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COUPLED FIXED POINT THEOREMS OF SOME CONTRACTION MAPS OF INTEGRAL TYPE ON CONE METRIC SPACES OVER BANACH ALGEBRAS

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Abstract. In this paper, we prove some coupled fixed point theorems satisfying some generalized contractive condition in a cone metric space over a Banach algebra. We also applied the results obtained to show coupled fixed point of some contractive mapping of integral type.

1. Introduction

Ever since S. Banach [13] introduced the well celebrated result commonly referred to as *Banach contraction principle* in 1921, fixed point theory have developed tremendously and have become an important field in mathematics. The author [13], introduced the concept of fixed point as a useful tool in solving problems in mathematics, economics and engineering. For instance, Most existence and uniqueness of solution of differential equations are shown using the theory of fixed point, the reader can consult [10, 11, 12, 24, 26, 27] and the references therein for further information; Afif et al. in [19] studied the solution of the stationary nonlinear model arising in the theory of growing cell population via fixed point theory; Shehu and Iyiola in [28] used the concept of variational inequality problem which can be converted to a fixed point problem to study the industrial electricity production model; very recently Okeke and Abass [27], Okeke [26] introduced the Picard-Krasnoselskii and Picard-Ishikawa hybrid iterative process respectively and showed that it converges to the solution of a delay differential equation; G. Viglialoro and J. Murcia in [33] believed that fixed point approach can be used to solve the direct problem related to the equilibrium analysis of a membrane with rigid and cable boundaries. For more applications of fixed point theory, see [20, 21, 24, 25, 27, 31, 32] and the references therein. Over the years several authors have focused their attention on single fixed point for different type of operator - J. Olilima et al. [34] showed

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that the modified Mann iteration converges strongly to the (single) fixed point for a uniformly L−Lipschitzian mapping of Gregus type in Banach space, for further information on single fixed points the reader may consult [22, 23, 29, 30] and the references therein. The discussion on coupled fixed point started when Guo and Lakshmikantham [4] in 1973 introduced the concept of coupled fixed point. In 2006 T. G. Bhaskar and V. Lakshmikantham in [16] introduced the concept of coupled fixed point in partially ordered metric spaces. The results of Bhaskar et al. in [16] inspired V. Lakshmikantham et al. [17] to work more on coupled fixed point in partially ordered set. For more results see the references therein.

In 2007 H Long-Guang and Z Xian [6] introduced the concept of cone metric space which is the generalization of metric spaces and they showed that there exists a unique fixed point for Banach, Kannan and Chatterhea's Contraction maps, these kind of maps can be found in [13], [8] and [15] respectively. Their results on cone metric spaces and the results of [4, 14, 16, 17] inspired E. Sabetghadam et al. [2] to state and prove the following theorem on cone metric space:

Theorem 1.1 (see [2]). *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x,y), F(u,v)) \preceq kd(x,u) + ld(y,v), \tag{1.1}$$

where k, l are nonnegative constants with $k + l < 1$.

$$d(F(x,y), F(u,v)) \preceq kd(F(x,y), x) + ld(F(u,v), u), \tag{1.2}$$

where k, l are nonnegative constants with $k + l < 1$.

$$d(F(x,y), F(u,v)) \preceq kd(F(x,y), u) + ld(F(u,v), x), \tag{1.3}$$

where k, l are nonnegative constants with $k + l < 1$. Then F has a unique coupled fixed point.

In 2012 Olaleru et al. [7] introduced the following definition and lemma,

Definition 1.2 (see [7]). *For a nondecreasing mapping $T : P \to P$, we define the following conditions which will be used in the sequel:*

- (T_1) *For every $\omega_n \in P$, $\omega_n \to 0$ if and only if $T\omega_n \to 0$;*
- (T_2) *For every $\omega_1, \omega_2 \in P$, $T(\alpha\omega_1 + \beta\omega_2) \preceq \alpha T(\omega_1) + \beta T(\omega_2)$ for $\alpha, \beta \in [0, 1)$.*

Lemma 1.3 (see [7]). *If a mapping $T : P \to P$ satisfies (T_1), then, for all $\omega \in P$, $T(\omega) = 0 \iff \omega = 0$.*

Using Definition 1.2 and Lemma 1.3 in [7], Olaleru et al. proved some coupled fixed point results for a given type of contractive map.

Theorem 1.4 (see [7]). *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition,*

$$T(d(F(x,y), F(u,v))) \preceq T(j), \quad \forall\, x, y, u, v \in X \tag{1.4}$$

where
$j = a_1 d(x,u) + a_2 d(y,v) + a_3 d(F(x,y), x) + a_4 d(F(u,v), u) + a_5 d(F(x,y), u) + a_6(F(u,v), x)$ *and* $a_1, a_2, a_3, a_4, a_5, a_6$ *are nonnegative constants with* $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 < 1$ *and* $T : P \to P$ *is a nondecreasing mapping satisfying* $(T_1) - (T_2)$*. Then* F *has a unique coupled fixed point.*

Also in [7], Olaleru et al. extended Theorem 1.4 to the integral version of couple fixed point.

In 2013 Liu et al. [5] introduced the concept of cone metric space with Banach algebra (Using an attribute of a cone, which is called normality), this result inspired Xu et al. in [9] to work on a similar concept without using the concept of normality, hence, they introduced a sequence called "*c-sequence*"

In this paper, we prove some coupled fixed point on cone metric spaces over Banach algebra extending the results of [5, 7, 9].

2. Preliminary

Definition 2.1. (see [18]) Let $\mathcal{A}$ be a Banach space in which the operation of multiplication is defined as follows: for all $x, y, z \in \mathcal{A}$, $\alpha \in \mathbb{F}$

1. $x(yz) = (xy)z$.
2. $x(y + z) = xy + xz$.
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$.
4. $\|xy\| \leq \|x\| \cdot \|y\|$.

If these properties are satisfied, then $\mathcal{A}$ is called a Banach algebra.

In this work we assume that $\mathcal{A}$ has a multiplicative identity e such that $\forall\, x \in \mathcal{A}$, $xe = x = ex$. Also, let the inverse of $x \in \mathcal{A}$ be denoted by x^{-1}.

Proposition 2.2 (see [9]). *Let $\mathcal{A}$ be a Banach algebra with identity e, and $x \in \mathcal{A}$. If the spectra radius of x, $r(x) < 1$ i.e.,*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1. \tag{2.1}$$

Then $(e - x)$ is invertible, i.e. $(e - x)^{-1} = \sum_{i=0}^{\infty} x^i$.

Proof. Notice that, if we let $r(x) \leq \|x\| \leq 1$, (2.1) will still be true. And in [18], it was shown that if $\|x\| < 1$ then $(e - x)$ is invertible. □

Let $\mathcal{A}$ be a Banach algebra and let P be a subset of $\mathcal{A}$, P is called a cone if and only if

(i) P is closed, nonempty and $\{0, e\} \subset P$.
(ii) For $\alpha, \beta \in \mathbb{R}, \alpha, \beta \geqslant 0$ then $\alpha P + \beta P \subset P$.
(iii) $P^2 = PP \subset P$.
(iv) $P \cap -P = \{0\}$.

Given a cone $P \subset \mathcal{A}$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ denotes $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P. If $\text{int}P \neq \emptyset$ then P is called a solid cone.
The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in \mathcal{A}$

$$0 \preceq x \preceq y \text{ implies } \|x\| \leq K\|y\|. \tag{2.2}$$

The least positive number satisfying equation (2.2) is called the normal constant [6].
In this paper, we assume that P is a solid cone in $\mathcal{A}$ where $\mathcal{A}$ is a Banach algebra and $\preceq$ is the partial ordering with respect to P.

Definition 2.3 (see [6])**.** Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathcal{A}$ satisfies

(d1) $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \iff x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \preceq d(x, z) + d(z, x)$ for all $x, y, z \in X$.

Then d is called a cone metric and (X, d) is called a cone metric space over a Banach Algebra.

Example 2.4. Let $\mathcal{A} = \mathbb{R}$ and we define the norm

$$\|x\| = |x|, \qquad \text{for all } x \in \mathcal{A}.$$

Then, $\mathcal{A}$ is a real Banach algebra with unit $e = 1$.
Let $P = \{x \in \mathcal{A} \mid x \geq 0\}$. Then $P \subset \mathcal{A}$ is a normal cone.
Let $X = \mathbb{R}$, and define the metric $d : X \times X \to \mathcal{A}$ by

$$d(x, y) = |x - y|, \qquad \text{for all } x, y \in \mathcal{A}.$$

Then, (X, d) is a cone metric over a Banach algebra.

Example 2.5 (see [9])**.** Let $\mathcal{A} = C^1_{\mathbb{R}}([0, 1])$ with the norm

$$\|f\| = \|f\|_\infty + \|f'\|_\infty, \quad \text{for all } x \in \mathcal{A}.$$

Let $X = \{1, 2, 3\}$. Define $d : X \times X \to \mathcal{A}$ by $d(1, 2)(t) = d(2, 1)(t) = d(2, 3)(t) = d(3, 2)(t) = e^t$, $d(1, 3)(t) = 2e^t, d(x, x)(t) = 0$. We see that (X, d) is a cone metric space over Banach algebra $\mathcal{A}$ without normality.

Definition 2.6 (see [6], [5], [9])**.** Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$, for $x \in X$ and the sequence x_n in X, then we define the following:

1. x_n is said to converge to $x \in X$ if for any $0 \ll c$ there is a natural number N_c such that $d(x_n, x) \ll c$ for all $n \geq N_c$. We denote this by either $\lim_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.
2. x_n is said to be Cauchy if for any $0 \ll c$ there is a natural number N_c such that $d(x_n, x_m) \ll c$ for all $n, m \geq N_c$.
3. (X, d) is said to be a complete cone metric space if every Cauchy sequence is convergent.

Let us state some lemmas that will be useful in the proof of our main results.

Lemma 2.7 (see [6]). *Let (X, d) be a cone metric space over a Banach algebra and let x_n be a sequence in X. Then we say that $x_n \to x$ as $n \to \infty$ if and only if*

$$\lim_{n\to\infty} d(x_n, x) = 0. \tag{2.3}$$

Corollary 2.8. *Let (X, d) be a cone metric space over a Banach algebra and let x_n be a sequence in X. Then we say that $x_n \to 0$ as $n \to \infty$ if and only if*

$$\|x_n\| \to 0 \text{ as } n \to \infty. \tag{2.4}$$

Lemma 2.9 (see [6]). *Let (X, d) be a cone metric space over a Banach algebra and let x_n be a sequence in X. Then we say that x_n is Cauchy if and only if*

$$\lim_{n\to\infty} d(x_n, x_m) = 0. \tag{2.5}$$

Lemma 2.10 (see [9]). *Let P be a solid cone in a Banach algebra $\mathcal{A}$ and let $\{x_n\}$ be a sequence in P. Then the following conditions are equivalent:*

1. *$\{x_n\} \to 0$ as $n \to \infty$.*
2. *for each $c \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $x_n \prec c$ for $n \geq n_0$.*
3. *for each $c \gg 0$ there exists $n_1 \in \mathbb{N}$ such that $x_n \preceq c$ for $n \geq n_0$.*

Lemma 2.11 (see [9]). *Let $\mathcal{A}$ be a Banach algebra and let x, y be vectors in $\mathcal{A}$. If x and y commute, then the following hold:*

(i) $r(xy) \leq r(x)r(y)$.
(ii) $r(x + y) \leq r(x) + r(y)$.
(iii) $|r(x) - r(y)| \leq r(x - y)$.

Lemma 2.12 (see [9]). *Let $\mathcal{A}$ be a Banach algebra and let k be a vector in $\mathcal{A}$. If $0 \leq r(k) < 1$, then we have $r((e - k)^{-1}) \leq (1 - r(k))^{-1}$.*

Remark 2.13. If $r(x) < 1$ for $x \in \mathcal{A}$, then it is easy to see that $\|x^n\| \to 0$ as $n \to \infty$. Also, from Lemma 2.7, $x^n \to 0$ as $n \to \infty$.

3. Main Results

Theorem 3.1. *Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$, and let P be the underlining solid cone with*

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10} \in P,$$

where $r(a_1)+r(a_2)+r(a_3)+r(a_4)+r(a_5)+2r(a_6)+r(a_7)+r(a_8)+2r(a_{10}) < 1$. *Suppose the map $F : X \times X \to X$ satisfies the contractive condition:*

$$T(d(F(x, y), F(u, v))) \preceq T(j), \quad \forall\, x, y, u, v \in X \tag{3.1}$$

where $j = a_1 d(x, u)+a_2 d(y, v)+a_3 d(F(x, y), x)+a_4 d(F(u, v), u)+a_5 d(F(x, y), u)+ a_6 d(F(u, v), x)+a_7 d(F(y, x), y)+a_8 d(F(v, u), v)+a_9 d(F(y, x), v)+a_{10} d(F(v, u), y)$ *and $T : P \to P$ is a nondecreasing mapping satisfying $(T_1) - (T_2)$. Then F has a unique coupled fixed point.*

Proof. Let x_0, y_0 be any points in X, set
$x_1 = F(x_0, y_0), y_1 = F(y_0, x_0)$ inductively we have that

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n) \text{ For all } n \in \mathbb{N}.$$

From the contractive condition in (3.1), we have the following:

$$\begin{aligned}
T(d(x_n, x_{n+1})) &= T(d(F(x_{n-1}, y_{n-1}), F(x_n, y_n))) \\
&\preceq T(a_1 d(x_{n-1}, x_n) + a_2 d(y_{n-1}, y_n) + a_3 d(F(x_{n-1}, y_{n-1}), \\
&\quad x_{n-1}) + a_4 d(F(x_n, y_n), x_n) + a_5 d(F(x_{n-1}, y_{n-1}), x_n) + \\
&\quad a_6 d(F(x_n, y_n), x_{n-1}) + a_7 d(F(y_{n-1}, x_{n-1}), y_{n-1}) + \\
&\quad a_8 d(F(y_n, x_n), y_n) + a_9 d(F(y_{n-1}, x_{n-1}), y_n) + \\
&\quad a_{10} d(F(y_n, x_n), y_{n-1})) \\
&= T(a_1 d(x_{n-1}, x_n) + a_2 d(y_{n-1}, y_n) + a_3 d(x_{n-1}, x_n) + \\
&\quad a_4 d(x_n, x_{n+1}) + a_5 d(x_n, x_n) + a_6 d(x_{n+1}, x_{n-1}) + \\
&\quad a_7 d(y_{n-1}, y_n) + a_8 d(y_n, x_{n+1}) + a_9 d(y_{n-1}, y_n) + \\
&\quad a_{10} d(y_{n+1}, y_{n-1})) \\
&= T[(a_1 + a_3 + a_6) d(x_{n-1}, x_n) + (a_4 + a_6) d(x_n, x_{n+1}) + \\
&\quad (a_2 + a_7 + a_{10}) d(y_{n-1}, y_n) + (a_8 + a_{10}) d(y_n, y_{n+1})] \\
&\preceq (a_1 + a_3 + a_6) T(d(x_{n-1}, x_n)) + (a_4 + a_6) T(d(x_n, x_{n+1})) + \\
&\quad (a_2 + a_7 + a_{10}) T(d(y_{n-1}, y_n)) + (a_8 + a_{10}) T(d(y_n, y_{n+1})).
\end{aligned}$$

Therefore,

$$\begin{aligned}
T(d(x_n, x_{n+1})) &\preceq (a_1 + a_3 + a_6) T(d(x_{n-1}, x_n)) + (a_4 + a_6) T(d(x_n, x_{n+1})) + \\
&\quad (a_2 + a_7 + a_{10}) T(d(y_{n-1}, y_n)) + (a_8 + a_{10}) T(d(y_n, y_{n+1})).
\end{aligned} \tag{3.2}$$

Similarly,

$$\begin{aligned}
T(d(y_n, y_{n+1})) &\preceq (a_1 + a_3 + a_6) T(d(y_{n-1}, y_n)) + (a_4 + a_6) T(d(y_n, y_{n+1})) + \\
&\quad (a_2 + a_7 + a_{10}) T(d(x_{n-1}, x_n)) + (a_8 + a_{10}) T(d(x_n, x_{n+1})).
\end{aligned} \tag{3.3}$$

Adding equations (3.2) and (3.3) we have the following

$$\begin{aligned} T(d(x_n, x_{n+1})) + T(d(y_n, y_{n+1})) &\preceq (a_1 + a_2 + a_3 + a_6 + a_7 + a_{10}) \cdot \\ &\quad [T(d(x_{n-1}, x_n)) + T(d(y_{n-1}, y_n))] + \\ &\quad (a_4 + a_6 + a_8 + a_{10})[T(d(x_n, x_{n+1})) \\ &\quad +T(d(y_n, y_{n+1}))], \end{aligned}$$

since $r(a_4+a_6+a_8+a_{10}) \leq r(a_4)+r(a_6)+r(a_8)+r(a_{10}) < 1$ then by Proposition 2.2, $(e - a_4 - a_6 - a_8 - a_{10})$ is invertible. Therefore, by Definition 1.2, we have the following,

$$\begin{aligned} T(d(x_n, x_{n+1}) + d(y_n, y_{n+1})) &\preceq T(d(x_n, x_{n-1})) + T(d(y_n, y_{n-1})) \\ &\preceq \frac{a_1 + a_2 + a_3 + a_6 + a_7 + a_{10}}{e - a_4 - a_6 - a_8 - a_{10}} T(d(x_n, x_{n-1})) \\ &\quad + \frac{a_1 + a_2 + a_3 + a_6 + a_7 + a_{10}}{e - a_4 - a_6 - a_8 - a_{10}} T(d(y_n, y_{n-1})) \\ &= \frac{a_1 + a_2 + a_3 + a_6 + a_7 + a_{10}}{e - a_4 - a_6 - a_8 - a_{10}} \cdot \\ &\quad \Big[T(d(x_n, x_{n-1})) + T(d(y_n, y_{n-1}))\Big]. \end{aligned}$$

Let $\lambda = \dfrac{a_1 + a_2 + a_3 + a_6 + a_7 + a_{10}}{e - a_4 - a_6 - a_8 - a_{10}}$. Therefore, for all $n \in \mathbb{N}$, we have that,

$$\begin{aligned} T(d(x_n, x_{n+1}) + d(y_n, y_{n+1})) &\preceq \lambda T(d(x_n, x_{n-1})) + T(d(y_n, y_{n-1})) \\ &\preceq \lambda^2 [T(d(x_{n-1}, x_{n-2})) + T(d(y_{n-1}, y_{n-2}))] \\ &\vdots \\ &\preceq \lambda^n [T(d(x_1, x_0)) + T(d(y_1, y_0))]. \end{aligned}$$

Now, we show that $r(\lambda) < 1$. By Lemmas 2.11 and 2.12, the following can be derived,

$$\begin{aligned} r(\lambda) &= r[(a_1 + a_2 + a_3 + a_6 + a_7 + a_{10})(e - a_4 - a_6 - a_8 - a_{10})^{-1}] \\ &< r(a_1 + a_2 + a_3 + a_6 + a_7 + a_{10}) \cdot r(e - a_4 - a_6 - a_8 - a_{10})^{-1} \\ &< \frac{r(a_1) + r(a_2) + r(a_3) + r(a_6) + r(a_7) + r(a_{10})}{1 - r(a_4) - r(a_6) - r(a_8) - r(a_{10})} < 1. \end{aligned}$$

We can now show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences respectively.

Case 1:: If $d(x_1, x_0)) + T(d(y_1, y_0) = 0 \Rightarrow (x_0, y_0)$ is a couple fixed point of F.

Case 2:: Suppose $d(x_1, x_0)) + T(d(y_1, y_0)) \neq 0$. Then for $n \geq m$, we have that

$$d(x_m, x_n) \preceq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)$$

and

$$d(y_m, y_n) \preceq d(y_m, y_{m+1}) + d(y_{m+1}, y_{m+2}) + \cdots + d(y_{n-1}, y_n).$$

Therefore,

$$\begin{aligned}
T(d(x_m,x_n)+d(y_m,y_n)) &\preceq T(d(x_m,x_{m+1})+d(x_{m+1},x_{m+2})+\cdots+\\
&\quad d(x_{n-1},x_n)+d(y_m,y_{m+1})+d(y_{m+1},y_{m+2})\\
&\quad +\cdots+d(y_{n-1},y_n))\\
&= T(d(x_m,x_{m+1})+d(y_m,y_{m+1})+\\
&\quad d(x_{m+1},x_{m+2})+d(y_{m+1},y_{m+2})+\cdots+\\
&\quad d(x_{n-1},x_n)+d(y_{n-1},y_n))\\
&\preceq T(d(x_m,x_{m+1})+d(y_m,y_{m+1}))+\\
&\quad T(d(x_{m+1},x_{m+2})+d(y_{m+1},y_{m+2}))+\cdots+\\
&\quad T(d(x_{n-1},x_n)+d(y_{n-1},y_n))\\
&\preceq \lambda^m[T(d(x_1,x_0))+T(d(y_1,y_0))]+\lambda^{m+1}\cdot\\
&\quad [T(d(x_1,x_0))+T(d(y_1,y_0))]+\cdots+\lambda^{n-1}\cdot\\
&\quad [T(d(x_1,x_0))+T(d(y_1,y_0))]\\
&= (e+\lambda+\lambda^2+\cdots+\lambda^{n-m-1})\lambda^m[T(d(x_1,x_0))\\
&\quad +T(d(y_1,y_0))]\\
&\preceq \left(\sum_{i=0}^{\infty}\lambda^i\right)\lambda^m[T(d(x_1,x_0))+T(d(y_1,y_0))]\\
&= (e-\lambda)^{-1}\lambda^m[T(d(x_1,x_0))+T(d(y_1,y_0))].
\end{aligned}$$

Since, $r(\lambda)<1$ by Remark 2.13, we can conclude that

$$\|(e-\lambda)^{-1}\lambda^m[T(d(x_1,x_0))+T(d(y_1,y_0))]\|\to 0,$$

therefore,

$$(e-\lambda)^{-1}\lambda^m[T(d(x_1,x_0))+T(d(y_1,y_0))]\ll c\Rightarrow T(d(x_m,x_n)+d(y_m,y_n))\ll c.$$

Hence, both x_n and y_n are Cauchy sequences in X, and since X is complete it implies that

$$x_n\to x^* \text{ and } y_n\to y^* \text{ as } n\to\infty.$$

Next, we are to show that (x^*, y^*) is a coupled fixed point i.e. $x^* = F(x^*, y^*)$, $y^* = F(y^*, x^*)$.

$$\begin{aligned}
T(d(F(x^*,y^*),x^*)) &\preceq T(d(F(x^*,y^*),x_{n+1}) + d(x_{n+1},x^*)) \\
&= T(d(F(x^*,y^*),F(x_n,y_n) + d(x_{n+1},x^*)) \\
&\preceq T(d(F(x^*,y^*),F(x_n,y_n)) + T(d(x_{n+1},x^*)) \\
&\preceq a_1T(d(x_n,x^*)) + a_2T(d(y_n,y^*)) + \\
&\quad a_3dT(d(F(x^*,y^*),x^*)) + a_4T(d(F(x_n,y_n),x_n)) + \\
&\quad a_5T(d(F(x^*,y^*),x_n)) + a_6T(d(F(x_n.y_n),x^*)) \\
&\quad +a_7T(d(F(y^*,x^*),y^*)) + a_8T(d(F(y_n,x_n),y_n)) \\
&\quad +a_9T(d(F(y^*,x^*),y_n)) + a_{10}T(d(F(y_n,x_n),y^*)) \\
&\quad +d(x_{n+1},x^*) \\
&= a_1T(d(x_n,x^*)) + a_2T(d(y_n,y^*)) + \\
&\quad a_3dT(d(F(x^*,y^*),x^*)) + a_4T(d(x_{n+1},x_n)) + \\
&\quad a_5T(d(F(x^*,y^*),x_n)) + a_6T(d(F(x_{n+1},x^*)) + \\
&\quad a_7T(d(F(y^*,x^*),y^*)) + a_8T(d(y_{n+1},y_n)) + \\
&\quad a_9T(d(F(y^*,x^*),y_n)) + a_{10}T(d(y_{n+1},y^*)) + \\
&\quad d(x_{n+1},x^*).
\end{aligned}$$

Hence

$$\begin{aligned}
T(d(F(x^*,y^*),x^*)) &\preceq a_1T(d(x_n,x^*)) + a_2T(d(y_n,y^*)) + \\
&\quad a_3dT(d(F(x^*,y^*),x^*)) + a_4T(d(x_{n+1},x_n)) + \\
&\quad a_5T(d(F(x^*,y^*),x_n)) + a_6T(d(x_{n+1},x^*)) + \\
&\quad a_7T(d(F(y^*,x^*),y^*)) + a_8T(d(y_{n+1},y_n)) + \\
&\quad a_9T(d(F(y^*,x^*),y_n)) + a_{10}T(d(y_{n+1},y^*)) \\
&\quad +d(x_{n+1},x^*).
\end{aligned} \tag{3.4}$$

From Lemma 2.7, if $x_n \to x^*$ then $\lim_{n\to\infty} d(x_n, x^*) = 0$ and by Lemma 1.3 $T(d(x_n, x^*)) = 0$. Therefore, we have the following as $n \to \infty$:

$$\begin{aligned}
T(d(F(x^*,y^*),x^*)) &\preceq (a_3 + a_5)T(d(F(x^*,y^*),x^*)) + \\
&\quad (a_7 + a_9)T(d(F(y^*,x^*),y^*)).
\end{aligned} \tag{3.5}$$

Similarly,

$$\begin{aligned}
T(d(F(y^*,x^*),y^*)) &\preceq (a_3 + a_5)T(d(F(y^*,x^*),y^*)) + \\
&\quad (a_7 + a_9)T(d(F(x^*,y^*),x^*)).
\end{aligned} \tag{3.6}$$

From (3.5) and (3.6) we have,

$$T(d(F(x^*,y^*),x^*))+T(d(F(y^*,x^*),y^*)) \preceq (a_3+a_5+a_7+a_9)\cdot \Big[T(d(F(x^*,y^*),x^*))+ T(d(F(y^*,x^*),y^*))\Big].$$

Since, $r(a_3+a_5+a_7+a_9)\le r(a_3)+r(a_5)+r(a_7)+r(a_9)<1 \Rightarrow (e-(c_3+c_5+c_7+c_9))^{-1}$ exists, then we can conclude that,

$$T(d(F(x^*,y^*),x^*)+d(F(y^*,x^*),y^*))=0. \tag{3.7}$$

Applying Lemma 1.3 on equation (3.7), we have that,

$$d(F(x^*,y^*),x^*)=0 \text{ and } d(F(y^*,x^*),y^*)=0. \tag{3.8}$$

Hence, (x^*,y^*) is the couple fixed point.
Next, we show that this coupled fixed point is unique.
Suppose there exists another coupled fixed point say $(x_0,y_0)\in X\times X$. Then

$$\begin{aligned}
T(d(x_0,x^*)) &= T(d(F(x_0,y_0),F(x^*,y^*)))\preceq T(a_1d(x_0,x^*)+a_2d(y_0,y^*)+\\
&\quad a_3d(F(x_0,y_0),x_0)+a_4d(F(x^*,y^*),x^*)+a_5d(F(x_0,y_0),x^*)\\
&\quad +a_6d(F(x^*,y^*),x_0)+a_7d(F(y_0,x_0),y_0)+a_8d(F(y^*,x^*),y^*)\\
&\quad +a_9d(F(y_0,x_0),y^*)+a_{10}d(F(y^*,x^*),y_0))\\
&= T(a_1d(x_0,x^*)+a_2d(y_0,y^*)+a_3d(x_0,x_0)+a_4d(x^*,x^*)+\\
&\quad a_5d(x_0,x^*)+a_6d(x^*,x_0))+a_7d(y_0,y_0)+a_8d(y^*,y^*)+\\
&\quad a_9d(y_0,y^*)+a_{10}d(y^*,y_0)\\
&= T((a_1+a_5+a_6)d(x_0,x^*)+(a_2+a_9+a_{10})d(y_0,y^*)).
\end{aligned}$$

Hence

$$T(d(x_0,x^*))\preceq T((a_1+a_5+a_6)d(x_0,x^*))+T((a_2+a_9+a_{10})d(y_0,y^*)). \tag{3.9}$$

Similarly,

$$T(d(y_0,y^*))\preceq T((a_1+a_5+a_6)d(y_0,y^*))+(a_2+a_9+a_{10})T(d(x_0,x^*)). \tag{3.10}$$

Therefore,

$$T(d(x_0,x^*)+d(y_0,y^*))\preceq (a_1+a_2+a_5+a_6+a_9+a_{10})T(d(x_0,x^*)+d(y_0,y^*)).$$

Let $\beta=a_1+a_2+a_5+a_6+a_9+a_{10}$. Hence, $T(d(x_0,x^*)+d(y_0,y^*))\preceq \beta T(d(x_0,x^*)+d(y_0,y^*))$. Since $r(\beta)<1$ then $T(d(x_0,x^*)+d(y_0,y^*))=0\Rightarrow x_0=x^*$ and $y_0=y^*$, therefore (x^*,y^*) is a unique coupled fixed point. This completes the proof. □

We can observe that if $T=I_d$, then we have the following corollary.

Corollary 3.2. *Let (X,d) be a complete cone metric space over Banach algebra $\mathcal{A}$, and let P be the underlining solid cone with* $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10} \in P$ *where* $r(a_1) + r(a_2) + r(a_3) + r(a_4) + r(a_5) + 2r(a_6) + r(a_7) + r(a_8) + 2r(a_{10}) < 1$. *Suppose the map* $F : X \times X \to X$ *satisfies the contractive condition:*

$$d(F(x,y), F(u,v)) \preceq j, \quad \forall\, x, y, u, v \in X \tag{3.11}$$

where

$j = a_1 d(x,u) + a_2 d(y,v) + a_3 d(F(x,y),x) + a_4 d(F(u,v),u) + a_5 d(F(x,y),u) + a_6 d(F(u,v),x) + a_7 d(F(y,x),y) + a_8 d(F(v,u),v) + a_9 d(F(y,x),v) + a_{10} d(F(v,u),y)$. *Then F has a unique coupled fixed point.*

Letting $r(a_1) = \cdots = r(a_5) = 2r(a_6) = \cdots = 2r(a_{10}) = r(a)$, we have the following corollary.

Corollary 3.3. *Let (X,d) be a complete cone metric space over Banach algebra $\mathcal{A}$, and let P be the underlining solid cone with* $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10} \in P$ *where* $r(a_1) = \cdots = r(a_5) = 2r(a_6) = \cdots = 2r(a_{10}) = r(a)$. *Suppose the map* $F : X \times X \to X$ *satisfies the contractive condition:*

$$T(d(F(x,y), F(u,v))) \preceq \left(\frac{a}{10}\right) T(j), \quad \forall\, x, y, u, v \in X \tag{3.12}$$

where $a \in [0,1)$ is a nonnegative constant and $T : P \to P$ is a nondecreasing mapping satisfying $(T_1) - (T_2)$. Then F has a unique coupled fixed point.

Theorem 3.4. *Let (X,d) be a complete cone metric space over Banach algebra $\mathcal{A}$, and let P be the underlining solid cone with $a_1, a_2, a_3, a_4, a_5, a_6 \in P$ where $r(a_1) + r(a_2) + r(a_3) + r(a_4) + r(a_5) + 2r(a_6) < 1$. Suppose the map $F : X \times X \to X$ satisfies the contractive condition:*

$$T(d(F(x,y), F(u,v))) \preceq T(j), \quad \forall\, x, y, u, v \in X \tag{3.13}$$

where

$j = a_1 d(x,u) + a_2 d(y,v) + a_3 d(F(x,y),x) + a_4 d(F(u,v),u) + a_5 d(F(x,y),u) + a_6 (F(u,v),x)$ and $T : P \to P$ *is a nondecreasing mapping satisfying* $(T_1) - (T_2)$. *Then F has a unique coupled fixed point.*

Proof. Let x_0, y_0 be any points in X, set $x_1 = F(x_0, y_0), y_1 = F(y_0, x_0)$ inductively we have that

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n)$$

For all $n \in \mathbb{N}$. from (3.13), we get

$$\begin{aligned}
T(d(x_n, x_{n+1})) &= T(d(F(x_{n-1}, y_{n-1}), F(x_n, y_n))) \\
&\preceq T(a_1 d(x_{n-1}, x_n) + a_2 d(y_{n-1}, y_n) + \\
&\quad a_3 d(F(x_{n-1}, y_{n-1}), x_{n-1}) + \\
&\quad a_4 d(F(x_n, y_n), x_n) + a_5 d(F(x_{n-1}, y_{n-1}), \\
&\quad x_n) + a_6 d(F(x_n, y_n), x_{n-1})) \\
&= T(a_1 d(x_{n-1}, x_n) + a_2 d(y_{n-1}, y_n) + \\
&\quad a_3 d(x_{n-1}, x_n) + a_4 d(x_n, x_{n+1}) + \\
&\quad a_5 d(x_n, x_n) + a_6 d(x_{n+1}, x_{n-1}) \\
&= T((a_1 + a_3 + a_6) d(x_{n-1}, x_n) + (a_4 + a_6) \cdot \\
&\quad d(x_n, x_{n+1}) + a_2 d(y_{n-1}, y_n)) \\
&\preceq (a_1 + a_3 + a_6) T(d(x_{n-1}, x_n)) + (a_4 + a_6) \cdot \\
&\quad T(d(x_n, x_{n+1})) + a_2 T(d(y_{n-1}, y_n)) \\
(e - (a_4 + a_6)) T(d(x_n, x_{n+1})) &\preceq (a_1 + a_3 + a_6) T(d(x_{n-1}, x_n)) + \\
&\quad a_2 T(d(y_{n-1}, y_n)),
\end{aligned}$$

since, $r(a_4 + a_6) \leq r(a_4) + r(a_6) < 1$ then, by Proposition 2.2, $(e - a_4 - a_6)$ is invertible. Therefore, we have

$$\begin{aligned}
T(d(x_n, x_{n+1})) &\preceq \frac{a_1 + a_3 + a_6}{e - a_4 - a_6} T(d(x_{n-1}, x_n)) + \\
&\quad \frac{a_2}{e - a_4 - a_6} T(d(y_n, y_{n-1})).
\end{aligned} \tag{3.14}$$

Similarly,

$$\begin{aligned}
T(d(y_n, y_{n+1})) &\preceq \frac{a_1 + a_3 + a_6}{e - a_4 - a_6} T(d(y_{n-1}, y_n)) + \\
&\quad \frac{a_2}{e - a_4 - a_6} T(d(x_n, x_{n-1})).
\end{aligned} \tag{3.15}$$

From Definition 1.2, we have the following,

$$\begin{aligned}
T(d(x_n, x_{n+1}) + d(y_n, y_{n+1})) &\preceq T(d(x_n, x_{n-1})) + T(d(y_n, y_{n-1})) \\
&\preceq \frac{a_1 + a_2 + a_3 + a_6}{e - a_4 - a_6} T(d(x_n, x_{n-1})) + \\
&\quad \frac{a_1 + a_2 + a_3 + a_6}{e - a_4 - a_6} T(d(y_n, y_{n-1})) \\
&= \frac{a_1 + a_2 + a_3 + a_6}{e - a_4 - a_6} \Big[T(d(x_n, x_{n-1})) + \\
&\quad T(d(y_n, y_{n-1})) \Big]
\end{aligned}$$

Let $\lambda = \dfrac{a_1 + a_2 + a_3 + a_6}{e - a_4 - a_6}$. Then

$$T(d(x_n, x_{n+1}) + d(y_n, y_{n+1})) \preceq \lambda T(d(x_n, x_{n-1})) + T(d(y_n, y_{n-1})). \tag{3.16}$$

Therefore, for all $n \in \mathbb{N}$, we have,

$$\begin{aligned}
T(d(x_n, x_{n+1}) + d(y_n, y_{n+1})) &\preceq \lambda T(d(x_n, x_{n-1})) + T(d(y_n, y_{n-1})) \\
&\preceq \lambda^2 [T(d(x_{n-1}, x_{n-2})) + T(d(y_{n-1}, y_{n-2}))] \\
&\vdots \\
&\preceq \lambda^n [T(d(x_1, x_0)) + T(d(y_1, y_0))]
\end{aligned}$$

Now, we show that $r(\lambda) < 1$. Lemma 2.11 and 2.12, makes the following to be true

$$\begin{aligned}
r(\lambda) &= r[(a_1 + a_2 + a_3 + a_6)(e - a_4 - a_6)^{-1}] \\
&\leq r(a_1 + a_2 + a_3 + a_6) \cdot r(e - a_4 - a_6)^{-1} \\
&< \frac{r(a_1) + r(a_2) + r(a_3) + r(a_6)}{1 - r(a_4) - r(a_6)} < 1.
\end{aligned}$$

Following the same procedure as Theorem 3.2, we can conclude that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X, and since X is complete it implies that $x_n \to x^*(n \to \infty)$ and $y_n \to y^*(n \to \infty)$.

Next, we are to show that (x^*, y^*) is a coupled fixed point i.e. $x^* = F(x^*, y^*)$, $y^* = F(y^*, x^*)$.

$$\begin{aligned}
T(d(F(x^*, y^*), x^*)) &\preceq T(d(F(x^*, y^*), x_{n+1}) + d(x_{n+1}, x^*)) \\
&= T(d(F(x^*, y^*), F(x_n, y_n) + d(x_{n+1}, x^*)) \\
&\preceq T(d(F(x^*, y^*), F(x_n, y_n)) + T(d(x_{n+1}, x^*)) \\
&\preceq a_1 T(d(x_n, x^*)) + a_2 T(d(y_n, y^*)) + \\
&\quad a_3 dT(d(F(x^*, y^*), x^*)) + a_4 T(d(F(x_n, y_n), x_n)) + \\
&\quad a_5 T(d(F(x^*, y^*), x_n)) + a_5 T(d(F(x_n . y_n), x^*)) \\
&\quad + d(x_{n+1}, x^*).
\end{aligned}$$

We know that if $x_n \to x^*$ then $\lim d(x_n, x^*) = 0$. Hence, by Lemma 1.3 $\lim T(d(x_n, x^*)) = 0$. Therefore, we have the following as $n \to \infty$:

$$\begin{aligned}
T(d(F(x^*, y^*), x^*)) &\preceq a_3 T(d(F(x^*, y^*), x^*)) + a_4 T(d(F(x^*, y^*), x^*)) + \\
&\quad a_5 T(d(F(x^*, y^*), x^*)) + a_6 T(d(F(x^*, y^*), x^*))
\end{aligned}$$

Therefore,

$$(e - (a_3 + a_4 + a_5 + a_6)) T(d(F(x^*, y^*), x^*)) \preceq 0 \Rightarrow T(d(F(x^*, y^*), x^*)) = 0,$$

Similarly,

$$T(d(F(y^*, x^*), y^*)) = 0.$$

Hence, (x^*, y^*) is the couple fixed point.
Next, we show that this coupled fixed point is unique.
Suppose there exists another coupled fixed point say $(x_0, y_0) \in X \times X$. Then

$$\begin{aligned} T(d(x_0, x^*)) &= T(d(F(x_0, y_0), F(x^*, y^*))) \preceq T(a_1 d(x_0, x^*) + a_2 d(y_0, y^*) + \\ &\quad a_3 d(F(x_0, y_0), x_0) + a_4 d(F(x^*, y^*), x^*) + a_5 d(F(x_0, y_0), x^*) \\ &\quad + a_6 d(F(x^*, y^*), x_0)) \\ &= T(a_1 d(x_0, x^*) + a_2 d(y_0, y^*) + a_3 d(x_0, x_0) + a_4 d(x^*, x^*) + \\ &\quad a_5 d(x_0, x^*) + a_6 d(x^*, x_0)) \\ &= T((a_1 + a_5 + a_6) d(x_0, x^*) + a_2 d(y_0, y^*)). \end{aligned}$$

Similarly,

$$T(d(y_0, y^*)) \preceq T((a_1 + a_5 + a_6) d(y_0, y^*) + a_2 d(x_0, x^*)).$$

Therefore,

$$T(d(x_0, x^*) + d(y_0, y^*)) \preceq (a_1 + a_2 + a_5 + a_6) T(d(x_0, x^*) + d(y_0, y^*)).$$

Let $\beta = a_1 + a_2 + a_5 + a_6$. Hence, $T(d(x_0, x^*) + d(y_0, y^*)) \preceq \beta T(d(x_0, x^*) + d(y_0, y^*))$. Since $r(\beta) < 1$ then $T(d(x_0, x^*) + d(y_0, y^*)) = 0 \Rightarrow x_0 = x^*$ and $y_0 = y^*$, therefore (x^*, y^*) is a unique coupled fixed point. □

Let T be the identity function. We have the following corollary,

Corollary 3.5. *Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$, and let P be the underlining solid cone with $a_1, a_2, a_3, a_4, a_5, a_6 \in P$ where $r(a_1) + r(a_2) + r(a_3) + r(a_4) + r(a_5) + 2r(a_6) < 1$. Suppose the map $F : X \times X \to X$ satisfies the contractive condition:*

$$d(F(x, y), F(u, v)) \preceq j, \quad \forall\, x, y, u, v \in X \tag{3.17}$$

where
$j = a_1 d(x, u) + a_2 d(y, v) + a_3 d(F(x, y), x) + a_4 d(F(u, v), u) + a_5 d(F(x, y), u) + a_6 (F(u, v), x)$. *Then F has a unique coupled fixed point.*

Remark 3.6. (i) If $\mathcal{A}$ is just a Banach space and we assume that $\sum_{i=1}^{6} a_i < 1$ then we have the following:
   (a) in Theorem 3.2: then Theorem 3.2 become [7, Theorem 2.1]
   (b) In Theorem 3.1: if $a_i = 0 \forall i = 7, 8, \ldots, 10$ we also obtain [7, Theorem 2.1]

(ii) In Corollary 3.1 if $a_i = 0\, \forall\, i = 3, 4 \ldots, 10$ we obtain [2, Theorem 2.2]. If $a_i = 0\, \forall\, i = 1, 2, 5 \ldots, 10$ we obtain [2, Theorem 2.5]. If $a_i = 0\, \forall\, i = 1, \ldots 4, 7, \ldots, 10$ we obtain [2, Theorem 2.6].

4. Some Coupled Fixed Point Satisfying Some Contractive Map of Integral Type

The concept of fixed point of contractive map of integral type was introduced by Branciari [1] in 2002. And several results followed; of important to this work is the result of F. Khojasteh et al. [3], where they introduced concept of subadditive cone integrable function φ, and then, showed that there exists a unique fixed point for the contractive map of integral type. We need some definitions which appear in work of Khojasteh et al. [3] in proving our results.

Definition 4.1 (see [3]). *Suppose that P is a normal cone in $\mathcal{A}$. Let $a, b \in \mathcal{A}$ and $a < b$. We define*

$$\begin{aligned} &[a,b] := \{x \in \mathcal{A} : \ x = tb + (1-t)a, \ \text{for some } t \in [0,1]\} \\ &[a,b) := \{x \in \mathcal{A} : \ x = tb + (1-t)a, \ \text{for some } t \in [0,1)\} \end{aligned} \tag{4.1}$$

Definition 4.2 (see [3]). *The set $\{a = x_0, x_1, \dots, x_n = b\}$ is called a partition for $[a,b]$ if and only if the sets $\{[x_{i-1}, x_i)\}_{i-1}^{n}$ are pairwise disjoint and $[a,b] = \{\bigcup_{i-1}^{n}[x_{i-1}, x_i)\} \cup \{b\}$.*

Definition 4.3 (see [3]). *For each partition Q of $[a,b]$ and each increasing function $\varphi : [a,b] \to P$, we define cone lower summation and cone upper summation as*

$$\begin{aligned} L_n^{Con}(\varphi, Q) = \textstyle\sum_{i=0}^{n-1} \varphi(x_i)\|x_i - x_{i+1}\|, \\ U_n^{Con}(\varphi, Q) = \textstyle\sum_{i=0}^{n-1} \varphi(x_{i+1})\|x_i - x_{i+1}\|, \end{aligned} \tag{4.2}$$

respectively.

Definition 4.4 (see [3]). *Suppose that P is a normal cone in $\mathcal{A}$. $\varphi : [a,b] \to P$ is called an integrable function on $[a,b]$ with respect to cone P or simply, cone integrable function, if and only if for all partition Q of $[a,b]$*

$$\lim_{n\to\infty} L_n^{Con}(\varphi, Q) = S^{Con} = \lim_{n\to\infty} U_n^{Con}(\varphi, Q), \tag{4.3}$$

where S^{Con} must be unique.
We show the common value S^{Con} by

$$\int_a^b \varphi(x) d_p(x) \text{ or simply } \int_a^b \varphi d_p. \tag{4.4}$$

Let $\mathcal{L}^1([a,b], P)$ denote the set of all cone integrable functions.

Lemma 4.5 (see [3]). *(1) If $[a,b] \subseteq [a,c]$, then $\int_a^b f d_p \preceq \int_a^c f d_p$, for $f \in \mathcal{L}^1([a,b], P)$.*
(2) $\int_a^b (\alpha f + \beta g) d_p = \alpha \int_a^b f d_p + \beta \int_a^b g d_p$, for $f, g \in \mathcal{L}^1([a,b], P)$ and $\alpha, \beta \in \mathbb{R}$.

Definition 4.6 (see [3]). The function $\varphi : P \to \mathcal{A}$ is called subadditive cone integrable function if and only if for all $a, b \in P$

$$\int_0^{a+b} \varphi(t)dt \preceq \int_0^a \varphi(t)dt + \int_0^b \varphi(t)dt. \tag{4.5}$$

Example 4.7 (see [3]). Let $\mathcal{A} = X = \mathbb{R}$, $d(x,y) = |x-y|$, $P = [0, +\infty)$, and $\varphi(t) = \dfrac{1}{t+1}$ for all $t > 0$. Then for all $a, b \in P$,

$$\int_0^{a+b} \frac{dt}{t+1} \preceq \int_0^a \frac{dt}{t+1} + \int_0^b \frac{dt}{t+1}.$$

It is sufficient to show that,

$$\ln(a+b+1) \preceq \ln(a+1) + \ln(b+1).$$

Observe that, $a+b+1 \preceq a+b+1+ab \preceq (a+1)(b+1)$. Therefore, $\ln(a+b+1) \preceq \ln[(a+1)(b+1)] = \ln(a+1) + \ln(b+1)$.

Theorem 4.8 (see [3]). *Let (X,d) be a complete cone metric space and let P be a normal cone. Suppose that $\varphi : P \to P$ is a nonvanishing map and a subadditive cone integrable on each $[a,b] \subset P$ such that for each $\epsilon \gg 0$, $0 \ll \int_0^\epsilon \varphi(t)dt$. If $T : X \to X$ is a map such that, for all $x, y \in X$,*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \preceq k \int_0^{d(x,y)} \varphi(t)dt \quad \forall\, x, y \in X, \tag{4.6}$$

for some $k \in [0,1)$, then T has a unique fixed point $x^* \in X$. And for each $x \in X$, $T^n(x) \to x^*$ as $n \to \infty$.

Theorem 4.1 extends the result of Branciari [1] to cone metric space. Now, using the idea of Khojasteh et al., Definition 2.4 and Lemma 2.6 Olaleru et al. [7] was able to show that the contractive map of integral type on a cone metric space has a unique coupled fixed point.

In this section, we extend the result of Olaleru et al. [7, Theorem 3.1] to cone metric space over Banach algebra.

Theorem 4.9. *Let (X,d) be a cone metric space over banach algebra and let P be a normal cone. Let $\varphi : P \to P$ be a nonvanishing map and a subbaditive cone integrable on each $[a,b]$. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition*

$$\int_0^{d(F(x,y),F(u,v))} \varphi(t)dt \preceq \int_0^{j(x,y,u,v)} \varphi(t)dt \quad \forall\, x, y, u, v \in X, \tag{4.7}$$

where

$$\begin{aligned} j(x,y,u,v) \quad = \quad & a_1 d(x,u) + a_2 d(y,v) + a_3 d(F(x,y),x) + a_4 d(F(u,v),u) + \\ & a_5 d(F(x,y),u) + a_6 d(F(u,v),x) + a_7 d(F(y,x),y) + \\ & a_8 d(F(v,u),v) + a_9 d(F(y,x),v) + a_{10} d(F(v,u),y), \end{aligned}$$

$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10} \in P$ *where* $r(a_1) + r(a_2) + r(a_3) + r(a_4) + r(a_5) + 2r(a_6) + r(a_7) + r(a_8) + 2r(a_{10}) < 1$. *Then* F *has a unique coupled fixed point.*

Proof. The proof of Theorem 4.9 follows from the method of proof of Theorem 3.1 if $T(j(x, y, u, v)) = \int_0^{j(x,y,u,v)}$. Thus, we conclude that F has a unique coupled fixed point. This ends the proof. □

Competing Interest

The authors declares that they have no competing interest.

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