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A REMARK ON WEAK MODULE-AMENABILITY IN BANACH ALGEBRAS

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Abstract. We define a new concept of module amenability which is compatible with original definition of amenability. For a module dual algebra $\mathcal{A}$, we will show that if every module derivation $D : \mathcal{A}^{**} \to {J_{\mathcal{A}^{**}}}^{\perp}$ is inner then $\mathcal{A}$ is weak module amenable. Moreover, we will prove that under certain conditions, weak module amenability of $\mathcal{A}^{**}$ implies weak module amenability of $\mathcal{A}$.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. A *derivation* $D : \mathcal{A} \to X$ is a bounded linear operator which satisfies the following,

$$D(ab) = a.D(b) + D(a).b \quad (a, b \in \mathcal{A}).$$

A derivation $D : \mathcal{A} \to X$ is called *inner* if there is some $x \in X$ such that

$$D(a) = a.x - x.a \quad (a \in \mathcal{A}).$$

Following B. E. Johnson [11], a Banach algebra $\mathcal{A}$ is said to be *amenable* if every derivation from $\mathcal{A}$ into a dual Banach $\mathcal{A}$-bimodule is inner.

A Banach algebra $\mathcal{A}$ is called *weakly amenable* [3] if every derivation $D : \mathcal{A} \to \mathcal{A}^*$ is inner. F. Gourdeau in [10] has shown that the amenability of $\mathcal{A}^{**}$ implies the amenability of $\mathcal{A}$. However, for weak amenability this result is not proved yet.

Problem. Let $\mathcal{A}^{**}$ be a weakly amenable Banach algebra, can we conclude that $\mathcal{A}$ is also weakly amenable?

The above problem has been solved in certain cases. For example, in each of the following cases the above problem has a positive answer.

(1) $\mathcal{A}$ is a dual Banach algebra [4, 8].
(2) $\mathcal{A}$ is a left ideal of $\mathcal{A}^{**}$ [9].
(3) $\mathcal{A}$ is a right ideal of $\mathcal{A}^{**}$ with $\mathcal{A}^{**}\Box\mathcal{A} = \mathcal{A}^{**}$ [7].

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(4) $\mathcal{A}$ is an Arens regular Banach algebra such that each continuous derivation from $\mathcal{A}$ to $\mathcal{A}^*$ is weakly compact.

The concept of module amenability for the class of Banach algebras that are modules over another Banach algebra was introduced in [1].

In Section 2, we will define the concept of module dual algebra. We will show that weak module amenability of $\mathcal{A}^{**}$ implies weak module amenability of $\mathcal{A}$ if the Banach algebra $\mathcal{A}$ is a module dual algebra. This result can be considered as an extension of [8, Theorem 2.2]. In [2], the authors proved that if $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with left trivial action and $\frac{\mathcal{A}}{J}$ is a dual algebra then weak module amenability of $\mathcal{A}^{**}$ implies weak module amenability of $\mathcal{A}$, where J is the closed ideal of $\mathcal{A}$ generated by $\{(\alpha.a)b - a(b.\alpha) : a, b \in \mathcal{A}, \alpha \in \mathfrak{A}\}$. We will show that if $\mathcal{A}$ is a module dual algebra, then we can omit the assumption of left triviality action of $\mathcal{A}$ in the above result.

For a Banach algebra $\mathcal{A}$ satisfying ${J_{\mathcal{A}}}^{\perp\perp} \subseteq \mathcal{A}$ and $\mathcal{A}^{**} \cdot \frac{\mathcal{A}}{{J_{\mathcal{A}}}^{\perp\perp}} \subseteq \frac{\mathcal{A}}{{J_{\mathcal{A}}}^{\perp\perp}}$, we prove that weak module amenability $\mathcal{A}^{**}$ implies weak module amenability of $\mathcal{A}$. This result can be considered as an extension of [9, Theorem 2.3].

2. Main results

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras and let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule such that

$$\alpha.(ab) = (\alpha.a).b \qquad (ab).\alpha = a(b.\alpha) \qquad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

If X is both a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule such that for all $a \in \mathcal{A}, x \in X, \alpha \in \mathfrak{A}$

(1)
$$\alpha.(a.x) = (\alpha.a).x \;\; (a.x).\alpha = a.(x.\alpha) \;\; x.(a.\alpha) = (x.a).\alpha \;\; x.(\alpha.a) = (x.\alpha).a,$$

then X is called an $\mathcal{A}$-$\mathfrak{A}$-*module*. If moreover,

$$\alpha.x = x.\alpha \qquad (\alpha \in \mathfrak{A}, x \in X),$$

then X is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module.

If X is a $\mathcal{A}$-$\mathfrak{A}$-module then so is X^*, with the following actions:

$$\langle \alpha.f, x\rangle = \langle f, x.\alpha\rangle \qquad \langle f.\alpha, x\rangle = \langle f, \alpha.x\rangle$$
$$\langle a.f, x\rangle = \langle f, x.a\rangle \qquad \langle f.a, x\rangle = \langle f, a.x\rangle \qquad (a \in \mathcal{A}, x \in X, \alpha \in \mathfrak{A}, f \in X^*).$$

Let X and Y be $\mathcal{A}$-$\mathfrak{A}$-modules and let $\phi : X \to Y$ be a linear map which satisfies the following conditions:

$$\phi(\alpha.x) = \alpha.\phi(x) \qquad \phi(x.\alpha) = \phi(x).\alpha$$
$$\phi(a.x) = a.\phi(x) \qquad \phi(x.a) = \phi(x).a \qquad (a \in \mathcal{A}, x \in X, \alpha \in \mathfrak{A}).$$

Then ϕ is called an $\mathcal{A}$-$\mathfrak{A}$-*module bi-homomorphism*. Let X be a commutative $\mathcal{A}$-$\mathfrak{A}$-module, then the projective tensor product $\mathcal{A}\hat{\otimes}X$ is also an $\mathcal{A}$-$\mathfrak{A}$-module with the following actions:

$$a.(b\otimes x)=(ab)\otimes x \qquad (b\otimes x).a=b\otimes(x.a)$$
$$\alpha.(b\otimes x)=(\alpha.b)\otimes x \qquad (b\otimes x).\alpha=b\otimes(x.\alpha) \qquad (a,b\in\mathcal{A},x\in X,\alpha\in\mathfrak{A}).$$

Now, let $\pi:\mathcal{A}\hat{\otimes}X\to X$ be defined by

$$\pi(a\otimes x)=a.x \qquad (a\in\mathcal{A},x\in X).$$

It follows from the definition that π is an $\mathcal{A}$-$\mathfrak{A}$-module bi-homomorphism.

Let I_X be the closed $\mathcal{A}$-$\mathfrak{A}$-submodule of the projective tensor product $A\hat{\otimes}X$ generated by

$$\{(a.\alpha)\otimes x-a\otimes(\alpha.x):a\in\mathcal{A},\alpha\in\mathfrak{A},x\in X\}.$$

Let J_X be the closed submodule of X generated by $\pi(I_X)$. That is

$$J_X=\overline{\langle\pi(I_X)\rangle}.$$

In special case, when $X=\mathcal{A}$, $J_{\mathcal{A}}$ is the closed ideal generated by $\{(a.\alpha)b-a(\alpha.b):a,b\in\mathcal{A},\alpha\in\mathfrak{A}\}$ and the quotient Banach algebra $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ is also an $\mathcal{A}$-$\mathfrak{A}$-module.

Remark 2.1. A discrete semigroup S is called inverse semigroup if for each $s\in S$ there is a unique element $s^*\in S$ such that $s^*ss^*=s^*$ and $ss^*s=s$. An element $e\in S$ is called idempotent if $e=e^*=e^2$. The set of idempotent of S is denote by E_S. It is easy to see that E_S is a commutative subsemigroup of S [10, Theorem V.1.2] with the natural order on E_S, defined by

$$e\le f \qquad\Longleftrightarrow\qquad ef=e \qquad (e,f\in E_S).$$

In particular $\ell^1(E_S)$ could be regarded as a subalgebra of $\ell^1(S)$ [10]. We also consider right and left actions (which doesn't has left trivial action) of $\ell^1(E_S)$ on $\ell^1(S)$ by

$$\delta_e.\delta_s=\delta_{s^*s}, \qquad \delta_s*\delta_e=\delta_{se} \qquad (e\in E_S,s\in S).$$

This actions make, $\ell^1(S)$ become a Banach $\ell^1(E_S)$-module. Therefore $J_{\ell^1(S)}$ is the closed submodule of $\ell^1(S)$ generated by

$$\Big\{\delta_{set}-\delta_{st^*t}:s,t\in S,e\in E_S\Big\}.$$

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module according to the conditions (1), a bounded linear map $D:\mathcal{A}\to X$ is called a *module derivation* if D satisfies the following:

$$D(ab)=D(a).b+a.D(b)$$
$$D(\alpha.a)=\alpha.D(a), \qquad D(a.\alpha)=D(a).\alpha \qquad (a,b\in\mathcal{A},\alpha\in\mathfrak{A}).$$

Lemma 2.2. *Let X be a commutative $\mathcal{A}$-$\mathfrak{A}$-module and let $D : \mathcal{A} \to X^*$ be a module derivation. Then $D(\mathcal{A}) \subseteq {J_X}^{\perp}$.*

Proof. Let $a, b \in \mathcal{A}, \alpha \in \mathfrak{A}$ *and* $x \in X$, then $(a.\alpha)x - a(\alpha.x) \in J_X$. Then

$$\langle D(b), (a.\alpha)x - a(\alpha.x)\rangle = \langle D(b).(a.\alpha) - (D(b).a).\alpha, x\rangle = 0.$$

□

Definition 2.3. *A Banach $\mathfrak{A}$-bimodule $\mathcal{A}$ is called* weak module amenable *(as an $\mathfrak{A}$-bimodule), if ${J_{\mathcal{A}}}^{\perp}$ is a commutative $\mathfrak{A}$-module and each linear module derivation $D : \mathcal{A} \to {J_{\mathcal{A}}}^{\perp}$ is inner.*

Corollary 2.4. *$\mathcal{A}$ is weak module amenable if $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ is weak amenable.*

Proof. Let $D : \mathcal{A} \to {J_{\mathcal{A}}}^{\perp}$ is a module derivation, then $\tilde{D} : \frac{\mathcal{A}}{J_{\mathcal{A}}} \to {J_{\mathcal{A}}}^{\perp}$ by $\tilde{D}(a + J_{\mathcal{A}}) = D(a)$ is deviation so it is inner. Therefore there exist $f \in {J_{\mathcal{A}}}^{\perp}$ such that

$$\tilde{D}(a + J_{\mathcal{A}}) = (a + J_{\mathcal{A}}).f - f.(a + J_{\mathcal{A}}),$$

therefore $D(a) = a \cdot f - f \cdot a$. i.e. D is inner. □

It is known that

$$(\frac{\mathcal{A}}{J_{\mathcal{A}}})^{**} \simeq \frac{\mathcal{A}^{**}}{{J_{\mathcal{A}}}^{\perp\perp}} \simeq ({J_{\mathcal{A}}}^{\perp})^*, \tag{2}$$

for every Banach algebra $\mathcal{A}$(see e.g. [5, Theorem 10.2]).

Remark 2.5. *Since $(\frac{\mathcal{A}}{J_{\mathcal{A}}})^* \simeq {J_{\mathcal{A}}}^{\perp}$, we have*

$$\langle \tilde{f}, a + J_{\mathcal{A}}\rangle = \langle f, a\rangle \qquad (a \in A), \tag{3}$$

where $f \in {J_{\mathcal{A}}}^{\perp}$ is the corresponding element to $\tilde{f} \in (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*$. Also since $(\frac{\mathcal{A}}{J_{\mathcal{A}}})^{**} \simeq \frac{\mathcal{A}^{**}}{{J_{\mathcal{A}}}^{\perp\perp}}$, we have

$$\langle \tilde{F}, \tilde{f}\rangle = \langle F, f\rangle \qquad (\tilde{f} \simeq f \in {J_{\mathcal{A}}}^{\perp}), \tag{4}$$

where $F + {J_{\mathcal{A}}}^{\perp\perp} \in \frac{\mathcal{A}^{**}}{{J_{\mathcal{A}}}^{\perp\perp}}$ is the corresponding element to $\tilde{F} \in (\frac{\mathcal{A}}{J_{\mathcal{A}}})^{**}$.

Note that $(\frac{\mathcal{A}^{**}}{J_{\mathcal{A}}})^*$ is a Banach $\mathcal{A}$-bimodule, where the actions of $\mathcal{A}$ on $(\frac{\mathcal{A}^{**}}{J_{\mathcal{A}}})^*$ are defined by

$$\langle \tilde{f}.a, b+J_{\mathcal{A}}\rangle = \langle \tilde{f}, ab+J_{\mathcal{A}}\rangle,\ \langle a.\tilde{f}, b+J_{\mathcal{A}}\rangle = \langle \tilde{f}, ba+J_{\mathcal{A}}\rangle \quad (a, b \in \mathcal{A}, \tilde{f} \in (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*). \tag{5}$$

Definition 2.6. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule. We say that $\mathcal{A}$ is a* module dual algebra *if there exists a closed $\mathcal{A}$-$\mathfrak{A}$-module X of ${J_{\mathcal{A}}}^{\perp}$ such that ${J_X}^{\perp} = \mathcal{A}$. In this case, X is called a pre-module dual of $\mathcal{A}$.*

Lemma 2.7. *Let $\mathcal{A}$ be a module dual algebra and let X be pre-module dual of $\mathcal{A}$. If $i : \frac{X}{J_X} \to (\frac{X}{J_X})^{**}$ is the canonical embedding, then i^* is an $\mathcal{A}$-$\mathfrak{A}$-module bi homomorphism. Moreover i^* is a $(\mathcal{A}^{**}, \Box)$-homomorphism.*

Proof. For each $a \in \mathcal{A}$, $\alpha \in \mathfrak{A}$ and $f + J_X \in \frac{X}{J_X}$, we have

$$\begin{aligned} i(a.(f + J_X)) &= i(a.f + J_X) = (a.\widehat{f + J}_X) = \widehat{(a.f)} + J_X \\ &= a.\hat{f} + J_X = a.(\hat{f} + J_X) = a.(\widehat{f + J_X}) = a.i(f + J_X). \end{aligned}$$

Hence

$$i(a.(f + J_X)) = a.i(f + J_X) \qquad (a \in \mathcal{A}, f \in X).$$

By applying a similar argument, we can prove that

$$i(\alpha.(f + J_X)) = \alpha.i(f + J_X)) \qquad (\alpha \in \mathfrak{A}).$$

Therefore i is an $\mathcal{A}$-$\mathfrak{A}$-module bi homomorphism. So that i^* is also an $\mathcal{A}$-$\mathfrak{A}$-module bi homomorphism. Moreover, for each $f + J_X \in \frac{X}{J_X}$ and $a \in \mathcal{A}$,

$$\langle i^*(\hat{a}), f + J_X\rangle = \langle \hat{a}, i(f + J_X)\rangle = \langle \hat{a}, f \hat{+} J_X\rangle = \langle f \hat{+} J_X, a\rangle = \langle a, f + J_X\rangle.$$

Hence

$$i^*(\hat{a}) = a. \tag{6}$$

In order to show that i^* is a $(\mathcal{A}^{**}, \Box)$-homomorphism, we use Goldstine's theorem to obtain bounded nets $\{a_\alpha\}$ and $\{b_\beta\}$ in $\mathcal{A}$ corresponding to $F, G \in \mathcal{A}^{**}$ such that w^*-$\lim_\alpha a_\alpha = F$ and w^*-$\lim_\beta b_\beta = G$. Then

$$\begin{aligned} i^*(F\Box G) &= i^*(w^* - \lim_\alpha \lim_\beta (a_\alpha \hat{} b_\beta)) \\ &= w^* - \lim_\alpha \lim_\beta i^*(a_\alpha \hat{} b_\beta) \qquad (i^* \text{ is } w^* - \text{continuous}) \\ &= w^* - \lim_\alpha \lim_\beta (a_\alpha b_\beta) \qquad \text{by (6)} \\ &= (w^* - \lim_\alpha a_\alpha)(w^* - \lim_\beta b_\beta) \\ &= (w^* - \lim_\alpha i^*(\hat{a}_\alpha))(w^* - \lim_\beta i^*(\hat{b}_\beta)) = i^*(F)i^*(G). \end{aligned}$$

Hence

$$i^*(F\Box G) = i^*(F)i^*(G). \tag{7}$$

□

Theorem 2.8. *Let $\mathcal{A}$ be a module dual algebra. Then the weak module amenability of $\mathcal{A}^{**}$ implies weak module amenability of $\mathcal{A}$.*

Proof. Let X be a closed submodule of $J_{\mathcal{A}}{}^{\perp}$ such that $J_X{}^{\perp} = \mathcal{A}$, and let i be the canonical embedding from $\frac{X}{J_X}$ to $\frac{X}{J_X}^{**}$.

First we show that $i^{**}\mid_{J_{\mathcal{A}}{}^{\perp}}(f) \in J_{\mathcal{A}^{**}}{}^{\perp}$ for each $f \in J_{\mathcal{A}}{}^{\perp}$. To do this, let $F = (a.\alpha).G - a.(\alpha.G)$ be a generating element of $J_{\mathcal{A}^{**}}$, where $G \in \mathcal{A}^{**}$, $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. Then

$$\begin{aligned}
\langle i^*(F), x + J_X\rangle &= \langle i^*\big((a.\alpha).G - a.(\alpha.G)\big), x + J_X\rangle \\
&= \langle (a.\alpha)i^*(G) - a.(\alpha.i^*(G)), x + J_X\rangle, \ (i^* \text{ is homomorphism}) \\
&= \langle i^*(G), (x + J_X)(a.\alpha)\rangle - \langle i^*(G), ((x + J_X).a).\alpha\rangle \\
&= \langle i^*(G), x.(a.\alpha) + J_X\rangle - \langle i^*(G), (x.a).\alpha + J_X)\rangle \\
&= \langle i^*(G), (x.(a.\alpha) - (x.a).\alpha) + J_X\rangle = 0,
\end{aligned}$$

since $x.(a.\alpha) = (x.a).\alpha$. Hence $i^*(F) \in (\frac{X}{J_X})^{\perp}$ whenever F belongs to a generating element of $J_{\mathcal{A}^{**}}$. It follows from continuity and linearity of i^* that $i^*(F) \in (\frac{X}{J_X})^{\perp}$ for each $F \in J_{\mathcal{A}^{**}}$. Since $i^*(F) \in (\frac{X}{J_X})^*$, we have $i^*(F) = 0$. Let $f \in J_{\mathcal{A}}{}^{\perp}$ and $F \in J_{\mathcal{A}^{**}}$, then

$$\langle i^{**}(f), F\rangle = \langle f, i^*(F)\rangle = 0.$$

It follows that $i^{**}|_{J_{\mathcal{A}}{}^{\perp}} \in J_{\mathcal{A}^{**}}{}^{\perp}$. Let $D : \mathcal{A} \to J_{\mathcal{A}}{}^{\perp}$ be a module derivation. Define

$$\bar{D} = i^{**} \circ D \circ i^* : \mathcal{A}^{**} \to J_{\mathcal{A}^{**}}{}^{\perp} \simeq (\frac{\mathcal{A}^{**}}{J_{\mathcal{A}^{**}}})^*$$

We will show that $\bar{D}$ is a module derivation. It is clear that $\bar{D}$ is a bounded linear map. For each $F, G, E \in \mathcal{A}^{**}$, by [8, Theorem 2.2]

$$\langle \bar{D}(F\Box G), E\rangle = \langle \bar{D}(F).G + F.\bar{D}(G), E\rangle.$$

Hence $\bar{D}(F\Box G) = \bar{D}(F).G + F.\bar{D}(G)$. Moreover, $\bar{D}$ is $\mathfrak{A}$-module derivation, since D and i are $\mathfrak{A}$-homomorphism. Thus $\bar{D} : \mathcal{A}^{**} \to J_{\mathcal{A}^{**}}{}^{\perp}$ is a module derivation. Since $\mathcal{A}^{**}$ is weakly module amenable, there exists some $P \in J_{\mathcal{A}^{**}}{}^{\perp}$ such that

$$\bar{D}(F) = F.P - P.F \qquad (F \in \mathcal{A}^{**}).$$

Define $\phi : \frac{\mathcal{A}}{J_{\mathcal{A}}} \to \frac{\mathcal{A}^{**}}{J_{\mathcal{A}^{**}}}$ by $\phi(a + J_{\mathcal{A}}) = \hat{a} + J_{\mathcal{A}^{**}}$ $(a \in \mathcal{A})$. We will show that ϕ is well defined. Let $a_1, a_2 \in \mathcal{A}$ and $a_1 + J_{\mathcal{A}} = a_2 + J_{\mathcal{A}}$. Therefore, we can find $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j} \in \mathcal{A}$ and $\alpha_{i,j} \in \mathfrak{A}$ such that

$$a_1 - a_2 = \lim_j \sum_{i=1}^{n} \Big(c_{i,j}\big((a_{i,j}.\alpha_{i,j})b_{i,j} - a_{i,j}(\alpha_{i,j}.b_{i,j})\big)d_{i,j}\Big).$$

Hence

$$\begin{aligned}\hat{a_1} - \hat{a_2} = (\widehat{a_1 - a_2}) &= \lim_j \sum_{i=1}^{n} \Big(\widehat{c_{i,j}\big((a_{i,j}.\alpha_{i,j})b_{i,j} - a_{i,j}(\alpha_{i,j}.b_{i,j})\big)d_{i,j}} \Big) \\ &= \lim_j \sum_{i=1}^{n} \Big(\hat{c}_{i,j}\big((\widehat{a_{i,j}.\alpha_{i,j}})\hat{b_{i,j}} - \hat{a_{i,j}}(\widehat{\alpha_{i,j}.b_{i,j}})\big)\hat{d}_{i,j} \Big) \\ &= \lim_j \sum_{i=1}^{n} \Big(\hat{c}_{i,j}\big((\hat{a_{i,j}}.\alpha_{i,j})\hat{b_{i,j}} - \hat{a_{i,j}}(\alpha_{i,j}.\hat{b_{i,j}})\big)\hat{d}_{i,j} \Big).\end{aligned}$$

Since $\Big(\hat{c}_{i,j}\big((\hat{a_{i,j}}.\alpha_{i,j})\hat{b_{i,j}} - \hat{a_{i,j}}(\alpha_{i,j}.\hat{b_{i,j}})\big)\hat{d}_{i,j} \Big) \in J_{\mathcal{A}^{**}}$ and $J_{\mathcal{A}^{**}}$ is a closed ideal, we have $\hat{a_1} - \hat{a_2} \in J_{\mathcal{A}^{**}}$. Hence

$$\phi(a_1 + J_{\mathcal{A}}) = \phi(a_2 + J_{\mathcal{A}}).$$

Now, we will show that D is inner. Let $a, b \in \mathcal{A}$, by [8, Theorem2.2]

$$\langle D(a), b\rangle = \langle P, \hat{ba}\rangle - \langle P, \hat{ab}\rangle.$$

It follows from (3) that $P \in {J_{\mathcal{A}^{**}}}^{\perp}$. Hence there exists some $\tilde{F} \in (\frac{\mathcal{A}^{**}}{J_{\mathcal{A}^{**}}})^*$ such that

$$\langle P, G\rangle = \langle \tilde{F}, G + J_{\mathcal{A}^{**}}\rangle \qquad (G \in \mathcal{A}^{**}).$$

Therefore for each $a, b \in \mathcal{A}$, we have

$$\begin{aligned}\langle D(a), b\rangle &= \langle \tilde{F}, \hat{ba} + J_{\mathcal{A}^{**}}\rangle - \langle \tilde{F}, \hat{ab} + J_{\mathcal{A}^{**}}\rangle \\ &= \langle \tilde{F}, \phi(ba + J_{\mathcal{A}})\rangle - \langle \tilde{F}, \phi(ab + J_{\mathcal{A}})\rangle \\ &= \langle \phi^*(\tilde{F}), ba + J_{\mathcal{A}}\rangle - \langle \phi^*(\tilde{F}), ab + J_{\mathcal{A}}\rangle \\ &= \langle \phi^*(\tilde{F}), b.(a + J_{\mathcal{A}})\rangle - \langle \phi^*(\tilde{F}), (a + J_{\mathcal{A}}).b\rangle \\ &= \langle (a + J_{\mathcal{A}}).\phi^*(\tilde{F}) - \phi^*(\tilde{F}).(a + J_{\mathcal{A}}), b\rangle \\ &= \langle a.\phi^*(\tilde{F}) - \phi^*(\tilde{F}).a, b\rangle \qquad \text{by (3)}\end{aligned}$$

□

Corollary 2.9. [8, Theorem 2.2] *Let Banach algebra $\mathcal{A}$ be a dual algebra, then weakly amenability $\mathcal{A}^{**}$ implies that of $\mathcal{A}$.*

Proof. Take $\mathfrak{A} = \mathbb{C}$ in Theorem 2.8. □

Remark 2.10. *In [2], it is shown that if $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with left trivial action and $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ is a dual algebra, then weak module amenability of $\mathcal{A}^{**}$ implies that of $\mathcal{A}$. According to Theorem 2.8, when $\mathcal{A}$ is a module dual algebra, the above conditions can be eliminated.*

For a Banach algebra $\mathcal{A}$, let $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ be a Banach $\mathcal{A}$-bimodule whose left and right module actions are

$$\pi_1 : \mathcal{A} \times \frac{\mathcal{A}}{J_{\mathcal{A}}} \to \frac{\mathcal{A}}{J_{\mathcal{A}}}, \quad \pi_1(a, b + J_{\mathcal{A}}) = ab + J_{\mathcal{A}}$$

and

$$\pi_2 : \frac{\mathcal{A}}{J_{\mathcal{A}}} \times \mathcal{A} \to \frac{\mathcal{A}}{J_{\mathcal{A}}}, \quad \pi_2(b + J_{\mathcal{A}}, a) = ba + J_{\mathcal{A}}$$

for $a, b \in \mathcal{A}$. We denote $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ with the above operations by $(\pi_1, \frac{\mathcal{A}}{J_{\mathcal{A}}}, \pi_2)$. Then $(\pi_2^{r*r}, (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*, \pi_1^*)$ is a Banach $\mathcal{A}$-bimodule [6], which is called the dual of $(\pi_1, \frac{\mathcal{A}}{J_{\mathcal{A}}}, \pi_2)$. Here $\pi_2^{r*r} : \mathcal{A} \times (\frac{\mathcal{A}}{J_{\mathcal{A}}})^* \to (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*$ and $\pi_1^* : (\frac{\mathcal{A}}{J_{\mathcal{A}}})^* \times \mathcal{A} \to (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*$ are given by

$$\pi_2^{r*r}(a, f) = a.f, \qquad \pi_1^*(f, a) = f.a \qquad (a \in \mathcal{A}, f \in (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*).$$

Since $(\pi_2^{r*r***}, (\frac{\mathcal{A}}{J_{\mathcal{A}}})^{***}, \pi_1^{****})$ is the second dual of $(\pi_2^{r*r}, (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*, \pi_1^*)$ (as a Banach $\mathcal{A}$-module), $(\frac{\mathcal{A}}{J_{\mathcal{A}}})^{***}$ is a $\mathcal{A}^{**}$-bimodule.

Lemma 2.11. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule, X be a $\mathcal{A}$-$\mathfrak{A}$-module and $D : \mathcal{A} \to X$ be a module derivation, then $D^{**} : \mathcal{A}^{**} \to X^{**}$ is a module derivation.*

Proof. It is clear that the adjoint of a module homomorphism is also a module homomorphism. □

Remark 2.12. [5, Theorem 10.2] *Let $\mathcal{A}$ be a Banach algebra, then*

$$({J_{\mathcal{A}}}^{\perp})^{**} \cong (\frac{\mathcal{A}}{J_{\mathcal{A}}})^{***} \cong (\frac{\mathcal{A}^{**}}{{J_{\mathcal{A}}}^{\perp\perp}})^*$$

Lemma 2.13. *For a Banach algebra $\mathcal{A}$, we have*

$$\widehat{(\frac{\mathcal{A}}{J_{\mathcal{A}}})^*} \cong \widehat{({J_{\mathcal{A}}}^{\perp})} \subseteq (\frac{\mathcal{A}^{**}}{J_{\mathcal{A}^{**}}})^* \cong {J_{\mathcal{A}^{**}}}^{\perp}.$$

Proof. If $f \in {J_{\mathcal{A}}}^{\perp}$, then $f \mid J_{\mathcal{A}} = 0$. We will show that $\hat{f} \in {J_{\mathcal{A}^{**}}}^{\perp}$. Take some $a^{**} \in \mathcal{A}^{**}$ and let $\{a_i\}$, be a bounded net in $\mathcal{A}$ such that $w^*\text{-}\lim_i a_i = a^{**}$ and let $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. Then

$$\begin{aligned} \langle \hat{f}, (a.\alpha).a^{**} - a.(\alpha.a^{**}) \rangle &= \langle (a.\alpha).a^{**} - a.(\alpha.a^{**}), f \rangle \\ &= \lim_i \langle (a.\alpha)\hat{a}_i - a.(\alpha.\hat{a}_i), f \rangle \\ &= \lim_i \langle (a.\alpha)\widehat{a_i - a.}(\alpha.a_i), f \rangle \\ &= \lim_i \langle f, (a.\alpha)a_i - a.(\alpha.a_i) \rangle = 0. \end{aligned}$$

Since $\hat{f}$ is linear and continuous, $\hat{f} \in {J_{\mathcal{A}^{**}}}^{\perp}$. □

Theorem 2.14. *Let $\mathcal{A}$ be a Banach algebra and a Banach $\mathfrak{A}$-bimodule Banach algebra. Suppose that for every module derivation $D : \mathcal{A} \to {J_{\mathcal{A}}}^{\perp}$*

$$(8) \qquad {J_{\mathcal{A}}}^{\perp\perp} \subseteq \mathcal{A} \quad \text{and} \quad D^{**}(\mathcal{A}^{**}) \subseteq (\frac{\mathcal{A}}{{J_{\mathcal{A}}}^{\perp\perp}})^{*}$$

or

$$(9) \qquad {J_{\mathcal{A}}}^{\perp\perp} \subseteq \mathcal{A} \quad \text{and} \quad \mathcal{A}^{**} \cdot \frac{\mathcal{A}}{{J_{\mathcal{A}}}^{\perp\perp}} \subseteq \frac{\mathcal{A}}{{J_{\mathcal{A}}}^{\perp\perp}}.$$

Then weak module amenability of $\mathcal{A}^{**}$ implies weak module amenability of $\mathcal{A}$.

Proof. Let $\phi : \frac{\mathcal{A}}{J_{\mathcal{A}}} \to \frac{\mathcal{A}^{**}}{{J_{\mathcal{A}}}^{\perp\perp}}$ be defined by $\phi(a + J_{\mathcal{A}}) = \hat{a} + {J_{\mathcal{A}}}^{\perp\perp}$ for each $a \in \mathcal{A}$. By applying the same argument that was used in the proof of Theorem 2.8, one can see that ϕ is well-defined.

Let $D : \mathcal{A} \to {J_{\mathcal{A}}}^{\perp}$ be a module derivation. By Remark 2.12 and Lemma 2.13, we may assume that $\phi^* \circ D^{**}$ is a function from $\mathcal{A}^{**}$ into $(\frac{\mathcal{A}^{**}}{J_{\mathcal{A}^{**}}})^*$. Hence, we have to show that is a module derivation. By Lemma 2.11, $D^{**} : \mathcal{A}^{**} \to ({J_{\mathcal{A}}}^{\perp})^{**}$ is a module derivation i.e.

$$D^{**}(a^{**}\Box b^{**}) = \pi_1^{****}(D^{**}(a^{**}), b^{**}) + \pi_2^{r*r***}(a^{**}, D^{**}(b^{**})) \qquad (a^{**}, b^{**} \in \mathcal{A}).$$

Hence for each $a^{**}, b^{**} \in \mathcal{A}$,

$$(10) \qquad \phi^* \circ D^{**}(a^{**}\Box b^{**}) = \phi^*(\pi_1^{****}(D^{**}(a^{**}), b^{**})) + \phi^*(\pi_2^{r*r***}(a^{**}, D^{**}(b^{**}))).$$

Let $\{a_i\}, \{b_j\}$ be bounded nets in $\mathcal{A}$ such that $w^*\text{-}\lim_i a_i = a^{**}$ and $w^*\text{-}\lim_j b_j = b^{**}$. Then for each $a + J_{\mathcal{A}} \in \frac{\mathcal{A}}{J_{\mathcal{A}}}$, we have

$$\begin{aligned}
\langle \phi^*(\pi_2^{r*r***}(a^{**}, D^{**}(b^{**}))), a + J_{\mathcal{A}} \rangle &= \langle \pi_2^{r*r***}(a^{**}, D^{**}(b^{**})), \phi(a + J_{\mathcal{A}}) \rangle \\
&= \lim_i \lim_j \langle \pi_2^{r*r}(\hat{a}_i, D^{**}(\hat{b}_j)), (\hat{a} + {J_{\mathcal{A}}}^{\perp\perp}) \rangle \\
&= \lim_i \lim_j \langle D^{**}(\hat{b}_j), \pi_2(\hat{a} + {J_{\mathcal{A}}}^{\perp\perp}, \hat{a}_i) \rangle \\
&= \lim_i \lim_j \langle D^{**}(\hat{b}_j), a\hat{a}_i + {J_{\mathcal{A}}}^{\perp\perp} \rangle \\
&= \lim_i \lim_j \langle D^{**}(\hat{b}_j), \phi(aa_i + J_{\mathcal{A}}) \rangle \\
&= \lim_i \lim_j \langle D^{**}(\hat{b}_j), \phi(\pi_2(a + J_{\mathcal{A}}, a_i)) \rangle \\
&= \lim_i \lim_j \langle \phi^*(D^{**}(\hat{b}_j)), \pi_2^r(a_i, a + J_A) \rangle \\
&= \lim_i \lim_j \langle \pi_2^{r*r}(a_i, \phi^*(D^{**}(\hat{b}_j))), a + J_{\mathcal{A}} \rangle \\
&= \langle \pi_2^{r*r***}(a^{**}, \phi^*(D^{**}(b^{**}))), a + J_{\mathcal{A}} \rangle.
\end{aligned}$$

Therefore

$$(11) \qquad \phi^*(\pi_2^{r*r***}(a^{**}, D^{**}(b^{**}))) = \pi_2^{r*r***}(a^{**}, \phi^*(D^{**}(b^{**}))).$$

Also

$$\begin{aligned}\langle \phi^*(\pi_1^{****}(D^{**}(a^{**}), b^{**})), a + J_{\mathcal{A}}\rangle &= \langle \pi_1^{****}(D^{**}(a^{**}), b^{**}), \phi(a + J_{\mathcal{A}})\rangle \\ &= \langle D^{**}(a^{**}), \pi_1^{***}(b^{**}, \hat{a} + J_{\mathcal{A}}{}^{\perp\perp}\rangle.\end{aligned}$$

If (9) holds $\pi_1^{***}(b^{**}, \hat{a} + J_{\mathcal{A}}{}^{\perp\perp}) = \phi(\pi_1^{***}(b^{**}, \hat{a} + J_{\mathcal{A}}{}^{\perp\perp}))$ and if (8) holds $D^{**}(a^{**}) = \phi^*(D^{**}(a^{**}))$. Therefore

$$\begin{aligned}\langle \phi^*(\pi_1^{****}(D^{**}(a^{**}), b^{**})), a + J_{\mathcal{A}}\rangle &= \langle \phi^*(D^{**}(a^{**})), \pi_1^{***}(b^{**}, \hat{a} + J_{\mathcal{A}}{}^{\perp\perp}\rangle \\ &= \langle \pi_1^{****} o \phi^*(D^{**}(a^{**}), b^{**}), \hat{a} + J_{\mathcal{A}}{}^{\perp\perp}\rangle.\end{aligned}$$

Thus

$$\phi^*(\pi_1^{****}(D^{**}(a^{**}), b^{**})) = \pi_1^{****}(\phi^*(D^{**}(a^{**}), b^{**})). \tag{12}$$

Hence by (11) and (12)

$$\phi^* \circ D^{**}(a^{**}\Box b^{**}) = \pi_1^{****}(\phi^*(D^{**}(a^{**}), b^{**})) + \pi_2^{r*r***}(a^{**}, \phi^*(D^{**}(b^{**})))$$

and so $\phi^* \circ D^{**}$ is a derivation. For each $\alpha \in \mathfrak{A}$ and $a^{**} \in \mathcal{A}^{**}$,

$$\phi^* \circ D^{**}(\alpha a^{**}) = \alpha \phi^* \circ D^{**}(a^{**})$$

and

$$\phi^* \circ D^{**}(a^{**}\alpha) = \phi^* \circ D^{**}(a^{**})\alpha.$$

Thus $\phi^* \circ D^{**} : \mathcal{A}^{**} \to (\frac{\mathcal{A}^{**}}{J_{\mathcal{A}^{**}}})^*$ is a module derivation. As $\mathcal{A}^{**}$ is weak module amenable, there is some $F \in (\frac{\mathcal{A}^{**}}{J_{\mathcal{A}^{**}}})^*$ such that $\phi^* \circ D^* = \delta_F$. Let $f = \phi^*(F) \in (\frac{\mathcal{A}^{**}}{J_{\mathcal{A}^{**}}})$ it follows that $D = \delta_f$; i.e D is inner. Therefore $\mathcal{A}$ is weak module amenable. □

Corollary 2.15. [9, Theorem 2.3] *Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}^{**}$ is weakly amenable, and suppose that $\mathcal{A}$ is a left ideal in $\mathcal{A}^{**}$. Then $\mathcal{A}$ is weakly amenable.*

Proof. Take $\mathfrak{A} = \mathbb{C}$ in Theorem 2.8. □

Corollary 2.16. [6, Corollary 7.5] *Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}^{**}$ is weakly amenable, and suppose that every derivation $D : \mathcal{A} \to \mathcal{A}^*$ is weakly compact. Then $\mathcal{A}$ is weakly amenable.*

Proof. Take $\mathfrak{A} = \mathbb{C}$ in Theorem 2.8. □

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