}e = O$. Thus O is a point of coincidence and e is a coincidence point of $\mathcal{R}$ and $\mathcal{K}$.

Uniqueness of O follows from the proof of Theorem 2.5 and condition (41). □

Theorem 2.7. *Let cone 2-metric space be (S, d) and for regular cone $\mathcal{P}$, $d(s, t, w) \in int\mathcal{P}$ with at least two of s, t, w are not equal for all $s, t, w \in S$. Let $\mathcal{R}, \mathcal{K} : S \to S$ are such that*

$$(46)\quad \psi\big(d(\mathcal{R}s, \mathcal{R}t, w)\big) \preccurlyeq \psi\Big(\frac{1}{2}\big(d(\mathcal{R}s, \mathcal{K}t, w)+d(\mathcal{R}t, \mathcal{K}s, w)\big)\Big)-\phi\big(d(\mathcal{K}s, \mathcal{K}t, w)\big)$$

for all $s, t, w \in S$, where ϕ and ψ are defined in Theorem 2.2 with given properties.
If $\mathcal{K}(S) \subseteq S$ is complete and $\mathcal{R}(S) \subseteq \mathcal{K}(S)$, then $\mathcal{R}$, $\mathcal{K}$ has unique point of coincidence.

Proof. The sequence $\{O_n\}$ is constructed in the same manner as in Theorem 2.5. Also, as showed in Theorem 2.5, we prove with the help of (46) that for all $w \in S$, the sequence $\{d(\mathcal{R}O_{n+1}, \mathcal{R}O_n, w)\}$ is monotonically decreasing and

$$(47)\qquad d(\mathcal{R}O_{n+1}, \mathcal{R}O_n, w) \to 0_{E_R} \text{ as } n \to \infty.$$

If $\{\mathcal{R}O_n\}$ is not a Cauchy sequence, then utilizing the same method as given in the proof of Theorem 2.5 that there are $\{\mathcal{R}O_{m(l)}\}$ and $\{\mathcal{R}O_{n(l)}\}$ of $\{\mathcal{R}O_n\}$, for which we can obtain

$$(48)\qquad \lim_{l\to\infty} d(\mathcal{R}O_{m(l)}, \mathcal{R}O_{n(l)}, w) = \phi(c),$$

and

$$(49)\qquad \lim_{l\to\infty} d(\mathcal{R}O_{m(l)+1}, \mathcal{R}O_{n(l)+1}, w) = \phi(c).$$

Now consider the following inequalities, that is

$$\begin{aligned}
&d(\mathcal{R}O_{m(l)},\mathcal{R}O_{n(l)},w)\\
&\quad\preccurlyeq d(\mathcal{R}O_{m(l)},\mathcal{R}O_{n(l)},\mathcal{R}O_{n(l)+1})+d(\mathcal{R}O_{m(l)},\mathcal{R}O_{n(l)+1},w)\\
&\quad+d(\mathcal{R}O_{n(l)+1},\mathcal{R}O_{n(l)},w),\\
&d(\mathcal{R}O_{m(l)},\mathcal{R}O_{n(l)+1},w)\\
&\quad\preccurlyeq d(\mathcal{R}O_{m(l)},\mathcal{R}O_{n(l)+1},\mathcal{R}O_{n(l)})+d(\mathcal{R}O_{m(l)},\mathcal{R}O_{n(l)},w)\\
&\quad+d(\mathcal{R}O_{n(l)},\mathcal{R}O_{n(l)+1},w),\\
&d(\mathcal{R}O_{m(l)},\mathcal{R}O_{n(l)},w)\\
&\quad\preccurlyeq d(\mathcal{R}O_{m(l)},\mathcal{R}O_{n(l)},\mathcal{R}O_{m(l)+1})+d(\mathcal{R}O_{m(l)},\mathcal{R}O_{m(l)+1},w)\\
&\quad+d(\mathcal{R}O_{m(l)+1},\mathcal{R}O_{n(l)},w),\\
&d(\mathcal{R}O_{m(l)+1},\mathcal{R}O_{n(l)},w)\\
&\quad\preccurlyeq d(\mathcal{R}O_{m(l)+1},\mathcal{R}O_{n(l)},\mathcal{R}O_{m(l)})+d(\mathcal{R}O_{m(l)+1},\mathcal{R}O_{m(l)},w)\\
&\quad+d(\mathcal{R}O_{m(l)},\mathcal{R}O_{n(l)},w).
\end{aligned}$$

Taking $l\to\infty$ in the four inequalities listed above and using (47) and (48), we get

$$\lim_{l\to\infty} d(\mathcal{R}O_{m(l)},\mathcal{R}O_{n(l)+1},w)=\phi(c), \tag{50}$$

and

$$\lim_{l\to\infty} d(\mathcal{R}O_{m(l)+1},\mathcal{R}O_{n(l)},w)=\phi(c). \tag{51}$$

Putting $s=O_{m(l)+1}$ and $t=O_{n(l)+1}$ in (46), we have

$$\begin{aligned}
&\psi\Big(d\big(\mathcal{R}O_{m(l)+1},\mathcal{R}O_{n(l)+1},w\big)\Big)\\
&\preccurlyeq \psi\Big(\frac{1}{2}\big(d(\mathcal{R}O_{m(l)+1},\mathcal{K}O_{n(l)+1},w)+d(\mathcal{R}O_{n(l)+1},\mathcal{K}O_{m(l)+1},w)\big)\Big)\\
&\quad-\phi\big(d(\mathcal{K}O_{m(l)+1},\mathcal{K}O_{n(l)+1},w)\big).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\psi\Big(d\big(\mathcal{R}O_{m(l)+1},\mathcal{R}O_{n(l)+1},w\big)\Big)\\
&\preccurlyeq \psi\Big(\frac{1}{2}\big(d(\mathcal{R}O_{m(l)+1},\mathcal{R}O_{n(l)},w)+d(\mathcal{R}O_{n(l)+1},\mathcal{R}O_{m(l)},w)\big)\Big)\\
&\quad-\phi\big(d(\mathcal{R}O_{m(l)},\mathcal{R}O_{n(l)},w)\big).
\end{aligned}$$

By letting $l\to\infty$ in above inequality, using (48), (49), (50), (51) and continuities of ϕ and ψ, we get

$$\psi\big(\phi(c)\big)\preccurlyeq\psi\big(\phi(c)\big)-\phi\big(\phi(c)\big) \tag{52}$$

which contradicts the fact that $0_{E_R}\lll c$, and therefore $\{\mathcal{R}O_n\}$ is a Cauchy sequence in $\mathcal{K}(S)$. As $\mathcal{K}(S)$ is complete, then there is $O\in\mathcal{K}(S)$ such that

$$\mathcal{R}O_n\to O \text{ as } n\to\infty. \tag{53}$$

Since $O \in \mathcal{K}(S)$, there should be a $e \in S$ such that $\mathcal{K}e = O$. Now, by replacing $s = O_{n+1}$ and $t = e$ in (46), we have

$$\psi\big(d(\mathcal{R}O_{n+1}, \mathcal{R}e, w)\big)$$
$$\preccurlyeq \psi\Big(\frac{1}{2}\big(d(\mathcal{R}O_{n+1}, \mathcal{K}e, w) + d(\mathcal{R}e, \mathcal{K}O_{n+1}, w)\big)\Big) - \phi\big(d(\mathcal{K}O_{n+1}, \mathcal{K}e, w)\big)$$

that is

$$\psi\big(d(\mathcal{R}O_{n+1}, \mathcal{R}e, w)\big)$$
$$\preccurlyeq \psi\Big(\frac{1}{2}\big(d(\mathcal{R}O_{n+1}, O, w) + d(\mathcal{R}e, \mathcal{R}O_n, w)\big)\Big) - \phi\big(d(\mathcal{R}O_n, O, w)\big).$$

As $n \to \infty$ in above inequality, using (53) and properties of ϕ and ψ, we get

$$\psi\big(d(O, \mathcal{R}e, w)\big) \preccurlyeq \psi\big(\frac{1}{2}d(\mathcal{R}e, O, w)\big),$$

that is

$$d(O, \mathcal{R}e, w) \preccurlyeq \frac{1}{2}d(\mathcal{R}e, O, w) \quad \text{(by monotonically increasing property of } \psi\text{)}$$

which is a contradiction unless $\mathcal{R}e = O$ and hence $\mathcal{R}e = \mathcal{K}e = O$. Thus, O is a point of coincidence and e is a coincidence point of $\mathcal{R}$ and $\mathcal{K}$.

Uniqueness of O follows from the proof of Theorem 2.5 and condition (46). □

Theorem 2.8. *If $\mathcal{R}$ and $\mathcal{K}$ are weakly compatible in Theorems 2.5, 2.6, 2.7, then $\mathcal{R}$ and $\mathcal{K}$ has a unique common fixed points.*

Proof. By seeing lemma 1.11, one can easily proof the results. □

Now, we here illustrate a non trivial example which validate our main result. i.e Theorem 2.2.

Example 2.9. Let $\mathcal{A} = \{(\mathfrak{u}, 0) \in \mathbb{R}^2 : \mathfrak{u} \in [0, 1]\}$, $\mathcal{B} = \{(0, \mathfrak{u}) \in \mathbb{R}^2 : \mathfrak{u} \in [0, 1]\}$ and $S = \mathcal{A} \cup \mathcal{B}$. Let $E_R = \mathbb{R}^2$ and $\mathcal{P} = \{(s, t) \in \mathbb{R}^2 : s, t \geq 0\}$ be a regular cone in E_R.
The mapping $d : S \times S \times S \to E_R$ is defined by $d(s, t, w) = \rho(\mathfrak{u}_1, \mathfrak{u}_2)$, where $s, t, w \in S$ and $\mathfrak{u}_1, \mathfrak{u}_2 \in \{s, t, w\}$ are such that

$$\|\mathfrak{u}_1 - \mathfrak{u}_2\| = \min\{\|s - t\|, \|t - w\|, \|s - w\|\}$$

and

$$\rho(\mathfrak{u}_1, \mathfrak{u}_2) = \left(\frac{5}{4}|\mathfrak{u}_1 - \mathfrak{u}_2|, |\mathfrak{u}_1 - \mathfrak{u}_2|\right) \tag{54}$$

$$\rho(\mathfrak{u}_1, \mathfrak{u}_2) = \left(|\mathfrak{u}_1 - \mathfrak{u}_2|, \frac{3}{4}|\mathfrak{u}_1 - \mathfrak{u}_2|\right) \tag{55}$$

$$\rho(\mathfrak{u}_1, \mathfrak{u}_2) = \rho(\mathfrak{u}_2, \mathfrak{u}_1) = \left(\frac{5}{4}\mathfrak{u}_1 + \mathfrak{u}_2, \mathfrak{u}_1 + \frac{3}{4}\mathfrak{u}_2\right). \tag{56}$$

Then (S, d) is a complete cone 2-metric space and for all $s, t, w \in S$, $d(s, t, w) \in \text{int}\mathcal{P}$ with atleast two of s, t, w are not equal. Let the mapping $\mathcal{D} : S \to S$ be defined by

$$\mathcal{D}(j, 0) = \left(0, \frac{1}{12}j\right) \tag{57}$$

$$\mathcal{D}(0, j) = \left(\frac{1}{18}j, 0\right). \tag{58}$$

Define $\phi : \text{int}\mathcal{P} \cup \{0\} \to \text{int}\mathcal{P} \cup \{0\}$ and $\psi : \mathcal{P} \to \mathcal{P}$ by

$$\psi(\xi) = (\xi_1, \xi_2) \text{ if } \xi \in \mathcal{P} \tag{59}$$

$$\phi(\xi) = \frac{1}{2}(\xi_1, \xi_2) \text{ if } \xi \in \text{int}\mathcal{P} \cup \{0\}. \tag{60}$$

Then ψ and ϕ satisfy all of properties mentioned in Theorem 2.2.
Without lose of generality, we assume that for all $s, t, w \in S$,

$$d(s, t, w) = \rho(s, t) \tag{61}$$

and

$$d(\mathcal{D}s, \mathcal{D}t, w) = \rho(\mathcal{D}s, \mathcal{D}t) \tag{62}$$

Now, we will discuss the following four cases.
Case 1. For $s, t \in \mathcal{A}$ and for all $w \in S$. Then

$$\begin{aligned}
\psi\big(d(\mathcal{D}s, \mathcal{D}t, w)\big) &= \psi\big(\rho(\mathcal{D}s, \mathcal{D}t)\big) = \psi\bigg(\rho\bigg((0, \frac{1}{12}s), (0, \frac{1}{12}t)\bigg)\bigg) \\
&= \left(\frac{1}{12}|s-t|, \frac{1}{16}|s-t|\right) \\
&\preccurlyeq \left(\left(\frac{5}{4} - \frac{5}{8}\right)|s-t|, \left(1 - \frac{1}{2}\right)|s-t|\right) \\
&= \left(\frac{5}{4}|s-t|, |s-t|\right) - \frac{1}{2}\left(\frac{5}{4}|s-t|, |s-t|\right) \\
&= \psi\big(\rho(s,t)\big) - \phi\big(\rho(s,t)\big) = \psi\big(d(s,t,w)\big) - \phi\big(d(s,t,w)\big).
\end{aligned}$$

Case 2. For $s, t \in \mathcal{B}$ and for all $w \in S$. Then

$$\begin{aligned}
\psi\big(d(\mathcal{D}s, \mathcal{D}t, w)\big) &= \psi\big(\rho(\mathcal{D}s, \mathcal{D}t)\big) = \psi\bigg(\rho\bigg((\frac{1}{18}s, 0), (\frac{1}{18}t, 0)\bigg)\bigg) \\
&= \left(\frac{5}{72}|s-t|, \frac{1}{18}|s-t|\right) \\
&\preccurlyeq \left(\left(1 - \frac{1}{2}\right)|s-t|, \left(\frac{3}{4} - \frac{3}{8}\right)|s-t|\right) \\
&= \left(|s-t|, \frac{3}{4}|s-t|\right) - \frac{1}{2}\left(|s-t|, \frac{3}{4}|s-t|\right) \\
&= \psi\big(\rho(s,t)\big) - \phi\big(\rho(s,t)\big) = \psi\big(d(s,t,w)\big) - \phi\big(d(s,t,w)\big).
\end{aligned}$$

Case 3. For $s \in \mathcal{A}$, $t \in \mathcal{B}$ and for all $w \in S$. Then

$$\begin{aligned}
\psi\big(d(\mathcal{D}s, \mathcal{D}t, w)\big) &= \psi\big(\rho(\mathcal{D}s, \mathcal{D}t)\big) = \psi\bigg(\rho\bigg((0, \frac{1}{12}s), (\frac{1}{18}t, 0)\bigg)\bigg) \\
&= \left(\frac{1}{12}s + \frac{5}{72}t, \frac{1}{16}s + \frac{1}{18}t\right) \\
&\preccurlyeq \left(\frac{5}{8}s + \frac{1}{2}t, \frac{1}{2}s + \frac{3}{8}t\right) \\
&= \left(\frac{5}{4}s + t, s + \frac{3}{4}t\right) - \frac{1}{2}\left(\frac{5}{4}s + t, s + \frac{3}{4}t\right) \\
&= \psi\big(\rho(s,t)\big) - \phi\big(\rho(s,t)\big) = \psi\big(d(s,t,w)\big) - \phi\big(d(s,t,w)\big).
\end{aligned}$$

Case 4. For $s \in \mathcal{B}$, $t \in \mathcal{A}$ and for all $w \in S$. Then

$$\begin{aligned}
\psi\big(d(\mathcal{D}s, \mathcal{D}t, w)\big) &= \psi\big(\rho(\mathcal{D}s, \mathcal{D}t)\big) = \psi\bigg(\rho\bigg((\frac{1}{18}s, 0), (0, \frac{1}{12}t)\bigg)\bigg) \\
&= \psi\bigg(\rho\bigg((0, \frac{1}{12}t), (\frac{1}{18}s, 0)\bigg)\bigg) \\
&= \left(\frac{1}{12}t + \frac{5}{72}s, \frac{1}{16}t + \frac{1}{18}s\right) \\
&\preccurlyeq \left(\frac{5}{8}t + \frac{1}{2}s, \frac{1}{2}t + \frac{3}{8}s\right) \\
&= \left(\frac{5}{4}t + s, t + \frac{3}{4}s\right) - \frac{1}{2}\left(\frac{5}{4}t + s, t + \frac{3}{4}s\right) \\
&= \psi\big(\rho(t,s)\big) - \phi\big(\rho(t,s)\big) = \psi\big(\rho(s,t)\big) - \phi\big(\rho(s,t)\big) \\
&= \psi\big(d(s,t,w)\big) - \phi\big(d(s,t,w)\big).
\end{aligned}$$

Hence, the conditions of Theorem 2.2 are satisfied.
Here, it is seen that $(0, 0)$ is the unique fixed point of the mapping $\mathcal{D}$.

3. Conclusion

Nowadays, the researchers in the subject area are working to produce more effective and generalized fixed point results. In this work we have generalized many of the results from metric space and cone metric space to the cone 2-metric space settings. Specifically, the work in [6] and [12] has been generalized to the context of cone 2-metric space. Also, an example is given to strengthen the main result.

4. Availability of data and material

Not Applicable.

5. Competing interests

The authors declare that they have no competing interest.

6. Authors' contributions

All authors contribute equally to the writing of this manuscript. All authors read and approve the final version.

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