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APPLICATIONS OF NANO TOPOLOGY VIA NANO OPERATIONS

Hariwan Z. Ibrahim

Abstract. The purpose of this paper is to define and study some new classes of sets by using nano operation namely, ζ-nano regular open, ζ-nano open, ζ-nano α-open, ζ-nano pre-open, ζ-nano semi-open, ζ-nano b-open and ζ-nano β-open in nano topology. Some properties and the relationships between these sets and the related concepts are investigated. Also, we found the deciding factors for the most common disease fever.

1. Introduction

General Topology is vast and has many different inventions and interactions with other fields of Mathematics and Science. The notion of approximations and boundary region of a set was originally proposed by Pawlak [10] in order to introduce the concept of rough set theory. An equivalence relation known as indiscernibility relation is the mathematical basis for the theory. A rough set can be described by a pair of definable sets called lower and upper approximations. The lower approximation is the greatest definable set contained in the given set of objects, while the upper approximation is the smallest definable set containing the given set. Thivagar et al [1, 2] introduced a nano topology which is defined in terms of the lower and upper approximations and the boundary region of a subset of an universe. Thivagar and Richard [5] established the weak forms of nano open sets namely nano α-open sets, nano pre-open sets and nano semi-open sets in a nano topological space and also they introduced nano-regular open sets. Revathy et al [7] and Parimala et al [8] respectively, introduced the notions nano β-open and nano b-open sets. Nano means something very small. It comes from the Greek word nanos which means 'dwarf', in

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its modern scientific sense, an order of magnitude-one billionth of something. But certain nano-terms are satisfied simply to mean "very small". Nanocar is an example. Ibrahim [9] introduced and discussed an operation of a family of all α-open sets in topological space. Ibrahim et al [3, 6, 11] continued studying the properties of such operations.

2. Preliminaries

Definition 2.1 ([4])**.** Let U be a nonempty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the *approximation space*. Let $X \subseteq U$.

(1) The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and its is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by x.
(2) The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$.
(3) The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not-X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2 ([1, 2])**.** Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then, $\tau_R(X)$ satisfies the following axioms:

(1) U and $\phi \in \tau_R(X)$.
(2) The union of the elements of any subcollection of $\tau_R(X)$ is in $\tau_R(X)$.
(3) The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ is a topology on U called the *nano topology* on U with respect to X. We call $(U, \tau_R(X))$ as the *nano topological space*. The elements of $\tau_R(X)$ are called as *nano open sets*. A subset F of U is *nano closed* if its complement is nano open.

Definition 2.3 ([1]). Let $(U, \tau_R(X))$ be a nano topological space with respect to X where $X \subseteq U$ and $K \subseteq U$, then the nano interior of K is defined as the union of all nano open subsets of K and it is denoted by $NInt(K)$. The nano closure of K is defined as the intersection of all nano closed sets containing K and it is denoted by $NCl(K)$.

Definition 2.4. A subset K of a nano topological space $(U, \tau_R(X))$ is called:
(1) *nano-regular open* if $NInt(NCl(K)) = K$ ([5]).
(2) *nano α-open* if $K \subseteq NInt(NCl(NInt(K)))$ ([5]).
(3) *nano pre-open* if $K \subseteq NInt(NCl(K))$ ([5]).
(4) *nano semi-open* if $K \subseteq NCl(NInt(K))$ ([5]).
(5) *nano b-open* if $K \subseteq NCl(NInt(K)) \cup NInt(NCl(K))$ ([8]).
(6) *nano β-open* if $K \subseteq NCl(NInt(NCl(K)))$ ([7]).

We denote by $NRO(U, X)$ (resp. $\tau_R^{\alpha}(X)$, $NPO(U, X)$, $NSO(U, X)$, $NbO(U, X)$ and $N\beta O(U, X)$) the family of nano-regular open (resp. nano α-open, nano pre-open, nano semi-open, nano b-open and nano β-open) sets in U.

Definition 2.5. Let $(U, \tau_R(X))$ be a nano topological space. A subset F of U is said to be *nano-regular closed* (resp. *nano α-closed, nano pre-closed, nano semi-closed, nano b-closed and nano β-closed*) if its complement is nano-regular open (resp. nano α-open, nano pre-open, nano semi-open, nano b-open and nano β-open).

Theorem 2.6 ([5]). *Any nano-regular open set is nano open.*

Theorem 2.7 ([5]). *If A is nano open in $(U, \tau_R(X))$, then it is nano α-open in U.*

Theorem 2.8 ([5]). $\tau_R^{\alpha}(X) \subseteq NPO(U, X)$ *in a nano topological space* $(U, \tau_R(X))$.

Theorem 2.9 ([5]). $\tau_R^{\alpha}(X) \subseteq NSO(U, X)$ *in a nano topological space* $(U, \tau_R(X))$.

Theorem 2.10 ([8]). *Every nano pre-open set is nano b-open.*

Theorem 2.11 ([8]). *Every nano semi-open set is nano b-open.*

Theorem 2.12 ([8]). *Every nano b-open set is nano β-open.*

3. New Types of Nano Open Sets

Definition 3.1. Let $\tau_R(X)$ be a nano topology on U with respect to X and $\zeta : N\beta O(U, X) \to P(U)$ be a nano operation satisfying that, $K \subseteq K^{\zeta}$ for each $K \in$

$N\beta O(U,X)$. Then, a nonempty subset L of U is called *ζ-nano regular open* (resp. *ζ-nano open, ζ-nano α-open, ζ-nano pre-open, ζ-nano semi-open, ζ-nano b-open and ζ-nano β-open*) if for each point $s \in L$, there exists a nano-regular open (resp. nano open, nano α-open, nano pre-open, nano semi-open, nano b-open and nano β-open) set D containing s such that $D^\zeta \subseteq L$. We suppose that the empty set is ζ-nano regular open (resp. ζ-nano open, ζ-nano α-open, ζ-nano pre-open, ζ-nano semi-open, ζ-nano b-open and ζ-nano β-open) for any operation $\zeta : N\beta O(U,X) \to P(U)$.

The family of all ζ-nano regular open (resp. ζ-nano open, ζ-nano α-open, ζ-nano pre-open, ζ-nano semi-open, ζ-nano b-open and ζ-nano β-open) sets denote by $NRO(U,X)_\zeta$ (resp. $\tau_R(X)_\zeta$, $\tau_R^\alpha(X)_\zeta$, $NPO(U,X)_\zeta$, $NSO(U,X)_\zeta$, $NbO(U,X)_\zeta$ and $N\beta O(U,X)_\zeta$). The complement of a ζ-nano regular open (resp. ζ-nano open, ζ-nano α-open, ζ-nano pre-open, ζ-nano semi-open, ζ-nano b-open and ζ-nano β-open) set is called *ζ-nano regular closed* (resp. *ζ-nano closed, ζ-nano α-closed, ζ-nano pre-closed, ζ-nano semi-closed, ζ-nano b-closed and ζ-nano β-closed*). The family of all ζ-nano regular closed (resp. ζ-nano closed, ζ-nano α-closed, ζ-nano pre-closed, ζ-nano semi-closed, ζ-nano b-closed and ζ-nano β-closed) sets denote by $NRC(U,X)_\zeta$ (resp. $\tau_R(X)_\zeta^C$, $\tau_R^\alpha(X)_\zeta^C$, $NPC(U,X)_\zeta$, $NSC(U,X)_\zeta$, $NbC(U,X)_\zeta$ and $N\beta C(U,X)_\zeta$).

Remark 3.2. From the following examples, it can be easily seen that the concept of ζ-nano regular open and nano-regular open are independent in general.

Example 3.3. Consider $U = \{r,m,n,o,p\}$ with $U/R = \{\{r,m\},\{n,p\},\{o\}\}$ and $X = \{r,o\}$. Then,

$$\tau_R(X) = \{U, \phi, \{o\}, \{r,m,o\}, \{r,m\}\}$$

and

$$NRO(U,X) = \{U, \phi, \{o\}, \{r,m\}\}.$$

Define a nano operation ζ on $N\beta O(U,X)$ by

(1)
$$K^\zeta = \begin{cases} U & \text{if } K = \{r,m\} \\ K & \text{otherwise.} \end{cases}$$
Then, $\{r,m\}$ is nano-regular open but not ζ-nano regular open.

(2) $K^\zeta = K$ for each $K \in N\beta O(U,X)$. Then, $\{r,m,o\}$ is ζ-nano regular open but not nano-regular open.

Theorem 3.4. *Let $(U, \tau_R(X))$ be nano topological space. Then,*

(1) Every ζ-nano open is nano open.
(2) Every ζ-nano α-open is nano α-open.
(3) Every ζ-nano pre-open is nano pre-open.
(4) Every ζ-nano semi-open is nano semi-open.
(5) Every ζ-nano b-open is nano b-open.
(6) Every ζ-nano β-open is nano β-open.

Proof. It is clear from the Definition 3.1. $\square$

Remark 3.5. The following examples show that all converses of Theorem 3.4, can not be reserved.

Example 3.6. Consider $U = \{r, m, n, o\}$ with $U/R = \{\{r\}, \{n\}, \{m, o\}\}$ and $X = \{r, m\}$. Then, $\tau_R(X) = \{U, \phi, \{r\}, \{r, m, o\}, \{m, o\}\}$. Define a nano operation ζ on $N\beta O(U, X)$ by

(1)

$$K^\zeta = \begin{cases} U & \text{if } K = \{m, o\} \\ K & \text{otherwise.} \end{cases}$$

Then, $\{m, o\}$ is nano open but not ζ-nano open.

(2)

$$K^\zeta = \begin{cases} U & \text{if } K = \{r, m, o\} \text{ or } K = \{m, o\} \\ K & \text{otherwise.} \end{cases}$$

Then, $\tau_R^\alpha(X) = \{U, \phi, \{r\}, \{m, o\}, \{r, m, o\}\}$ and $\{r, m, o\}$ is nano α-open but not ζ-nano α-open.

(3)

$$K^\zeta = \begin{cases} U & \text{if } K = \{m\} \\ K & \text{otherwise.} \end{cases}$$

Then, $NPO(U, X) = \{U, \phi, \{r\}, \{m\}, \{o\}, \{r, m\}, \{r, o\}, \{m, o\}, \{r, m, n\}, \{r, m, o\}, \{r, n, o\}\}$ and $\{m\}$ is nano pre-open but not ζ-nano pre-open.

(4)

$$K^\zeta = \begin{cases} U & \text{if } K = \{r, n\} \\ K & \text{otherwise.} \end{cases}$$

Then, $NSO(U, X) = \{U, \phi, \{r\}, \{r, n\}, \{m, o\}, \{r, m, o\}, \{m, n, o\}\}$ and $\{r, n\}$ is nano semi-open but not ζ-nano semi-open.

(5)

$$K^\zeta = \begin{cases} U & \text{if } K = \{o\} \\ K & \text{if } K \neq \{o\}. \end{cases}$$

Then, $NbO(U,X) = \{U, \phi, \{r\}, \{m\}, \{o\}, \{r,m\}, \{r,n\}, \{r,o\}, \{m,o\}, \{r,m,n\}, \{r,m,o\}, \{m,n,o\}, \{r,n,o\}\}$ and $\{o\}$ is nano b-open but not ζ-nano b-open.

(6)
$$K^{\zeta} = \begin{cases} U & \text{if } o \in K \\ K & \text{if } o \notin K. \end{cases}$$
Then, $N\beta O(U,X) = \{U, \phi, \{r\}, \{m\}, \{o\}, \{r,m\}, \{r,n\}, \{r,o\}, \{m,o\}, \{m,n\}, \{n,o\}, \{r,m,n\}, \{r,m,o\}, \{m,n,o\}, \{r,n,o\}\}$ and $\{n,o\}$ is nano β-open but not ζ-nano β-open.

Remark 3.7. Suppose the identity nano operation $id : N\beta O(U,X) \to P(U)$ is defined by $K^{id} = K$ for any $K \in N\beta O(U,X)$. Then, a subset L of U is

(1) an id-nano regular open if L is nano-regular open. Therefore, we have that $NRO(U,X) \subseteq NRO(U,X)_{id}$.
(2) an id-nano open if and only if L is nano open. Therefore, we have that $\tau_R(X)_{id} = \tau_R(X)$.
(3) an id-nano α-open if and only if L is nano α-open. Therefore, we have that $\tau_R^{\alpha}(X)_{id} = \tau_R^{\alpha}(X)$.
(4) an id-nano pre-open if and only if L is nano pre-open. Therefore, we have that $NPO(U,X)_{id} = NPO(U,X)$.
(5) an id-nano semi-open if and only if L is nano semi-open. Therefore, we have that $NSO(U,X)_{id} = NSO(U,X)$.
(6) an id-nano b-open if and only if L is nano b-open. Therefore, we have that $NbO(U,X)_{id} = NbO(U,X)$.
(7) an id-nano β-open if and only if L is nano β-open. Therefore, we have that $N\beta O(U,X)_{id} = N\beta O(U,X)$.

Theorem 3.8. *Let $(U, \tau_R(X))$ be nano topological space and $L \subseteq U$. Then,*

(1) If L is ζ-nano regular open, then L is ζ-nano open.
(2) If L is ζ-nano open, then L is ζ-nano α-open.
(3) If L is ζ-nano α-open, then L is ζ-nano pre-open.
(4) If L is ζ-nano α-open, then L is ζ-nano semi-open.
(5) If L is ζ-nano pre-open, then L is ζ-nano b-open.
(6) If L is ζ-nano semi-open, then L is ζ-nano b-open.
(7) If L is ζ-nano b-open, then L is ζ-nano β-open.

Proof. (1) Follows from Theorem 2.6.

(2) Follows from Theorem 2.7.
(3) Follows from Theorem 2.8.
(4) Follows from Theorem 2.9.
(5) Follows from Theorem 2.10.
(6) Follows from Theorem 2.11.
(7) Follows from Theorem 2.12.

□

Remark 3.9. The converse of the above theorem need not be true in general as it is shown below.

Example 3.10. (1) Consider $U = \{r, m, n\}$ with $U/R = \{\{r\}, \{m, n\}\}$ and $X = \{r\}$. Then,
$\tau_R(X) = \{U, \phi, \{r\}\}$,
$NRO(U, X) = \{U, \phi\}$, and
$\tau_R^{\alpha}(X) = \{U, \phi, \{r\}, \{r, m\}, \{r, n\}\}$.
Define a nano operation ζ on $N\beta O(U, X)$ by $K^{\zeta} = K$ for each $K \in N\beta O(U, X)$. Then, $\{r\}$ is ζ-nano open but not ζ-nano regular open and $\{r, m\}$ is ζ-nano α-open but not ζ-nano open.

(2) Consider $U = \{r, m, n, o\}$ with $U/R = \{\{r\}, \{n\}, \{m, o\}\}$ and $X = \{r, m\}$. Then, $\tau_R(X) = \{U, \phi, \{r\}, \{r, m, o\}, \{m, o\}\}$. Define a nano operation ζ on $N\beta O(U, X)$ by

- $$K^{\zeta} = \begin{cases} K & \text{if } K = \{o\} \\ U & \text{otherwise.} \end{cases}$$

  Then, $\{o\}$ is ζ-nano pre-open but not ζ-nano α-open.

- $$K^{\zeta} = \begin{cases} K & \text{if } K = \{r, n\} \\ U & \text{otherwise.} \end{cases}$$

  Then, $\{r, n\}$ is ζ-nano semi-open but not ζ-nano α-open.

- $$K^{\zeta} = \begin{cases} K & \text{if } K = \{m, n, o\} \text{ or } K = \{m\} \\ U & \text{otherwise.} \end{cases}$$

  Then, $\{m, n, o\}$ is ζ-nano b-open but not ζ-nano pre-open and $\{m\}$ is ζ-nano b-open but not ζ-nano semi-open.

- $$K^{\zeta} = \begin{cases} K & \text{if } K = \{m, n\} \\ U & \text{otherwise.} \end{cases}$$

  Then, $\{m, n\}$ is ζ-nano β-open but not ζ-nano b-open.

Proposition 3.11. *The concept of ζ-nano pre-open and ζ-nano semi-open are independent.*

Proof. Follows from that nano pre-open and nano semi-open are independent [5]. $\square$

Example 3.12. From Example 3.6, if we define a nano operation ζ on $N\beta O(U, X)$ by

(1)
$$K^{\zeta} = \begin{cases} K & \text{if } K = \{m, n, o\} \\ U & \text{if } K \neq \{m, n, o\}. \end{cases}$$

Then, $\{m, n, o\}$ is ζ-nano semi-open but not ζ-nano pre-open.

(2)
$$K^{\zeta} = \begin{cases} K & \text{if } K = \{m\} \\ U & \text{if } K \neq \{m\}. \end{cases}$$

Then, $\{m\}$ is ζ-nano pre-open but not ζ-nano semi-open.

Remark 3.13. From Theorems 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 3.4 and 3.8, we obtain the following diagram of implications:

$$\begin{tikzcd}
 & \zeta\text{-nano regular open} \arrow[d] & \text{nano-regular open} \arrow[d] & \\
 & \zeta\text{-nano open} \arrow[r] \arrow[d] & \text{nano open} \arrow[d] & \\
 & \zeta\text{-nano } \alpha\text{-open} \arrow[r] \arrow[dl] \arrow[dd] & \text{nano } \alpha\text{-open} \arrow[dr] \arrow[dd] & \\
\zeta\text{-nano semi-open} \arrow[rrr] \arrow[ddr] & & & \text{nano semi-open} \arrow[ddl] \\
 & \zeta\text{-nano pre-open} \arrow[r] \arrow[d] & \text{nano pre-open} \arrow[d] & \\
 & \zeta\text{-nano } b\text{-open} \arrow[r] \arrow[d] & \text{nano } b\text{-open} \arrow[d] & \\
 & \zeta\text{-nano } \beta\text{-open} \arrow[r] & \text{nano } \beta\text{-open} &
\end{tikzcd}$$

where $A \to B$ represents that A implies B.

Theorem 3.14. *Let $\{L_i\}_{i \in J}$ be a collection of ζ-nano open sets in a nano topological space $(U, \tau_R(X))$, then $\cup_{i \in J} L_i$ is ζ-nano open.*

Proof. Let $s \in \cup_{i \in J} L_i$, then $s \in L_i$ for some $i \in J$. Since L_i is a ζ-nano open set, implies that there exists a nano open set D containing s such that $D^\zeta \subseteq L_i \subseteq \cup_{i \in J} L_i$. Thus, $\cup_{i \in J} L_i$ is a ζ-nano open set of $(U, \tau_R(X))$. □

Theorem 3.15. *Let $\{L_i\}_{i \in J}$ be a collection of ζ-nano regular open (resp. ζ-nano α-open, ζ-nano pre-open, ζ-nano semi-open, ζ-nano b-open and ζ-nano β-open) sets in a nano topological space $(U, \tau_R(X))$, then $\cup_{i \in J} L_i$ is ζ-nano regular open (resp. ζ-nano α-open, ζ-nano pre-open, ζ-nano semi-open, ζ-nano b-open and ζ-nano β-open).*

Proof. The proof is similar to that of Theorem 3.14. □

Theorem 3.16. *If $(U, \tau_R(X))$ is a nano topological space and $X \subseteq U$, then*

(1) the intersection of two ζ-nano open sets is ζ-nano open.

(2) the intersection of two ζ-nano regular open sets is ζ-nano regular open.

Proof. Follows from that $L_R(X) \subseteq U_R(X)$, $B_R(X) \subseteq U_R(X)$ and $L_R(X) \cap B_R(X) = \phi$. □

Remark 3.17. From Theorems 3.14 and 3.16 (1), we notice that the family of all ζ-nano open is a nano topology.

Remark 3.18. From Theorems 3.15 and 3.16 (2), we notice that the family of all ζ-nano regular open is a nano topology.

Remark 3.19. If D and K are two ζ-nano α-open (resp. ζ-nano pre-open, ζ-nano semi-open, ζ-nano b-open and ζ-nano β-open) sets in $(U, \tau_R(X))$, then the following examples shows that $D \cap K$ need not be ζ-nano α-open (resp. ζ-nano pre-open, ζ-nano semi-open, ζ-nano b-open and ζ-nano β-open).

Example 3.20. Consider $U = \{r, m, n\}$ with $U/R = \{\{r\}, \{m, n\}\}$ and $X = \{r\}$. Then, $\tau_R(X) = \{U, \phi, \{r\}\}$. Define a nano operation ζ on $N\beta O(U, X)$ by

$$K^\zeta = \begin{cases} K & \text{if } K = \{r, m\} \text{ or } K = \{r, n\} \\ U & \text{otherwise.} \end{cases}$$

Then, $D = \{r, m\}$ and $K = \{r, n\}$ are ζ-nano α-open sets but $D \cap K = \{r\}$ is not ζ-nano α-open.

Example 3.21. Consider $U = \{r, a, b, o\}$ with $U/R = \{\{r\}, \{b\}, \{a, o\}\}$ and $X = \{r, a\}$. Then, $\tau_R(X) = \{U, \phi, \{r\}, \{r, a, o\}, \{a, o\}\}$. Define a nano operation ζ on $N\beta O(U, X)$ by

(1)
$$K^\zeta = \begin{cases} U & \text{if } K = \{a\} \\ K & \text{if } K \neq \{a\}. \end{cases}$$
Then, $D = \{r, a, b\}$ and $K = \{r, b, o\}$ are ζ-nano pre-open sets but $D \cap K = \{r, b\}$ is not ζ-nano pre-open.

(2)
$$K^\zeta = \begin{cases} U & \text{if } K = \{r\} \\ K & \text{if } K \neq \{r\}. \end{cases}$$
Then, $D = \{r, b\}$ and $K = \{a, b, o\}$ are ζ-nano semi-open sets but $D \cap K = \{b\}$ is not ζ-nano semi-open.

(3)
$$K^\zeta = \begin{cases} U & \text{if } K = \{a\} \\ K & \text{if } K \neq \{a\}. \end{cases}$$
Then, $D = \{r, a\}$ and $K = \{a, o\}$ are ζ-nano b-open sets but $D \cap K = \{a\}$ is not ζ-nano b-open.

(4)
$$K^\zeta = \begin{cases} K & \text{if } K = \{r, b\} \text{ or } K = \{a, b\} \\ U & \text{otherwise.} \end{cases}$$
Then, $D = \{r, b\}$ and $K = \{a, b\}$ are ζ-nano β-open sets but $D \cap K = \{b\}$ is not ζ-nano β-open.

Remark 3.22. From Theorem 3.15 and Remark 3.19, we notice that the family of all ζ-nano α-open (resp. ζ-nano pre-open, ζ-nano semi-open, ζ-nano b-open and ζ-nano β-open) is a nano supratopology need not be a nano topology in general.

Definition 3.23. An operation ζ on $N\beta O(U, X)$ is called *ζ-regular* if for every two nano α-open (resp. nano pre-open, nano semi-open, nano b-open and nano β-open) L and D subsets of U containing $s \in U$, there exists a nano α-open (resp. nano pre-open, nano semi-open, nano b-open and nano β-open) set K such that $s \in K^\zeta \subseteq L^\zeta \cap D^\zeta$.

Theorem 3.24. *Let ζ be a ζ-regular operation on $N\beta O(U, X)$. If L and D are two ζ-nano α-open sets in U, then $L \cap D$ is also a ζ-nano α-open set.*

Proof. Let $s \in L \cap D$, then $s \in L$ and $s \in D$. Since L and D are ζ-nano α-open sets, then there exist nano α-open sets V_1 and V such that $s \in V_1 \subseteq V_1^{\zeta} \subseteq L$ and $s \in V \subseteq V^{\zeta} \subseteq D$. Since ζ is ζ-regular, then there exists a nano α-open set K such that $s \in K \subseteq K^{\zeta} \subseteq V_1^{\zeta} \cap V^{\zeta} \subseteq L \cap D$. This implies that $L \cap D$ is ζ-nano α-open set. □

Remark 3.25. By the above theorem, if ζ is a ζ-regular operation on $N\beta O(U, X)$. Then, $\tau_R^{\alpha}(X)_{\zeta}$ form a nano topology on U.

Proposition 3.26. *Let ζ be a ζ-regular operation on $N\beta O(U, X)$.*

(1) If L and D are ζ-nano pre-open sets in U, then $L \cap D$ is also a ζ-nano pre-open set.

(2) If L and D are ζ-nano semi-open sets in U, then $L \cap D$ is also a ζ-nano semi-open set.

(3) If L and D are ζ-nano b-open sets in U, then $L \cap D$ is also a ζ-nano b-open set.

(4) If L and D are ζ-nano β-open sets in U, then $L \cap D$ is also a ζ-nano β-open set.

Proof. The proof is similar to that of Theorem 3.24. □

Remark 3.27. By the above proposition, if ζ is a ζ-regular operation on $N\beta O(U, X)$. Then, $NPO(U, X)_{\zeta}$ (resp. $NSO(U, X)_{\zeta}$, $NbO(U, X)_{\zeta}$ and $N\beta O(U, X)_{\zeta}$) form a nano topology on U.

Theorem 3.28. *Let $(U, \tau_R(X))$ be nano topological space and $L \subseteq U$. Then, the set L is ζ-nano open if and only if for each $s \in L$, there exists a ζ-nano open set D such that $s \in D \subseteq L$.*

Proof. Let L be ζ-nano open. Then, for each $s \in L$, put $K = L$ is a ζ-nano open set such that $s \in K \subseteq L$.
Conversely, suppose that for each $s \in L$, there exists a ζ-nano open set K such that $s \in K \subseteq L$, thus $L = \cup K_s$ where $K_s \in \tau_R(X)_{\zeta}$ for each s. Hence, L is a ζ-nano open set. □

Proposition 3.29. *Let $(U, \tau_R(X))$ be nano topological space and $L \subseteq U$. Then, the set L is*

(1) ζ-nano regular open if and only if for each $s \in L$, there exists a ζ-nano regular open set D such that $s \in D \subseteq L$.

(2) *ζ-nano α-open if and only if for each $s \in L$, there exists a ζ-nano α-open set D such that $s \in D \subseteq L$.*
(3) *ζ-nano pre-open if and only if for each $s \in L$, there exists a ζ-nano pre-open set D such that $s \in D \subseteq L$.*
(4) *ζ-nano semi-open if and only if for each $s \in L$, there exists a ζ-nano semi-open set D such that $s \in D \subseteq L$.*
(5) *ζ-nano b-open if and only if for each $s \in L$, there exists a ζ-nano b-open set D such that $s \in D \subseteq L$.*
(6) *ζ-nano β-open if and only if for each $s \in L$, there exists a ζ-nano β-open set D such that $s \in D \subseteq L$.*

Proof. The proof is similar to that of Theorem 3.28. □

Definition 3.30. A nano topological space $(U, \tau_R(X))$ is called *ζ-regular* if for each nano-regular open (resp. nano open, nano α-open, nano pre-open, nano semi-open, nano b-open and nano β-open) set L in U containing $s \in U$, there exist a nano-regular open (resp. nano open, nano α-open, nano pre-open, nano semi-open, nano b-open and nano β-open) set D in U containing s such that $D^\gamma \subseteq L$.

Theorem 3.31. *Let $(U, \tau_R(X))$ be a nano topological space. Then, the following statements are equivalent.*

(1) $\tau_R(X)_\zeta = \tau_R(X)$.
(2) $(U, \tau_R(X))$ is ζ-regular.
(3) For every $s \in U$ and every nano open set L of U containing s there exists a ζ-nano open set D of U such that $s \in D$ and $D \subseteq L$.

Proof. (1) $\Rightarrow$ (2): Let $s \in U$ and L be a nano open set containing s. Then by assumption, L is a ζ- nano open set, this implies that for each $s \in L$, there exists a nano open set D containing s such that $D^\zeta \subseteq L$. Thus, $(U, \tau_R(X))$ is ζ-regular.
(2) $\Rightarrow$ (3): Let $s \in U$ and L be any nano open set containing s. Then by (2), there is a nano open set D such that $s \in D \subseteq D^\gamma \subseteq L$. Applying (2) to D, then D is ζ-nano open. Hence, D is a ζ-nano open set containing s such that $D \subseteq L$.
(3) $\Rightarrow$ (1): By (3) and Theorem 3.28, it follows that every nano open set is ζ-nano open, that is, $\tau_R(X) \subseteq \tau_R(X)_\zeta$. Also from Theorem 3.4 (1), $\tau_R(X)_\zeta \subseteq \tau_R(X)$. Hence, we have the result. □

Corollary 3.32. *Let $(U, \tau_R(X))$ be a nano topological space. Then, the following statements are equivalent.*

(1) $NRO(U,X) \subseteq NRO(U,X)_\zeta$.

(2) $(U, \tau_R(X))$ *is* ζ*-regular.*

(3) For every $s \in U$ *and every nano-regular open set* L *of* U *containing* s *there exists a* ζ*-nano regular open set* D *of* U *such that* $s \in D \subseteq L$.

Proof. (1) $\Rightarrow$ (2): Let $s \in U$ and L be a nano-regular open set containing s. Then, L is a ζ- nano regular open set, this implies that for each $s \in L$, there exists a nano-regular open set D containing s such that $D^\zeta \subseteq L$. Thus, $(U, \tau_R(X))$ is ζ-regular.
(2) $\Rightarrow$ (3): Let $s \in U$ and L be any nano-regular open set containing s. Then by (2), there is a nano-regular open set D such that $s \in D \subseteq D^\gamma \subseteq L$. Applying (2) to D, then D is ζ-nano regular open. Hence, D is a ζ-nano regular open set containing s such that $D \subseteq L$.
(3) $\Rightarrow$ (1): By (3) and Proposition 3.29, it follows that every nano-regular open set is ζ-nano regular open, that is, $NRO(U,X) \subseteq NRO(U,X)_\zeta$. □

The proof of the following result is easy and hence it is omitted.

Proposition 3.33. *Let* $(U, \tau_R(X))$ *be a nano topological space. Then, the following statements are equivalent.*

(1) $\tau_R^\alpha(X)_\zeta = \tau_R^\alpha(X)$ *(resp.* $NPO(U,X)_\zeta = NPO(U,X)$, $NSO(U,X)_\zeta = NSO(U,X)$, $NbO(U,X)_\zeta = NbO(U,X)$ *and* $N\beta O(U,X)_\zeta = N\beta O(U,X)$*).*

(2) $(U, \tau_R(X))$ *is* ζ*-regular.*

(3) For every $s \in U$ *and every nano* α*-open (resp. nano pre-open, nano semi-open, nano b-open and nano* β*-open) set* L *of* U *containing* s *there exists a* ζ*-nano* α*-open (resp.* ζ*-nano pre-open,* ζ*-nano semi-open,* ζ*-nano b-open and* ζ*-nano* β*-open) set* D *of* U *such that* $s \in D$ *and* $D \subseteq L$.

4. Real Life Application of Nano Topology

In this section an algorithm is developed to find the deciding factors or core to pick the minimum number of attributes necessary for the classification of objects.

Definition 4.1. Let U be the universe and R be an equivalence relation on U. Let $\tau_R(X)$ be the nano topology on U and $K^\zeta = K$ for each $K \in N\beta O(U,X)$. A subset C of A, the set of attributes is called the *Core of* R if $\tau_R(X)_\zeta \neq \tau_{R-(r)}(X)$ for every r in C. That is, a core of R is a subset of attributes which is such that none of its elements can be removed without affecting the classification power of attributes.

The procedure applied in the Example 4.2, can be put in the form of an algorithm as follows:

Step 1: Given a finite universe U, a finite set A of attributes that is divided into two classes, A_1 of condition attributes and A_2 of decision attribute, an equivalence relation R on U corresponding to A_1 and a subset X of U, represent the data as an information table, columns of which are labeled by attributes, rows by objects and entries of the table are attribute values.

Step 2: Find the lower approximation, upper approximation and the boundary region of X with respect to R.

Step 3: Generate the nano topology $\tau_R(X)$ on U and also $\tau_R(X)_\zeta$.

Step 4: Remove an attribute x from A_1 and find the lower and upper approximations and the boundary region of X with respect to the equivalence relation on $A_1 - \{x\}$.

Step 5: Generate the nano topology $\tau_{R-(x)}(X)$ on U.

Step 6: Repeat steps 3 and 4 for all attributes in A_1.

Step 7: Those attributes in A_1 for which $\tau_{R-(x)}(X) \neq \tau_R(X)_\zeta$ form the $Core(R)$.

Example 4.2. In this example nano topology concepts is applied to find the key factors for the fever. Fever is an important sign of inflammation recognized by health care practitioners and family caregivers. Fever often occurs in response to infection, inflammation and trauma. However, this view of fever is merely an oversimplification as a growing body of evidence now suggests that fever represents a complex adaptive response of the host to various immune challenges whether infectious or non-infectious. Although elevated body temperature is an indispensable component of the febrile response, it is not synonymous with fever.

The following table gives information about patients those who are having Headache (H), Weakness (W), Sweating (S), Temperature (T) and Irritability (I).

Patients	**H**	**W**	**S**	**T**	**I**	**Fever**
u_1	No	Yes	No	Low	No	Yes
u_2	Yes	Yes	Yes	Low	Yes	Yes
u_3	No	Yes	No	High	Yes	Yes
u_4	Yes	No	Yes	Low	Yes	No
u_5	No	Yes	No	Low	No	No
u_6	No	Yes	No	Normal	Yes	No
u_7	No	Yes	No	High	Yes	Yes
u_8	No	No	No	High	No	No

The columns of the table represent the attributes (the symptoms for fever) and the rows represent the objects (the patients). Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$, then

Case I: Let $X = \{u_1, u_2, u_3, u_7\}$ be the set of patient having fever and R be the equivalence relation on U with respect to the set of all condition attributes. The set of equivalence classes corresponding to R is given by
$U/R = \{\{u_8\}, \{u_3, u_7\}, \{u_1, u_5\}, \{u_2\}, \{u_4\}, \{u_6\}\}$, therefore the nano topology on U with respect to X is given by $\tau_R(X) = \{U, \phi, \{u_2, u_3, u_7\}, \{u_2, u_3, u_7, u_1, u_5\}, \{u_1, u_5\}\}$. Define a nano operation ζ on $N\beta O(U, X)$ by $K^\zeta = K$ for each $K \in N\beta O(U, X)$, then $\tau_R(X)_\zeta = \tau_R(X)$.

If Headache is removed from the set of condition attributes, then

$U/R - (H) = \{\{u_2\}, \{u_1, u_5\}, \{u_3, u_7\}, \{u_4\}, \{u_6\}, \{u_8\}\}$. Hence, $\tau_{R-(H)}(X) = \{U, \phi, \{u_2, u_3, u_7\}, \{u_2, u_3, u_7, u_1, u_5\}, \{u_1, u_5\}\} = \tau_R(X)$.

If Weakness is removed from the set of condition attributes, then

$U/R - (W) = \{\{u_1, u_5\}, \{u_2, u_4\}, \{u_3, u_7\}, \{u_6\}, \{u_8\}\}$. Hence, $\tau_{R-(W)}(X) = \{U, \phi, \{u_3, u_7\}, \{u_1, u_2, u_3, u_4, u_5, u_7\}, \{u_1, u_2, u_4, u_5\}\} \neq \tau_R(X)$.

If Sweating is removed from the set of condition attributes, then

$U/R - (S) = U/R$ and hence $\tau_{R-(S)}(X) = \tau_R(X)$.

If Temperature is removed from the set of condition attributes, then

$U/R - (T) = \{\{u_1, u_5\}, \{u_2\}, \{u_3, , u_6, u_7\}, \{u_4\}, \{u_8\}\}$. Hence, $\tau_{R-(T)}(X) = \{U, \phi, \{u_2\}, \{u_1, u_2, u_3, u_5, u_6, u_7\}, \{u_1, u_3, u_5, u_6, u_7\}\} \neq \tau_R(X)$.

If Irritability is removed from the set of condition attributes, then

$U/R - (I) = U/R$ and hence $\tau_{R-(I)}(X) = \tau_R(X)$.

From Case I, we get $Core(R) = \{W, T\}$.

Case II: Let $X = \{u_4, u_5, u_6, u_8\}$ be the set of patient not having fever. Then, $U/R = \{\{u_8\}, \{u_3, u_7\}, \{u_1, u_5\}, \{u_2\}, \{u_4\}, \{u_6\}\}$, therefore the nano topology on U with respect to X is given by $\tau_R(X) = \{U, \phi, \{u_4, u_6, u_8\}, \{u_1, u_4, u_5, u_6, u_8\}, \{u_1, u_5\}\}$. Define a nano operation ζ on $N\beta O(U, X)$ by $K^\zeta = K$ for each $K \in N\beta O(U, X)$, then $\tau_R(X)_\zeta = \tau_R(X)$.

If the attribute Headache is removed, then

$U/R - (H) = \{\{u_2\}, \{u_1, u_5\}, \{u_3, u_7\}, \{u_4\}, \{u_6\}, \{u_8\}\}$. Thus, $\tau_{R-(H)}(X) = \{U, \phi, \{u_4, u_6, u_8\}, \{u_1, u_4, u_5, u_6, u_8\}, \{u_1, u_5\}\} = \tau_R(X)$.

If the attribute Weakness is removed, then

$U/R - (W) = \{\{u_1, u_5\}, \{u_2, u_4\}, \{u_3, u_7\}, \{u_6\}, \{u_8\}\}$. Thus, $\tau_{R-(W)}(X) = \{U, \phi, \{u_6, u_8\}, \{u_1, u_2, u_4, u_5, u_6, u_8\}, \{u_1, u_2, u_4, u_5\}\} \neq \tau_R(X)$.

If the attribute Sweating is removed, then

$U/R - (S) = U/R$ and thus $\tau_{R-(S)}(X) = \tau_R(X)$.

If the attribute Temperature is removed, then

$U/R - (T) = \{\{u_1, u_5\}, \{u_2\}, \{u_3, , u_6, u_7\}, \{u_4\}, \{u_8\}\}$.

Thus, $\tau_{R-(T)}(X) = \{U, \phi, \{u_4, u_8\}, \{u_1, u_3, u_4, u_5, u_6, u_7, u_8\}, \{u_1, u_3, u_5, u_6, u_7\}\} \neq \tau_R(X)$.

If the attribute Irritability is removed, then

$U/R - (I) = U/R$ and hence $\tau_{R-(I)}(X) = \tau_R(X)$.

From Case II, we get $Core(R) = \{W, T\}$.

Observation. From the Core of R, we conclude that "Weakness" and "Temperature" are the key attributes necessary to say that a patient has fever.

5. Conclusion

In this work, the properties of some new types of nano open sets via nano operations are discussed. Also, we introduced an application example in nano topology. Thus, it is advantageous to use nano topology in real life situations.

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Department of Mathematics, Faculty of Education, University of Zakho, Kurdistan-Region, Iraq
Email address: `hariwan_math@yahoo.com`