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RELATIVE $(p, q) - \varphi$ ORDER BASED SOME GROWTH ANALYSIS OF COMPOSITE p-ADIC ENTIRE FUNCTIONS

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Abstract. Let $\mathbb{K}$ be a complete ultrametric algebraically closed field and $\mathcal{A}(\mathbb{K})$ be the $\mathbb{K}$-algebra of entire function on $\mathbb{K}$. For any p-adic entire functions $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. In this paper we study some growth properties of composite p-adic entire functions on the basis of their relative (p, q)-φ order where p, q are any two positive integers and $\varphi(r) : [0, +\infty) \to (0, +\infty)$ is a non-decreasing unbounded function of r.

1. Introduction, Definitions and Notations

Let us consider an algebraically closed field $\mathbb{K}$ of characteristic zero complete with respect to a p-adic absolute value $|\cdot|$ (example $\mathbb{C}_p$). For any $\alpha \in \mathbb{K}$ and $R \in]0, +\infty[$, the closed disk $\{x \in \mathbb{K} : |x - \alpha| \leq R\}$ and the open disk $\{x \in \mathbb{K} : |x - \alpha| < R\}$ are denoted by $d(\alpha, R)$ and $d(\alpha, R^-)$ respectively. Also $C(\alpha, r)$ denotes the circle $\{x \in \mathbb{K} : |x - \alpha| = r\}$. Moreover $\mathcal{A}(\mathbb{K})$ represent the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$ i.e. the set of power series with an infinite radius of convergence. For the most comprehensive study of analytic functions inside a disk or in the whole field $\mathbb{K}$, we refer the reader to the books [14–16, 18]. During the last several years the ideas of p-adic analysis have been studied from different aspects and many important results were gained (see [10] to [11], [12, 13]).

Let $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. Moreover, if f is not a constant, the $|f|(r)$ is strictly increasing function of r and tends to $+\infty$ with r, therefore there exists its inverse function $\widehat{|f|} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim_{s\to\infty} \widehat{|f|}(s) = \infty$.

For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ and $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. Throughout the paper, log denotes the Neperian logarithm. Further we assume that throughout the present paper p, q and m always denote positive

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integers. Taking this into account the (p,q)-th order and (p,q)-th lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ are define as follows:

Definition 1.1. [4] Let $f \in \mathcal{A}(\mathbb{K})$. Then the (p,q)-th order $\rho^{(p,q)}(f)$ and (p,q)-th lower order $\lambda^{(p,q)}(f)$ of f are respectively defined as:

$$\rho^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} |f|(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} |f|(r)}{\log^{[q]} r}$$

Definition 1.1 avoids the restriction $p \geq q$ of the original definition of (p,q)-th order (respectively (p,q)-th lower order) of entire functions introduced by Juneja et al. [17] in complex context.

When $q = 1$, we get the definitions of generalized order and generalized lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ which symbolize as $\rho^{(p)}(f)$ and $\lambda^{(p)}(f)$ respectively. If $p = 2$ and $q = 1$ then we write $\rho^{(2,1)}(f) = \rho(f)$ and $\lambda^{(2,1)}(f) = \lambda(f)$ where $\rho(f)$ and $\lambda(f)$ are respectively known as order and lower order of $f \in \mathcal{A}(\mathbb{K})$ introduced by Boussaf et al. [10].

The concepts of (p,q)-φ order and (p,q)-φ lower order of entire functions in complex context were introduced by Shen et al. [19] where $p \geq q \geq 1$ and $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. For details about (p,q)-φ order and (p,q)-φ lower order, one may see [19]. Considering the ideas developed by Shen et al. [19], one can define the (p,q)-φ order and (p,q)-φ lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ respectively in the following way:

Definition 1.2. [2] Let $f \in \mathcal{A}(\mathbb{K})$. Also let $\varphi(r) : [0,+\infty) \to (0,+\infty)$ is a non-decreasing unbounded function of r. The (p,q)-φ order $\rho^{(p,q)}(f,\varphi)$ and (p,q)-φ lower order $\lambda^{(p,q)}(f,\varphi)$ of f are respectively defined as:

$$\rho^{(p,q)}(f,\varphi) = \limsup_{r\to\infty} \frac{\log^{[p]} |f|(r)}{\log^{[q]} \varphi(r)} \text{ and } \lambda^{(p,q)}(f,\varphi) = \liminf_{r\to\infty} \frac{\log^{[p]} |f|(r)}{\log^{[q]} \varphi(r)}$$

If $\varphi(r) = r$, then Definition 1.1 is a special case of Definition 1.2. If $q = 1$, then Definition 1.2 reduces to the definitions of generalized φ order and generalized φ lower order of $f \in \mathcal{A}(\mathbb{K})$ and in this case we simplify to denote $\rho^{(p,1)}(f,\varphi)$ and $\lambda^{(p,1)}(f,\varphi)$ by $\rho^{[p]}(f,\varphi)$ and $\lambda^{[p]}(f,\varphi)$ respectively.

The notion of relative order was first introduced by Bernal [1]. In order to make some progress in the study of p-adic analysis, Biswas [3] introduce the definition of relative order $\rho_g(f)$ and relative lower order $\lambda_g(f)$ of entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ in the following way:

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log \widehat{|g|}(|f|(r))}{\log r} \text{ and } \lambda_g(f) = \liminf_{r\to\infty} \frac{\log \widehat{|g|}(|f|(r))}{\log r}$$

In the case of relative order, it therefore seems reasonable to define suitably the (p,q)-th relative order of entire function belonging to $\mathcal{A}(\mathbb{K})$. With this in view one may introduce the definition of (p,q)-th relative order $\rho_g^{(p,q)}(f)$ and (p,q)-th relative lower order $\lambda_g^{(p,q)}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$, in the light of index-pair which are as follows:

Definition 1.3. [4] Let $f,\ g \in \mathcal{A}(\mathbb{K})$. The (p,q)-th relative order $\rho_g^{(p,q)}(f)$ and (p,q)-th relative lower order $\lambda_g^{(p,q)}(f)$ of f with respect to g are defined as

$$\begin{aligned}\rho_g^{(p,q)}(f) &= \limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|g|}(|f|(r))}{\log^{[q]} r} = \limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|g|}(r)}{\log^{[q]}\widehat{|f|}(r)},\\ \lambda_g^{(p,q)}(f) &= \liminf_{r\to\infty}\frac{\log^{[p]}\widehat{|g|}(|f|(r))}{\log^{[q]} r} = \liminf_{r\to\infty}\frac{\log^{[p]}\widehat{|g|}(r)}{\log^{[q]}\widehat{|f|}(r)}.\end{aligned}$$

Extending this notion, the definition of relative (p,q)-φ order $\rho_g^{(p,q)}(f,\varphi)$ and relative (p,q)-φ lower order $\lambda_g^{(p,q)}(f,\varphi)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$.are given in [2] which are as follows:

Definition 1.4. [2] Let $f,\ g \in \mathcal{A}(\mathbb{K})$. Also let $\varphi(r) : [0,+\infty) \to (0,+\infty)$ is a non-decreasing unbounded function of r. The relative (p,q)-φ order denoted as $\rho_g^{(p,q)}(f,\varphi)$ and relative (p,q)-φ lower order denoted by $\lambda_g^{(p,q)}(f,\varphi)$ of f with respect to g are defined as

$$\begin{aligned}\rho_g^{(p,q)}(f,\varphi) &= \limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|g|}(|f|(r))}{\log^{[q]}\varphi(r)} = \limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|g|}(r)}{\log^{[q]}\varphi\left(\widehat{|f|}(r)\right)},\\ \lambda_g^{(p,q)}(f,\varphi) &= \liminf_{r\to\infty}\frac{\log^{[p]}\widehat{|g|}(|f|(r))}{\log^{[q]}\varphi(r)} = \liminf_{r\to\infty}\frac{\log^{[p]}\widehat{|g|}(r)}{\log^{[q]}\varphi\left(\widehat{|f|}(r)\right)}.\end{aligned}$$

If $\varphi(r) = r$, then Definition 1.3 is a special case of Definition 1.4.If $q = 1$, then Definition 1.4 reduces to the definitions of generalized relative φ order and generalized relative φ lower order of $f \in \mathcal{A}(\mathbb{K})$ with respect to $g \in \mathcal{A}(\mathbb{K})$ and in this case we simplify to denote $\rho_g^{(p,1)}(f,\varphi)$ and $\lambda_g^{(p,1)}(f,\varphi)$ by $\rho_g^{[p]}(f,\varphi)$ and $\lambda_g^{[p]}(f,\varphi)$ respectively.

The main aim of this paper is to establish some results related to the growth rates of p-adic entire functions on the basis of relative (p,q)-φ order and relative (p,q)-φ lower order where $\varphi(r) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function of r.

2. Lemma

In this section we present the following lemma which can be found in [10] or [9] and will be needed in the sequel.

Lemma 2.1. *Let $f,\ g \in \mathcal{A}(\mathbb{K})$. Then for all sufficiently large positive numbers of r the following equality holds*

$$|f \circ g|(r) = |f|(|g|(r)).$$

3. Main Results

In this section we state the main results of the paper.

Theorem 3.1. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$ be such that*

$$(i)\quad \limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|g|\left(r\right)\right)}{\left(\log^{[q]}\varphi\left(r\right)\right)^{\alpha}} = A,\ \text{a real number } > 0,$$

$$(ii)\quad \liminf_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f|\left(r\right)\right)}{\left(\log^{[p]}\widehat{|h|}\left(r\right)\right)^{\beta+1}} = B,\ \text{a real number } > 0$$

for any pair of α,β satisfying $0<\alpha<1$, $\beta>0$ and $\alpha\left(\beta+1\right)>1$. Then

$$\rho_h^{(p,q)}\left(f\circ g,\varphi\right)=\infty.$$

Proof. From (i), we get for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]}\widehat{|h|}\left(|g|\left(r\right)\right)\geq\left(A-\varepsilon\right)\left(\log^{[q]}\varphi\left(r\right)\right)^{\alpha} \tag{1}$$

and from (ii), it follows for all sufficiently large positive numbers of r that

$$\log^{[p]}\widehat{|h|}\left(|f|\left(r\right)\right)\geq\left(B-\varepsilon\right)\left(\log^{[p]}\widehat{|h|}\left(r\right)\right)^{\beta+1}.$$

As $|g|\left(r\right)$ is continuous, increasing and unbounded function of r, we obtain from above for all sufficiently large positive numbers of r that

$$\log^{[p]}\widehat{|h|}\left(|f|(|g|\left(r\right))\right)\geq\left(B-\varepsilon\right)\left(\log^{[p]}\widehat{|h|}\left(|g|\left(r\right)\right)\right)^{\beta+1}. \tag{2}$$

Since $\widehat{|h|}\left(r\right)$ is an increasing function of r, we get from Lemma 2.1, (1) and (2) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)\geq\left(B-\varepsilon\right)\left(\log^{[p]}\widehat{|h|}\left(|g|\left(r\right)\right)\right)^{\beta+1}$$

$$i.e.,\ \log^{[p]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)\geq\left(B-\varepsilon\right)\left(\left(A-\varepsilon\right)\left(\log^{[q]}\varphi\left(r\right)\right)^{\alpha}\right)^{\beta+1}$$

$$i.e.,\ \frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log^{[q]}\varphi\left(r\right)}\geq\frac{\left(B-\varepsilon\right)\left(A-\varepsilon\right)^{\beta+1}\left(\log^{[q]}\varphi\left(r\right)\right)^{\alpha(\beta+1)}}{\log^{[q]}\varphi\left(r\right)}$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log^{[q]}\varphi\left(r\right)}$$

$$\geq\liminf_{r\to\infty}\frac{\left(B-\varepsilon\right)\left(A-\varepsilon\right)^{\beta+1}\left(\log^{[q]}\varphi\left(r\right)\right)^{\alpha(\beta+1)}}{\log^{[q]}\varphi\left(r\right)}.$$

As $\varepsilon\left(>0\right)$ is arbitrary and $\alpha\left(\beta+1\right)>1$, therefore it follows from above that

$$\rho_h^{(p,q)}\left(f\circ g,\varphi\right)=\infty.$$

Thus the theorem follows. □

In the line of Theorem 3.1, one may state the following two theorems without their proofs:

THEOREM 3.2. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$ be such that*

$$(i) \quad \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|g|(r))}{\left(\log^{[q]} \varphi(r)\right)^{\alpha}} = A, \text{ a real number } > 0,$$

$$(ii) \quad \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f|(r))}{\left(\log^{[p]} \widehat{|h|}(r)\right)^{\beta+1}} = B, \text{ a real number } > 0$$

for any pair of α, β satisfying $0 < \alpha < 1$, $\beta > 0$ and $\alpha(\beta+1) > 1$. Then

$$\rho_h^{(p,q)}(f \circ g, \varphi) = \infty.$$

THEOREM 3.3. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$ be such that*

$$(i) \quad \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|g|(r))}{\left(\log^{[q]} \varphi(r)\right)^{\alpha}} = A, \text{ a real number } > 0,$$

$$(ii) \quad \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f|(r))}{\left(\log^{[p]} \widehat{|h|}(r)\right)^{\beta+1}} = B, \text{ a real number } > 0$$

for any pair of α, β satisfying $0 < \alpha < 1$, $\beta > 0$ and $\alpha(\beta+1) > 1$. Then

$$\lambda_h^{(p,q)}(f \circ g, \varphi) = \infty.$$

THEOREM 3.4. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$ be such that*

$$(i) \quad \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|g|(r))}{\left(\log^{[q]} r\right)^{\alpha}} = A, \text{ a real number } > 0,$$

$$(ii) \quad \liminf_{r\to\infty} \frac{\log\left(\frac{\log^{[p]} \widehat{|h|}(|f|(r))}{\log^{[p]} \widehat{|h|}(r)}\right)}{\left(\log^{[p]} \widehat{|h|}(r)\right)^{\beta}} = B, \text{ a real number } > 0$$

for any pair of α, β satisfying $\alpha > 1$, $0 < \beta < 1$ and $\alpha\beta > 1$. Then

$$\rho_h^{(p,q)}(f \circ g, \varphi) = \infty.$$

Proof. From (i), we obtain for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} \widehat{|h|}(|g|(r)) \geq (A - \varepsilon)\left(\log^{[q]} r\right)^{\alpha} \tag{3}$$

and from (ii), we get for all sufficiently large positive numbers of r that

$$\log\left(\frac{\log^{[p]} \widehat{|h|}(|f|(r))}{\log^{[p]} \widehat{|h|}(r)}\right) \geq (B - \varepsilon)\left(\log^{[p]} \widehat{|h|}(r)\right)^{\beta}$$

$$i.e., \quad \frac{\log^{[p]} \widehat{|h|}(|f|(r))}{\log^{[p]} \widehat{|h|}(r)} \geq \exp\left((B - \varepsilon)\left(\log^{[p]} \widehat{|h|}(r)\right)^{\beta}\right).$$

As $|g|(r)$ is continuous, increasing and unbounded function of r, we have from above for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]} \widehat{|h|}(|f|(|g|(r)))}{\log^{[p]} \widehat{|h|}(|g|(r))} \geq \exp\left((B-\varepsilon)\left(\log^{[p]} \widehat{|h|}(|g|(r))\right)^{\beta}\right). \tag{4}$$

Since $\widehat{|h|}(r)$ is an increasing function of r, we get from Lemma 2.1, (3) and (4) for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[q]} \varphi(r)} = \frac{\log^{[p]} \widehat{|h|}(|f|(|g|(r)))}{\log^{[p]} \widehat{|h|}(|g|(r))} \cdot \frac{\log^{[p]} \widehat{|h|}(|g|(r))}{\log^{[q]} \varphi(r)}$$

$$i.e., \ \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[q]} \varphi(r)} \geq$$

$$\exp\left((B-\varepsilon)\left(\log^{[p]} \widehat{|h|}(|g|(r))\right)^{\beta}\right) \cdot \frac{(A-\varepsilon)\left(\log^{[q]} r\right)^{\alpha}}{\log^{[q]} \varphi(r)}$$

$$i.e., \ \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[q]} \varphi(r)} \geq$$

$$\exp\left((B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[q]} r\right)^{\alpha\beta}\right) \cdot \frac{(A-\varepsilon)\left(\log^{[q]} r\right)^{\alpha}}{\log^{[q]} \varphi(r)}$$

$$i.e., \ \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[q]} \varphi(r)} \geq$$

$$\exp\left((B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[q]} r\right)^{\alpha\beta-1}\left(\log^{[q]} r\right)\right) \cdot \frac{(A-\varepsilon)\left(\log^{[q]} r\right)^{\alpha}}{\log^{[q]} \varphi(r)}$$

$$i.e., \ \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[q]} \varphi(r)} \geq$$

$$\left(\log^{[q-1]} r\right)^{(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[q]} r\right)^{\alpha\beta-1}} \cdot \frac{(A-\varepsilon)\left(\log^{[q]} r\right)^{\alpha}}{\log^{[q]} \varphi(r)}$$

$$i.e., \ \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[q]} \varphi(r)} \geq$$

$$\liminf_{r\to\infty} \left(\log^{[q-1]} r\right)^{(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[q]} r\right)^{\alpha\beta-1}} \cdot \frac{(A-\varepsilon)\left(\log^{[q]} r\right)^{\alpha}}{\log^{[q]} \varphi(r)}.$$

Since $\varepsilon (> 0)$ is arbitrary and $\alpha > 1$, $\alpha\beta > 1$, therefore, the conclusion of the theorem follows from above. $\square$

In the line of Theorem 3.4, one may also state the following two theorems without their proofs :

THEOREM 3.5. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$ be such that*

$$(i) \qquad \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|g|(r))}{\left(\log^{[q]} r\right)^{\alpha}} = A, \text{ a real number } > 0,$$

$$(ii) \ \limsup_{r\to\infty} \frac{\log\left(\frac{\log^{[p]} \widehat{|h|}(|f|(r))}{\log^{[p]} \widehat{|h|}(r)}\right)}{\left(\log^{[p]} \widehat{|h|}(r)\right)^{\beta}} = B, \text{ a real number } > 0$$

for any pair of α, β with $\alpha > 1$, $0 < \beta < 1$ and $\alpha\beta > 1$. Then

$$\rho_h^{(p,q)}(f \circ g, \varphi) = \infty.$$

THEOREM 3.6. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$ be such that*

$$(i) \qquad \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|g|(r))}{\left(\log^{[q]} r\right)^{\alpha}} = A, \text{ a real number } > 0,$$

$$(ii) \ \liminf_{r\to\infty} \frac{\log\left(\frac{\log^{[p]} \widehat{|h|}(|f|(r))}{\log^{[p]} \widehat{|h|}(r)}\right)}{\left(\log^{[p]} \widehat{|h|}(r)\right)^{\beta}} = B, \text{ a real number } > 0$$

for any pair of α, β satisfying $\alpha > 1$, $0 < \beta < 1$ and $\alpha\beta > 1$. Then

$$\lambda_h^{(p,q)}(f \circ g, \varphi) = \infty.$$

THEOREM 3.7. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_h^{(p,q)}(g, \varphi) \le \rho_h^{(p,q)}(g, \varphi) < \infty$ and*

$$\lim_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f|(r))}{\log^{[p]} \widehat{|h|}(r)} = A, \text{ a real number } < \infty.$$

Then

$$\rho_h^{(p,q)}(f \circ g, \varphi) = A \cdot \rho_h^{(p,q)}(g, \varphi) \text{ and } \lambda_h^{(p,q)}(f \circ g, \varphi) = A \cdot \lambda_h^{(p,q)}(g, \varphi).$$

Proof. Since $\widehat{|h|}(r)$ is an increasing function of r, it follows from Lemma 2.1 for all sufficiently large positive numbers of r that

$$(5) \qquad \widehat{|h|}(|f \circ g|(r)) = \widehat{|h|}(|f|(|g|(r))).$$

Therefore from (5) we get for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[q]} \varphi(r)} = \frac{\log^{[p]} \widehat{|h|}(|f|(|g|(r)))}{\log^{[p]} \widehat{|h|}(|g|(r))} \cdot \frac{\log^{[p]} \widehat{|h|}(|g|(r))}{\log^{[q]} \varphi(r)}$$

$$i.e., \ \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[q]} \varphi(r)}$$

$$= \limsup_{r\to\infty} \left(\frac{\log^{[p]} \widehat{|h|}(|f|(|g|(r)))}{\log^{[p]} \widehat{|h|}(|g|(r))} \cdot \frac{\log^{[p]} \widehat{|h|}(|g|(r))}{\log^{[q]} \varphi(r)} \right)$$

$$i.e., \ \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[q]} \varphi(r)}$$

$$= \lim_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f|\left(|g|\left(r\right)\right)\right)}{\log^{[p]} \widehat{|h|}\left(|g|\left(r\right)\right)} \cdot \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|g|\left(r\right)\right)}{\log^{[q]} \varphi\left(r\right)}$$

$$(6) \qquad i.e., \ \rho_h^{(p,q)}\left(f\circ g, \varphi\right) = A \cdot \rho_h^{(p,q)}\left(g, \varphi\right).$$

Similarly one can easily verify that

$$\lambda_h^{(p,q)}\left(f\circ g, \varphi\right) = A \cdot \lambda_h^{(p,q)}\left(g, \varphi\right).$$

Hence the theorem follows. □

Theorem 3.8. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$ be such that $\rho_h^{[p]}\left(f, \varphi_1\right) > 0$ and $\lambda^{[m]}\left(g, \varphi_1\right) > 0$. Also let $\varphi_1(r) = \log^{[q-2]} r \cdot \varphi(r)$ where $m \geq q$. Then*

$$\rho_h^{[p]}\left(f\circ g, \varphi_1\right) = \infty.$$

Proof. Since $\widehat{|h|}(r)$ is an increasing function of r, it follows from (5) for a sequence of positive numbers of r tending to infinity,

$$\log^{[p]} \widehat{|h|}\left(|f\circ g|\left(r\right)\right) \geq \left(\rho_h^{[p]}\left(f, \varphi_1\right) - \varepsilon\right)\left(\log^{[q-1]} |g|\left(r\right) + \log\varphi\left(|g|\left(r\right)\right)\right)$$

$$i.e., \ \log^{[p]} \widehat{|h|}\left(|f\circ g|\left(r\right)\right) \geq$$

$$\left(\rho_h^{[p]}\left(f, \varphi_1\right) - \varepsilon\right)\left(\exp^{[m-q]}\left(\log^{[q-2]} r\cdot\varphi\left(r\right)\right)^{\left(\lambda^{[m]}\left(g,\varphi_1\right)-\varepsilon\right)} + \log\varphi\left(|g|\left(r\right)\right)\right)$$

$$i.e., \ \frac{\log^{[p]} \widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log\varphi_1\left(r\right)} \geq$$

$$\frac{\left(\rho_h^{[p]}\left(f, \varphi_1\right) - \varepsilon\right)\left(\exp^{[m-q]}\left(\log^{[q-2]} r\cdot\varphi\left(r\right)\right)^{\left(\lambda^{[m]}\left(g,\varphi_1\right)-\varepsilon\right)} + \varphi\left(|g|\left(r\right)\right)\right)}{\log^{[q-1]} r + \log\varphi\left(r\right)}$$

$$i.e., \ \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log\varphi_1\left(r\right)} \geq$$

$$\liminf_{r\to\infty} \frac{\left(\rho_h^{[p]}\left(f, \varphi_1\right) - \varepsilon\right)\left(\exp^{[m-q]}\left(\log^{[q-2]} r\cdot\varphi\left(r\right)\right)^{\left(\lambda^{[m]}\left(g,\varphi_1\right)-\varepsilon\right)} + \varphi\left(|g|\left(r\right)\right)\right)}{\log^{[q-1]} r + \log\varphi\left(r\right)}$$

$$i.e., \ \rho_h^{[p]}\left(f\circ g, \varphi_1\right) = \infty.$$

Hence the theorem follows. □

Corollary 3.9. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$ be such that $\lambda_h^{[p]}\left(f, \varphi_1\right) > 0$ and $\rho^{[m]}\left(g, \varphi_1\right) > 0$. Also let $\varphi_1(r) = \log^{[q-2]} r \cdot \varphi(r)$ where $m \geq q$. Then*

$$\rho_h^{[p]}\left(f\circ g, \varphi_1\right) = \infty.$$

The proof of Corollary 3.9 is omitted as it can be carried out in the line of Theorem 3.8.

In the line of Theorem 3.8 one can easily prove the following theorem:

Theorem 3.10. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$ be such that $\lambda_h^{[p]}\left(f, \varphi_1\right) > 0$ and $\lambda^{[m]}\left(g, \varphi_1\right) > 0$. Also let $\varphi_1(r) = \log^{[q-2]} r \cdot \varphi(r)$ where $m \geq q$. Then*

$$\lambda_h^{[p]}\left(f\circ g, \varphi_1\right) = \infty.$$

THEOREM 3.11. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$ be such that $\rho_h^{[p]}(f,\varphi_1) > 0$ and $\lambda^{[m]}(g,\varphi_1) > 0$. Also let $\varphi_1(r) = \log^{[q-2]} r \cdot \varphi(r)$ where $m \geq q$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(r))} = \infty.$$

Proof. In view of Theorem 3.8, we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(r))} \geq$$

$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log \varphi_1(r)} \cdot \liminf_{r\to\infty} \frac{\log \varphi_1(r)}{\log^{[p]} \widehat{|h|}(|f|(r))}$$

$$i.e., \ \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(r))} \geq \rho_h^{[p]}(f \circ g,\varphi_1) \cdot \frac{1}{\rho_h^{[p]}(f,\varphi_1)}$$

$$i.e., \ \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(r))} = \infty.$$

Thus theorem follows. □

THEOREM 3.12. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$ be such that $\lambda_h^{[p]}(f,\varphi_1) > 0$ and $\lambda^{[m]}(g,\varphi_1) > 0$. Also let $\varphi_1(r) = \log^{[q-2]} r \cdot \varphi(r)$ where $m \geq q$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(r))} = \infty.$$

Proof of Theorem 3.12 is omitted as it can be carried out in the line of Theorem 3.11 and in view of Theorem 3.10.

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