REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM WITH SPECIAL STRUCTURE TENSOR FIELD

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Abstract. Let $M$ be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. In this paper, we prove that if $(\nabla_X \phi)Y + (\nabla_Y \phi)X = 0$ holds on $M$, then $M$ is a Hopf hypersurface, where $\phi$ is the tangential projection of the complex structure of $M_n(c)$. We characterize such Hopf hypersurfaces of $M_n(c)$.

1. Introduction

A complex $n$-dimensional Kaehlerian manifold of constant holomorphic sectional curvature $c$ is called a complex space form, which is denoted by $M_n(c)$. It is well-known that a complete and simply connected complex space form is complex analytically isometric to a complex projective space $\mathbb{P}^n\mathbb{C}$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n\mathbb{C}$, according to $c > 0$, $c = 0$ or $c < 0$.

In this paper, we consider a real hypersurface $M$ in a complex space form $M_n(c)$, $c \neq 0$. Then $M$ has an almost contact metric structure $(\phi, g, \xi, \eta)$ induced from the Kaehler metric and complex structure $J$ on $M_n(c)$. The Reeb vector field $\xi$ is said to be principal if $A\xi = \alpha\xi$ is satisfied, where $A$ is the shape operator of $M$ and $\alpha = \eta(A\xi)$. In this case, it is known that $\alpha$ is locally constant [2] and that $M$ is called a Hopf hypersurface.

Typical examples of Hopf hypersurfaces in $\mathbb{P}^n\mathbb{C}$ are homogeneous ones, and these real hypersurfaces are given as orbits under the subgroup of the projective unitary group $PU(n + 1)$. Takagi [9] completely classified all such hypersurfaces into six model spaces: $A_1$, $A_2$, $B$, $C$, $D$ and $E$. 

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On the other hand, real hypersurfaces in $H_n\mathbb{C}$ have been investigated by Berndt [1], Montiel and Romero [6] and so on. Berndt [1] categorized all homogeneous Hopf hypersurfaces in $H_n\mathbb{C}$ as four model spaces which are said to be $A_0$, $A_1$, $A_2$ and $B$.

If a real hypersurface $M$ is of $A_1$ or $A_2$ in $P_n\mathbb{C}$ or of $A_0$, $A_1$, $A_2$ in $H_n\mathbb{C}$, then $M$ is said to be type $A$ for simplicity.

The following theorem is a typical characterization of real hypersurfaces of type $A$ due to Okumura [8] for $c > 0$ and Montiel and Romero [6] for $c < 0$.

**Theorem A.** Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $A\phi - \phi A = 0$ on $M$ if and only if $M$ is locally congruent to one of the model spaces of type $A$.

We define a structure tensor field $(\nabla_X^\phi)Y$ on $M$ in $M_n(c)$ by

$$(\nabla_X^\phi)Y = \nabla_X(\phi Y) - \phi \nabla_X Y = \eta(Y)AX - g(AX,Y)\xi,$$

by using the tangential projection and parallelism of $J$.

Many geometricians have studied real hypersurfaces with the conditions of the structure tensor field and obtain some results on the classification of real hypersurfaces in complex space form $M_n(c)$ [4, 5, 6, 8, etc].

For the Codazzi type of structure tensor field, Lim and Kim [3] have proved the following theorem;

**Theorem B.** There exists no real hypersurface of $M_n(c)$, $c \neq 0$, whose structure tensor field is Codazzi type.

In this paper, we shall study a real hypersurface in a nonflat complex space form $M_n(c)$, with special conditions of structure tensor field, and give some characterizations of such a real hypersurface in $M_n(c)$.

All manifolds are assumed to be connected and of class $C^\infty$ and real hypersurfaces supposed to be orientable.

2. Preliminaries

Let $M$ be a real hypersurface immersed in a complex space form $M_n(c)$, and $N$ be a unit normal vector field of $M$. By $\tilde{\nabla}$, we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\tilde{g}$ of $M_n(c)$. Then the Gauss and
Weingarten formulas are given by
\[ \tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX \]
respectively, where \( X \) and \( Y \) are any vector fields tangent to \( M \), \( g \) denotes the Riemannian metric tensor of \( M \) induced from \( \tilde{g} \), and \( A \) is the shape operator of \( M \) in \( M_n(c) \). For any vector field \( X \) on \( M \), we put
\[ JX = \phi X + \eta(X)N, \quad JN = -\xi, \]
where \( J \) is the almost complex structure of \( M_n(c) \). And we see that \( M \) induces an almost contact metric structure \( (\phi, g, \eta) \), that is,
\[ \phi^2 X = -X + \eta(X)\xi, \quad \phi \xi = 0, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \]
for any vector fields \( X \) and \( Y \) on \( M \). Since the almost complex structure \( J \) is parallel, we can verify the followings from the Gauss and Weingarten formulas:
\[ \nabla_X \xi = \phi AX, \]
\[ (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \]

Let \( X, Y \) and \( Z \) be vector fields on \( M \) and \( R \) denote the Riemannian curvature tensor of \( M \). As the ambient space has holomorphic sectional curvature \( c \), the equations of Gauss and Codazzi are given by
\[ R(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \]
\[ \quad - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \]
\[ (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \]
respectively.

Let \( \Omega \) be an open subset of \( M \) defined by
\[ \Omega = \{ p \in M \mid A\xi - \alpha\xi \neq 0 \} \]
where \( \alpha = \eta(A\xi) \). We put
\[ A\xi = \alpha\xi + \mu W, \]
where \( W \) be a unit vector field orthogonal to \( \xi \) and \( \mu \) does not vanish on \( \Omega \).
3. Some Lemmas

In this section, we assume that $\Omega$ is not empty and recall some well known results in [7] which will be used to prove our results.

**Lemma 3.1.** Let $M$ be a Hopf hypersurface in a nonflat complex space form $M_n(c)$. If $X$ is a unit vector field such that $AX = \lambda X$, Then

$$\lambda - \frac{\alpha}{2} A\phi X = \frac{1}{2}(\alpha \lambda + \frac{c}{2}) \phi X.$$  

**Lemma 3.2.** The B type hypersurface in $H_n C$ has three distinct principal curvatures, $\frac{1}{r} \coth u$, $\frac{1}{r} \tanh u$ of multiplicity $n - 1$ and $\alpha = \frac{2}{r} \tanh 2u$ of multiplicity 1. On the other hand, in $P_n C$, the type B hypersurface also has three distinct principal curvatures, $-\frac{1}{r} \tan u$ of multiplicity $2p$, $\frac{1}{r} \cot u$ of multiplicity $2q$ and $\alpha = \frac{2}{r} \cot 2u$ of multiplicity 1, where $p > 0$, $q > 0$, and $p + q = n - 1$.

**Lemma 3.3.** Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$ and $\xi$ be a principal curvature vector with corresponding principal curvature $\alpha$. If $X$ and $\phi X$ are principal vector fields with principal curvatures $\{\lambda, \mu\} = \{0, -\frac{c}{2\alpha}\}$, then $M$ does not exist in $M_n(c)$.

Using Lemmas above, we get the following important tool of this paper;

**Lemma 3.4.** Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$. If $M$ satisfies $(\nabla_X \phi)Y + (\nabla_Y \phi)X = 0$, then $M$ is a Hopf hypersurface in $M_n(c)$.

**Proof.** We assume that $(\nabla_X \phi)Y + (\nabla_Y \phi)X = 0$ for any vector fields $X$ and $Y$. By using (2.1) and symmetric properties of the shape operator, we have

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = \eta(Y) AX - g(AX, Y) \xi + \eta(X) AY - g(AY, X) \xi$$

$$= \eta(Y) AX + \eta(X) AY - 2g(AX, Y) \xi.$$  

From the our assumption and the above equation, it follows that

$$\eta(Y) AX + \eta(X) AY = 2g(AX, Y) \xi.$$  

If we put $Y = \xi$ into (3.2) and make use of (2.2), then we have

$$AX = \{\alpha \eta(X) + 2\mu g(X, W)\} \xi - \mu \eta(X) W.$$  

If we substitute $X = W$ into (3.3), then we obtain

$$AW = 2\mu \xi.$$
Taking inner product of (3.4) with $\xi$ and using (2.2), we have $\mu = 0$ on $\Omega$, then it is a contradiction. Thus the set $\Omega$ is empty and hence $M$ is a Hopf hypersurface. \qed

4. Non-existence of Real Hypersurfaces

In this section, we shall study a real hypersurface $M$ in $M_n(c)$ which satisfies $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0$.

**Theorem 4.1.** Let $M$ be a real hypersurface in $M_n(c)$, $c \neq 0$. If $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0$, then we obtain $\lambda = 0$ and $\mu = -\frac{c}{2\alpha}$.

**Proof.** By Lemma 3.4, the real hypersurface $M$ satisfying $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0$ is a Hopf hypersurface in $M_n(c)$, that is, $A\xi = \alpha\xi$. Since $\xi$ is a Reeb vector field, the assumption $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0$ is given by

$$\eta(Y)AX + \eta(X)AY = 2g(AX, Y)\xi. \quad (4.1)$$

To find the principal curvatures, we can divide equation (4.1) into three cases.

In the first case, if we put $Y = \xi$ into (4.1), then we have

$$AX + \eta(X)A\xi = 2\eta(AX)\xi. \quad (4.2)$$

For any vector field $X \perp \xi$ on $M$ such that $AX = \lambda X$, the principal value $\lambda = 0$ follows from (4.2). From the equation (3.1), we obtain

$$-\frac{\alpha}{2}A\phi X = \frac{c}{4}\phi X. \quad (4.3)$$

If $\alpha = 0$, then $c = 0$, and there is no real hypersurface. Now, we assume that $\alpha$ is not zero. Then it follows from (4.3) that $\phi X$ is a principal direction, say $A\phi X = -\frac{c}{2\alpha}\phi X$. Therefore, we see that the principal curvatures are constant $\alpha, \lambda = 0$ and $\mu = -\frac{c}{2\alpha}$.

In the second case, if we substitute $X = \xi$ into (4.1), then we obtain the principal curvature $\{\lambda, \mu\} = \{0, -\frac{c}{2\alpha}\}$ in a similar way to the first case.

In the last case, if any vector field $X = Y$ is orthogonal to $\xi$ on $M$ and $AX = \lambda X$, then we get the principal value $\{\lambda, \mu\} = \{0, -\frac{c}{2\alpha}\}$ by using (4.2) and (4.3).

From the above three cases, we conclude that the principal curvatures are $\lambda = 0$ and $\mu = -\frac{c}{2\alpha}$.

**Theorem 4.2.** Let $M$ be a real hypersurface in $M_n(c)$, $c \neq 0$. If $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0$, then $M$ does not exist in $M_n(c)$.\qed
Proof. By Lemma 3.4 and Theorem 4.1, we know that the real hypersurface $M$ is a Hopf hypersurface and the principal curvatures have values of 0 and $-\frac{c}{2a}$. By Lemma 3.3, $M$ does not exist in $M_n(c)$ and the proof is completed. □

REFERENCES


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