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# THE LINE ELEMENT APPROACH FOR THE GEOMETRY OF POINCARÉ DISK

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**Abstract.** The geometry of Poincaré disk itself is interpreted without any mapping to different spaces. Our approach might be one of the shortest and is intended for educational contribution.

## 1. Introduction

For vectors $v_p = (v_1, v_2)$ and $w_p = (w_1, w_2)$ in 2-dimensional Euclidean space $\mathbb{R}^2$, the norm of a vector $|v_p| = \sqrt{v_p \cdot v_p} = \sqrt{v_1^2 + v_2^2}$ which is the Pythagorean theorem is defined by the dot product

$$v_p \cdot w_p = v_1 w_1 + v_2 w_2$$

and the angle $\theta$ formed by two vector $v, w \in \mathbb{R}^2$ is given by $\cos\theta = \frac{v \cdot w}{|v||w|}$. The arc length of a differentiable curve $\alpha(t)$ in $\mathbb{R}^2$ from $\alpha(0)$ to $\alpha(1)$ is given by

$$\int_0^1 |\alpha'(t)|\, dt = \int_0^1 \sqrt{\alpha'(t) \cdot \alpha'(t)}\, dt$$

and the arc length of a piecewise differentiable curve is the sum of the arc length of differentiable parts. The distance from $\alpha(0)$ to $\alpha(1)$ is defined by the shortest arc length among all curves. We can easily show that the straight line from $\alpha(0)$ to $\alpha(1)$ is the shortest arc length when the dot product is given on $\mathbb{R}^2$. Thus by considering an inner product $g(v, w)$ on a vector space $V \subset \mathbb{R}^2$ and defining the arc length of a curve $\alpha(t)$ by

$$\int_0^1 |\alpha'(t)|\, dt = \int_0^1 \sqrt{g(\alpha'(t), \alpha'(t))}\, dt,$$

we can have a distance different from Euclidean geometry. A geometry where four Euclidean postulates except for the Parallel one hold is known as absolute geometry ([7]). A non-Euclidean geometry with an inner product $g$ on the Poincaré disk $D_P = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\}$ satisfying the following three observations is going to be determined.

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• First, if $g(v_p, w_p) = f(p)(v_p \cdot w_p)$, then the angle defined by an inner product is equal to the angle defined by the dot product

$$\begin{aligned}\cos\theta &= \frac{g(v_p, w_p)}{\sqrt{g(v_p, v_p)}\sqrt{g(w_p, w_p)}} \\ &= \frac{f(p)(v_p \cdot w_p)}{\sqrt{f(p)(v_p \cdot v_p)}\sqrt{f(p)(w_p \cdot w_p)}} \\ &= \frac{v_p \cdot w_p}{|v_p||w_p|} \\ &= \cos\theta\,.\end{aligned}$$

• Second, the geometry of the Poincaré disk $D_P$ is assumed to be rotationally symmetric, that is, the geometry of a neighborhood at $p$ is isometric to that of a neighborhood at any point $q$ related to $p$ by rotation of any angle. It means that a function $f(r, \theta) = f(x, y)$ in $g(v_p, w_p) = f(p)(v_p \cdot w_p)$ depends only on $r$ for the polar coordinates $p = (x, y) = (r\cos\theta, r\sin\theta) \in D_P$.

• Third, the Euclidean norm of a vector $v_p = (v_1, v_2)$ at $p = (p_1, p_2) \in D_P$ with $(p_1 + v_1, p_2 + v_2) \in D_P$ must be scaled to infinity as $p$ goes to the boundary of $D_P$, since the boundary is considered to be a circle of radius $\infty$.

Under these three assumptions, one of the simplest candidates for an inner product $g$ on the Poincaré disk $D_P$ could be

$$g(v_p, w_p) = \frac{2(v_p \cdot w_p)}{1 - (x^2 + y^2)}$$

for all points $p = (x, y) \in D_P$ and scaling constant 2. The line element $ds^2$ of the Poincaré disk is

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - (x^2 + y^2))^2}.$$

Let $\alpha(t) = (x(t), y(t))$ be a differentiable curve from $\alpha(0)$ to $\alpha(1)$ in $D_P$. The arc length of $\alpha(t)$ from $\alpha(0)$ to $\alpha(1)$ is

$$\int_0^1 \sqrt{g(\alpha'(t), \alpha'(t))}\,dt = \int_0^1 \frac{2|\alpha'(t)|}{1 - |\alpha(t)|^2}\,dt,$$

where $|\alpha'(t)| = \sqrt{\alpha'(t) \cdot \alpha'(t)}$ and $|\alpha(t)|^2 = x(t)^2 + y(t)^2$. The distance $d(\alpha(0), \alpha(1))$ is the shortest arc length among all curve from $\alpha(0)$ to $\alpha(1)$

$$d(\alpha(0), \alpha(1)) = \inf_\alpha \int_0^1 \frac{2|\alpha'(t)|}{1 - |\alpha(t)|^2}\,dt\ .$$

We are going to find a shortest path between any two points in the Poincaré disk by using a distance-preserving biholomorphic mapping on the Poincaré disk or a linear fractional transformation which preserves the cross ratio and the distance. We also show that the Poincaré Disk is isometric to one connected component of two-sheeted hyperboloid $-x^2 + y^2 + z^2 = -1$ in 3-dimensional Minkowski space-time and the sum of the interior angles of a triangle, a Saccheri quadrilateral on the Poincaré disk is less than $\pi$, $2\pi$, respectively.

There are plenty lecture notes, papers([1],[6], [8]) and books ([2] [4],[5], [7]) on the hyperbolic geometry. The picture of the hyperbolic geometry is well-known. Here we suggest intuitive and direct approaches for the effective understanding of the hyperbolic geometry. In the last section, we can find the shortest arc length in the Poincaré disk numerically by using Python language.

## 2. A distance-preserving biholomorphic mapping on the Poincaré disk

A function $f : D_P \longrightarrow D_P \subset \mathbb{C}$ is said to be holomorphic if $f(x, y) = u(x, y) + iv(x, y)$ satisfies the Cauchy Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

A biholomorphic function is a holomorphic function $f$ which is bijective and whose inverse $f^{-1}$ is also holomorphic. Let $\alpha(t) = x(t)+iy(t)$ be a differentiable curve from $\alpha(0) = z_1$ to $\alpha(1) = z_2$ in $D_P$. For $f(z_1) = (f \circ \alpha)(0)$ and $f(z_2) = (f \circ \alpha)(1)$, the distance $d(f(z_1), f(z_2))$ is

$$d(f(z_1), f(z_2)) = \inf_{f(\alpha)} \int_0^1 \frac{2|(f \circ \alpha)'(t)|}{1 - |(f \circ \alpha)(t)|^2}\, dt\ .$$

We show that if $f$ is a holomorphic function on $D_P$, then $d(f(z_1), f(z_2)) \leq d(z_1, z_2)$.

**Schwarz lemma** For a holomorphic function $f : D_P \longrightarrow \mathbb{C}$ such that $f(0) = 0$ and $f(z) \leq 1$ on $D_P$, it satisfies that $\big|f(z)\big| \leq |z|$ for all $z \in D_P$ and $\big|f'(0)\big| \leq 1$.

**Schwarz-Pick theorem** For a holomorphic function $f : D_P \longrightarrow D_P$, it holds that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}\ .$$

*Proof.* Define $g, h : D_P \longrightarrow D_P$

$$g(z) = \frac{z_1 - z}{1 - \overline{z_1} z}, \quad h(z) = \frac{f(z_1) - z}{1 - \overline{f(z_1)} z} \quad z_1 \in D_P.$$

Since $g^{-1}(0) = z_1$, we have $h\Big(f(g^{-1}(0))\Big) = 0$. Using the Schwarz lemma, we get

$$\Big|h\Big(f(g^{-1}(z))\Big)\Big| \leq |z|.$$

Hence $\left|h\Big(f(z)\Big)\right| = \left|h\Big(f(g^{-1}(g(z)))\Big)\right| \leq |g(z)|$ and so

$$\left|\frac{f(z_1)-f(z)}{1-\overline{f(z_1)}f(z)}\right| \leq \left|\frac{z_1-z}{1-\overline{z_1}z}\right|$$

$$\lim_{z_1\to z}\left|\frac{\frac{f(z_1)-f(z)}{z_1-z}}{1-\overline{f(z_1)}f(z)}\right| \leq \lim_{z_1\to z}\left|\frac{1}{1-\overline{z_1}z}\right| .$$

So we have

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2} .$$

□

For $z_1=\alpha(0)$, $z_2=\alpha(1)$ and $f(z_1)=(f\circ\alpha)(0)$, $f(z_2)=(f\circ\alpha)(1)$, we get

$$\inf_{f(\alpha)}\int_0^1 \frac{2|f'(\alpha(t))||\alpha'(t)|}{1-|f(\alpha(t))|^2}\,dt \leq \inf_\alpha \int_0^1 \frac{2|\alpha'(t)|}{1-|\alpha(t)|^2}\,dt = d(z_1,z_2)$$

by Schwarz-Pick Theorem. So we get ([6])

$$d(f(z_1),f(z_2)) \leq d(z_1,z_2). \tag{1}$$

If $f^{-1}$ is a holomorphic function on $D_P$, then we have

$$d(z_1,z_2) = d(f^{-1}(f(z_1)),f^{-1}(f(z_2))) \leq d(f(z_1),f(z_2))$$

by applying the above (1). Hence we obtain

$$d(f(z_1),f(z_2)) = d(z_1,z_2) \tag{2}$$

for a biholomorphic function $f$ on $D_P$.

## 3. A shortest path between any two points in the Poincaré disk

(Case 1) We show that a straight line connecting the origin and an arbitrary point $p\in D_P$ is a shortest path. Let $\alpha(t)=(x(t),y(t))=(r(t)\cos\theta(t),r(t)\sin\theta(t))$ be a differentiable curve from $\alpha(0)=(0,0)$ to $\alpha(1)=(x(1),y(1))=p$ for the polar coordinates. Calculations show that

$$\alpha'(t) = (r'(t)\cos\theta(t)-r(t)\sin\theta(t)\theta'(t),\ r'(t)\sin\theta(t)+r(t)\cos\theta(t)\theta'(t))$$

$$\begin{aligned}\int_0^1\sqrt{g(\alpha'(t),\alpha'(t))}\,dt &= \int_0^1 \frac{2\sqrt{(r'(t))^2+(r(t))^2(\theta'(t))^2}}{1-r(t)^2}\,dt\\ &\geq \int_0^1\frac{2\sqrt{(r'(t))^2}}{1-r(t)^2}\,dt\\ &= \int_0^1\frac{2|r'(t)|}{1-r(t)^2}\,dt.\end{aligned}$$

It means that the length of an arbitrary curve connecting the origin and an arbitrary point $p\in D_P$ is greater than equal to the length of a straight line which is the case of a constant $\theta_0 = \theta(t)$ connecting the origin and an arbitrary point $p \in D_P$. Fix two constants $\cos\theta(t) = a$ and $\sin\theta(t) = b$. Let $\alpha(t) = (at, bt)$ be a straight line from $\alpha(0) = (0,0)$ to $\alpha(1) = (a,b)$ for $c = \sqrt{a^2+b^2} < 1$. Then we have

$$\begin{aligned}
\int_0^1 \sqrt{g(\alpha'(t),\alpha'(t))}\,dt &= \int_0^1 \frac{2\sqrt{a^2+b^2}}{1-(a^2+b^2)t^2}\,dt \\
&= \int_0^1 \frac{2c}{1-c^2t^2}\,dt \\
&= \int_0^1 \frac{c}{1+ct} - \frac{-c}{1-ct}\,dt \\
&= \ln\frac{1+c}{1-c}\,. \qquad (3)
\end{aligned}$$

Also let $\alpha(t) = (0, ct)$ be a straight line from $\alpha(0) = (0,0)$ to $\alpha(1) = (0,c)$ for $c < 1$. Then we have

$$\begin{aligned}
\int_0^1 \sqrt{g(\alpha'(t),\alpha'(t))}\,dt &= \int_0^1 \frac{2c}{1-c^2t^2}\,dt \\
&= \int_0^1 \frac{c}{1+ct} - \frac{-c}{1-ct}\,dt \\
&= \ln\frac{1+c}{1-c}\,.
\end{aligned}$$

A function $f : \mathbb{C} \longrightarrow \mathbb{C}$ of the form

$$f(z) = \frac{az+b}{cz+d} \quad (ad - bc \neq 0)$$

is called a linear fractional transformation. Note that a Möbius transformation

$$f(z) = \frac{z+a}{\bar{a}z+1}$$

for $|a|^2 = a\bar{a} < 1$ and $a \in \mathbb{C}$ is a linear fractional transformation which is a bijective mapping on $D_P$, since

$$|z+a|^2 - |\bar{a}z+1|^2 = -(1-|a|^2)(1-|z|^2) < 0$$

and $f(0) = a$. It is clear that

$$f(z) = \frac{z+a}{\bar{a}z+1} = \frac{1}{\bar{a}} + \frac{a-\frac{1}{\bar{a}}}{\bar{a}z+1}$$

is biholomorphic, since $f(z) = az+b$ for $a, b \in \mathbb{C}$ and $f(z) = \frac{1}{z}$ are holomorhic.

Let $C$ be a circle with center $z_0 = (x_0, y_0)$ and radius $r$ on the complex plane $\mathbb{C}$, that is,

$$(z-z_0)\overline{(z-z_0)} = r^2$$

$$z\bar{z} - \overline{z_0}z - z_0\bar{z} + z_0\overline{z_0} - r^2 = 0. \qquad (4)$$

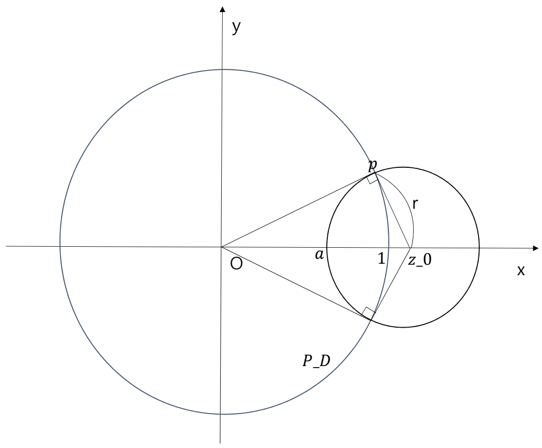


FIGURE 1. A shortest path through a point $(a, 0)$ on the Poincaré disk

(Case 2) We show that a part of $C$ in $D_P$ which meets orthogonally at two points of the boundary of the Poincaré disk is a shortest path. Rewrite (4) as

$$z\bar{z} + \delta z + \bar{\delta}\bar{z} + \gamma = 0 \quad \text{for } \delta = -\overline{z_0} \text{ and } |z_0|^2 - r^2 = \gamma\ . \tag{5}$$

Let $L$ be a line through the origin on the complex plane $\mathbb{C}$

$$L = \{(x, y) \mid cx + dy = 0\}. \tag{6}$$

Rewrite (6) as

$$\beta z + \bar{\beta}\bar{z} = 0 \quad \text{for } \beta = \frac{c - id}{2} \text{ and } z = x + iy\,. \tag{7}$$

We find the image of $y-$axis (that is, $\bar{\beta} = \beta$ by (7)) in $D_P$ by a biholomorphic function

$$f(z) = \frac{z + a}{az + 1} \quad \text{for a real } a \in D_P.$$

Recall that $d\,(f(z_1), f(z_2)) = d\,(z_1, z_2)$ (2). $\bar{\beta} = \beta$ implies $d = 0$. So we get $x = 0$. Put $w = \frac{z+a}{az+1}$. Then we get $z = \frac{w-a}{-aw+1}$. From the equation (7)

$$\beta\frac{w - a}{-aw + 1} + \beta\overline{\Big(\frac{w - a}{-aw + 1}\Big)} = 0,$$

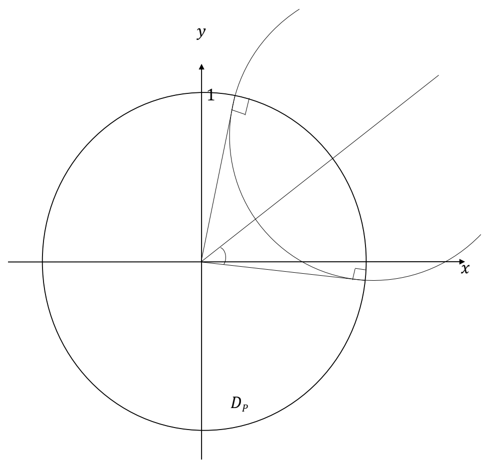


FIGURE 2. A rotation on the Poincaré disk

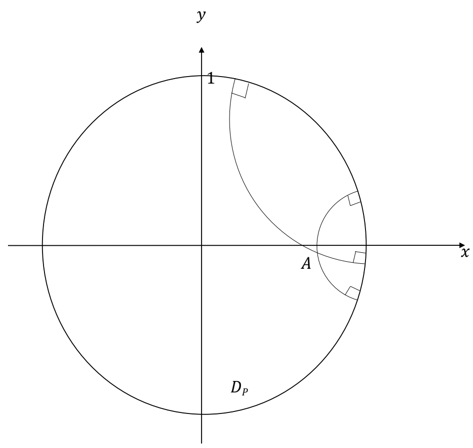


FIGURE 3. Two parallels to y-axis through a point $A$ on the Poincaré disk

we get a circle equation (5)

$$(-2a\beta)w\bar{w} + (\beta + a^2\beta)w + (\beta + a^2\beta)\bar{w} + (-2a\beta) = 0$$
$$w\bar{w} - \frac{1}{2}\Big(a + \frac{1}{a}\Big)w - \frac{1}{2}\Big(a + \frac{1}{a}\Big)\bar{w} + 1 = 0$$

with

$$-\delta = \overline{z_0} = z_0 = \frac{1}{2}\Big(a + \frac{1}{a}\Big) > \sqrt{a\,\frac{1}{a}} = 1 \tag{8}$$

and

$$|z_0|^2 - r^2 = \gamma = 1. \tag{9}$$

Two circles meet at two points with $x = \frac{1}{z_0}$, since

$$x^2 + y^2 = 1, \quad (x - z_0)^2 + y^2 = r^2$$

$$1 - x^2 = r^2 - (x - z_0)^2 = r^2 - x^2 + 2z_0 x - z_0^2 = -1 - x^2 + 2z_0 x$$

or

$$x^2 + y^2 = 1, \quad (x - z_0)^2 + y^2 = z_0^2 - 1$$

$$1 - x^2 = z_0^2 - 1 - (x - z_0)^2, \quad 2 = 2z_0 x.$$

As $a$ goes to 0, $\frac{1}{z_0}$ goes to 0 by (8). The equations (8) and (9) imply that the center and radius of a circle depending on $a$ are on the hyperbola $|z_0|^2 - r^2 = 1$. As $a$ goes to 1, $\frac{1}{z_0}$ goes to 1 and the norm of radius of a circle goes to zero by $|z_0|^2 - r^2 = 1$.

Two circles meet orthogonally at two points by the following two facts. The equation $|z_0|^2 = r^2 + 1$ implies that the triangle consisting of three points (the origin, $z_0$ and one of the two meeting points) is a right triangle. Note that the position vector from the origin to a point $p$ of the boundary of the Poincaré disk is always orthogonal to the tangent vector at a point $p$ (Fig. 1).

Finally, a rotation $f(z) = e^{i\theta} z$ on $D_P$ is also holomorphic, it preserves the distance by (2) (Fig. 2).

## 4. The cross ratio and the distance between two points on the Poincaré disk

Let $\alpha(t) = (t, 0) \subset D_P$ be a differentiable curve from $\alpha(0) = (0, 0)$ to $\alpha(x)$. The shortest arc length of $\alpha(t)$ from $\alpha(0)$ to $\alpha(x)$ is

$$\begin{aligned} \int_0^x \sqrt{g(\alpha'(t), \alpha'(t))}\, dt &= \int_0^x \frac{2}{1 - t^2}\, dt \\ &= \int_0^x \frac{1}{1 + t} - \frac{-1}{1 - t}\, dt \\ &= \ln \frac{1 + x}{1 - x}\,. \end{aligned} \tag{10}$$

The cross ratio $[z_0, z_1, w_1, w_0]$ for four points $z_0, z_1, w_1, w_0 \in \mathbb{C}$ is defined by

$$[z_0, z_1, w_1, w_0] = \frac{(z_0 - w_1)(z_1 - w_0)}{(z_1 - w_1)(z_0 - w_0)}.$$

Put

$$T(z) = [z, z_1, w_1, w_0] = \frac{(z - w_1)(z_1 - w_0)}{(z_1 - w_1)(z - w_0)}.$$

Then we see

$$T(z_1) = 1, \quad T(w_1) = 0, \quad T(w_0) = \infty.$$

For a real number $0 < x < 1$, we have

$$T(x) = [x, 0, 1, -1] = \frac{(x - 1)(0 - (-1))}{(0 - 1)(x + 1)} = \frac{1 - x}{1 + x} \tag{11}$$

$$T(0) = 1, \quad T(1) = 0, \quad T(-1) = \infty. \tag{12}$$

From (10), (11) and (12), we can define the distance from $x$ to the origin $O$ of $D_P$ by

$$d(x, O) = \Big| \ln \frac{1 - x}{1 + x} \Big|.$$

For $0 < iy < i$, we have

$$T(iy) = [iy, 0, i, -i] = \frac{(iy - i)(0 - (-i))}{(0 - i)(iy + i)} = \frac{1 - y}{1 + y}$$

So we get

$$d(iy, O) = \Big| \ln \frac{1 - y}{1 + y} \Big|.$$

We find the image of $y-$axis by a linear fractional transformation

$$f(z) = \frac{z + a}{az + 1}$$

for a real $a$ with $|a| < 1$. We get the same results as in section 3. A linear fractional transformation $f$ preserves the cross ratio

$$[f(z_0), f(z_1), f(w_1), f(w_0)] = [z_0, z_1, w_1, w_0].$$

So we can define the distance by

$$d(z_1, z_2) = d(f(z_1), f(z_2)) = \Big| \ln[z_1, z_2, w_1, w_0] \Big|,$$

where $w_1, w_0 \in \partial D$. It is easy to check that

$$d(z_1, z_2) = d(z_2, z_1), \quad d(z_1, z_2) + d(z_2, z_3) \geq d(z_1, z_3)$$

$$d(z_1, z_2) \geq 0 \quad \text{and} \quad d(z_1, z_2) = 0 \text{ if and only if } z_1 = z_2$$

for all $z_1, z_2, z_3 \in D_P$.

**Remark.** Let us denote by $\mathbb{H}$ the Poincaré upper half plane. Take a bijective mapping $h : \mathbb{H} \to D_P$

$$h(z) = \frac{z - i}{iz - 1}$$

with the inverse $g(w) = \frac{i-w}{1-iw}$. Then we have $g'(w) = \frac{-2}{(1-iw)^2}$ and the imaginary part of $g(w)$ is

$$\operatorname{Im}(g(w)) = \frac{1-|w|^2}{|1-iw|^2}.$$

Since $|w^2|^2 = ww\bar{w}\bar{w} = w\bar{w}w\bar{w} = |w|^4$, we have $|w^2| = |w|^2$. So we obtain the following relation

$$\int_0^1 \frac{2|\alpha'(t)|}{1-|\alpha(t)|^2}\,dt = \int_0^1 \frac{|\frac{-2}{(1-i\alpha(t))^2}||\alpha'(t)|}{\frac{1-|\alpha(t)|^2}{|1-i\alpha(t)|^2}}\,dt = \int_0^1 \frac{|(g\circ\alpha)'(t)|}{\operatorname{Im}(g(\alpha(t)))}\,dt.$$

Therefore we can define an inner product on $\mathbb{H}$ as

$$g(v_p, w_p) = \frac{1}{y}(v_p \cdot w_p)$$

for all $p \in \mathbb{H}$.

## 5. The line element of a surface

Let $S$ be a regular surface in 3-dimensional Euclidean space $\mathbb{R}^3$. The first fundamental form

$$I_p : T_pS \times T_pS \longrightarrow \mathbb{R}, \quad I_p(v,w) = v \cdot w$$

is the inner product on the tangent space $T_pS$ at $p \in S$ of a surface $S$ induced by the dot product of $\mathbb{R}^3$. Let $\mathbf{X} : U \subseteq \mathbb{R}^2 \longrightarrow S \subset \mathbb{R}^3$ be a coordinate chart of a surface $S$, that is,

$$\mathbf{X}(u,v) = (x(u,v), y(u,v), z(u,v)).$$

Let $\alpha : [t_0, t_1] = I \longrightarrow S \subset \mathbb{R}^3$ be a curve in $S$ such that

$$\alpha(t) = \mathbf{X}(u(t), v(t))$$

for a curve $c(t) = (u(t), v(t)) \subset U \subseteq \mathbb{R}^2$. The length of a curve from $\alpha(t_0)$ to $\alpha(t)$ is

$$s(t) = \int_{t_0}^{t} |\alpha'(r)|\,dr = \int_{t_0}^{t} \sqrt{I_{\alpha(r)}\big(\alpha'(r), \alpha'(r)\big)}\,dr.$$

Since $\alpha'(r) = \mathbf{X}_u \frac{du}{dr} + \mathbf{X}_v \frac{dv}{dr}$ and

$$\begin{aligned} I_{\alpha(r)}\big(\alpha'(r), \alpha'(r)\big) &= I_{\alpha(r)}\Big(\mathbf{X}_u \frac{du}{dr} + \mathbf{X}_v \frac{dv}{dr},\ \mathbf{X}_u \frac{du}{dr} + \mathbf{X}_v \frac{dv}{dr}\Big) \\ &= \Big(\mathbf{X}_u \cdot \mathbf{X}_u\Big)\Big(\frac{du}{dr}\Big)^2 + 2\Big(\mathbf{X}_u \cdot \mathbf{X}_v\Big)\Big(\frac{du}{dr}\Big)\Big(\frac{dv}{dr}\Big) + \Big(\mathbf{X}_v \cdot \mathbf{X}_v\Big)\Big(\frac{dv}{dr}\Big)^2 \\ &= E\Big(\frac{du}{dr}\Big)^2 + 2F\Big(\frac{du}{dr}\Big)\Big(\frac{dv}{dr}\Big) + G\Big(\frac{dv}{dr}\Big)^2, \end{aligned}$$

where we put $E = \mathbf{X}_u \cdot \mathbf{X}_u$, $F = \mathbf{X}_u \cdot \mathbf{X}_v$ and $G = \mathbf{X}_v \cdot \mathbf{X}_v$, we get

$$\frac{ds}{dt} = \sqrt{E\Big(\frac{du}{dt}\Big)^2 + 2F\Big(\frac{du}{dt}\Big)\Big(\frac{dv}{dt}\Big) + G\Big(\frac{dv}{dt}\Big)^2}.$$

Therefore we have the so-called line element

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

Note that the line element $ds^2$ of 2-dimensional Euclidean space $\mathbb{R}^2$ with the dot product is

$$ds^2 = dx^2 + dy^2.$$

For a vector $v = (v_1, v_2) \in \mathbb{R}^2$, the notations

$ds^2(v) = |v|^2,\ dx^2(v) = dx(v)dx(v) = v_1 v_1 = v_1^2,\ dy^2(v) = dy(v)dy(v) = v_2 v_2 = v_2^2$

imply the Pythagorean Theorem. Using the polar coordinates, we get

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta\,,$$

since $(x, y) = (r\cos\theta, r\sin\theta)$ and

$$(dx, dy) = (dr\cos\theta - r\sin\theta d\theta,\ dr\sin\theta + r\cos\theta d\theta).$$

The line element $ds^2$ of the Poincaré upper half plane, the Poincaré disk is

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \quad ds^2 = \frac{4(dx^2 + dy^2)}{(1 - (x^2 + y^2))^2},$$

respectively. Let us find the line element $ds^2$ of the Poincaré disk with respect to the polar coordinates. By (3) and (10), we can put for $0 \le r < 1$

$$\bar{r} = \ln\frac{1 + r}{1 - r}.$$

Then we have

$$\sinh\bar{r} = \frac{e^{\bar{r}} - e^{-\bar{r}}}{2} = \frac{\frac{1+r}{1-r} - \frac{1-r}{1+r}}{2} = \frac{2r}{1 - r^2}.$$

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - r^2)^2} = \frac{4(dr^2 + r^2 d\theta^2)}{(1 - r^2)^2} = \Big(\frac{2}{1 - r^2}\Big)^2 dr^2 + \Big(\frac{2r}{1 - r^2}\Big)^2 d\theta^2$$

It follows from $\bar{r}(r) = \ln\frac{1+r}{1-r}$ that

$$d\bar{r} = \frac{2}{1 - r^2}dr.$$

So we get

$$ds^2 = d\bar{r}^2 + \Big(\frac{2r}{1 - r^2}\Big)^2 d\theta^2 = d\bar{r}^2 + \sinh^2\bar{r}d\theta^2.$$

A rotation $f(z) = e^{i\theta}z$ on $D_P$ preserves the isometry. Rewrite it on $D_P$

$$ds^2 = dr^2 + \sinh^2 r d\theta^2.$$

So

$$E = 1, \quad F = 0, \quad G = \sinh^2 r.$$

Then we get Gaussian curvature $K = -1$, where

$$K = \frac{-1}{2\sqrt{EG}}\Big(\frac{\partial}{\partial\theta}\Big(\frac{\frac{\partial E}{\partial\theta}}{\sqrt{EG}}\Big) + \frac{\partial}{\partial r}\Big(\frac{\frac{\partial G}{\partial r}}{\sqrt{EG}}\Big)\Big).$$

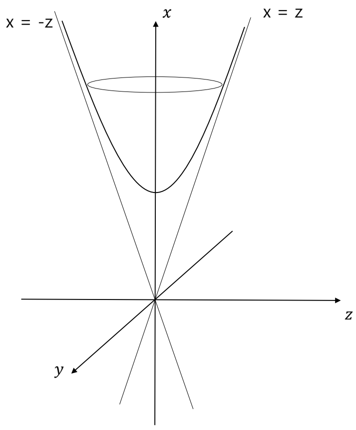


FIGURE 4. One connected component of the two-sheeted hyperboloid

By the Gauss-Bonnet theorem, the sum of the interior angles of a triangle on $D_P$ is less than $\pi$. A quadrilateral $\square ABCD$ which has two right angles $\angle DAB \cong \angle CBA$ and two congruent sides $\overline{AD} \cong \overline{BC}$ without assuming the Parallel postulate is called a Saccheri quadrilateral. $\overline{AB}$ is called the base of the quadrilateral. Consider a Saccheri quadrilateral $\square YOXP$ with the base $\overline{OX}$ on the Poincaré disk $D_P$, where $X$, $Y$ is a point in $x$-axis, $y$-axis, respectively. Since we have two triangles $\triangle YOX$ and $\triangle XYP$, the sum of the interior angles of a Saccheri quadrilateral on the Poincaré disk is less than $2\pi$.

### 5.1. The line element of a surface of revolution

Let $\mathbb{R}^3_1$ be Minkowski space-time with metric

$$g(v, w) = -v_1 w_1 + v_2 w_2 + v_3 w_3.$$

Consider the surface of revolution of a curve $\alpha(r) = (\cosh r, 0, \sinh r)$ of the hyperbola $x^2 - z^2 = 1$ in the $xz$-plane with a coordinate chart

$$\mathbf{X}(r, \theta) = (\cosh r, \sinh r \cos\theta, \sinh r \sin\theta) \subset \mathbb{R}^3_1,$$

which satisfies $-x^2 + y^2 + z^2 = -1$ (Fig. 4). So we have

$$\mathbf{X}_r = (\sinh r, \cosh r \cos\theta, \cosh r \sin\theta), \quad \mathbf{X}_\theta = (0, -\sinh r \sin\theta, \sinh r \cos\theta)$$

$$E = g(\mathbf{X}_r, \mathbf{X}_r) = -\sinh^2 r + \cosh^2 r = 1$$

$$F = g(\mathbf{X}_r, \mathbf{X}_\theta) = 0,\ G = g(\mathbf{X}_\theta, \mathbf{X}_\theta) = \sinh^2 r.$$

Hence we get

$$ds^2 = dr^2 + \sinh^2 r d\theta^2.$$

So the Poincaré disk is isometric to the above surface of revolution of a curve $\alpha(r) = (\cosh r, 0, \sinh r)$ of the hyperbola $x^2 - z^2 = 1$ in the $xz$-plane in Minkowski space-time $\mathbb{R}^3_1$.

## 6. Numerical calculations of the length of of a curve in the Poincaré disk

In this section, Example 2,3 (Figure 5,6) show numerically that the shortest path connecting $P$ and $Q$ in $D_P$ is the part of $C_1$ which meets orthogonally at the two boundary points of $D_P$ compared with a bigger or smaller circle $C_2$ which does not meet orthogonally at the two boundary points of $D_P$. Example 1 indicates that the length of the straight line connecting $P$ and $Q$ independent of $\theta$(Figure 5) is not the shortest.

Note that the Riemannian integral $\int_a^b f\,dx$ is given by the limit of the Riemannian sum. Let $P$ be a partition of an integration interval $I = [a, b]$ denoted by

$$P : a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k < \cdots < x_n = b$$

for $k \in \{1, 2, 3, \ldots, n\}$ and $f$ be the integrand. First, we give Python code of numerical integration which is the sum of the areas of trapezoids consisting of four points $x_{k-1}, x_k, f(x_{k-1}), f(x_k)$ as a approximation value when the maximum $x_k - x_{k-1}$ for all $k$ is small enough. The area of the $k$-th trapezoid is

$$\frac{1}{2}\Big(\big(f\big(a + (k-1)\frac{b-a}{n}\big)\big) + \big(f\big(a + k\frac{b-a}{n}\big)\big)\Big)\Big(\frac{b-a}{n}\Big)$$

where $x_k - x_{k-1} = \frac{b-a}{n}$ for all $k$ and a sufficiently large positive integer $n$ ([3]).

```
import math
from math import *
import numpy as np
def integraltrapezoid(f,a,b):
    sum=0
    n=10000
    for k in range(1, 10000+1):
        sum =sum+((1/2)*(f( a+(k-1)*((b-a)/n))+f( a+k*((b-a)/n))))*((b-a)/n)
    return sum
```

Second, we give Python code of numerical differentiation.

```
def diff(f,x):
    h=1e-5
    d=(f(x+h)-f(x-h))/(2*h)
    return d
```

Third, we can calculate the Euclidean length of a curve $\alpha(t) = (-2+\sqrt{3}\cos(t), \sqrt{3}\sin(t))$ by using Python lambda function.

```
f=lambda x:-2+((3)**(1/2))*np.cos(x)

g=lambda x:((3)**(1/2))*np.sin(x)

def length(f,g,a,b):
    fg=lambda x:(diff(f,x)**2 +diff(g,x)**2)**(1/2)
    return integraltrapezoid(fg,a,b)

length(f,g,-pi/12,pi/12)
```

0.9068996821024429 which is a approximation value of

$\int_a^b \sqrt{\alpha'(t)\cdot\alpha'(t)}\,dt = \int_a^b \sqrt{3}\,dt = \sqrt{3}(b-a)$

```
((3)**(1/2))*(pi/12-(-pi/12))
```

0.9068996821171088

Consider a geodesic $\alpha(t) = (-2+\sqrt{3}\cos(t), \sqrt{3}\sin(t))$ for $-\pi/12 \le t \le \pi/12$ which is the part of circle $(x+2)^2+y^2 = 3$ in the Poincaré disk. Let us calculate the length of $\alpha(t)$

$$\int_a^b \frac{2\sqrt{\alpha'(t)\cdot\alpha'(t)}}{1-(\alpha_1(t)^2+\alpha_2(t)^2)}\,dt$$

by using the inner product $g(v_p, w_p) = \frac{2(v_p\cdot w_p)}{1-(x^2+y^2)}$.

```
def lengthpoincare(f,g,a,b):
    fg=lambda x:2*(1/(1-(f(x)**2+g(x)**2)))*(diff(f,x)**2 +diff(g,x)**2)**(1/2)
    return integraltrapezoid(fg,a,b)

f=lambda x:-2+((3)**(1/2))*np.cos(x)

g=lambda x:((3)**(1/2))*np.sin(x)

lengthpoincare(f,g,-pi/12,pi/12)
```

2.151268379413075

Let us compare the length of a curve (a line segment in the sense of Euclidean geometry)

$$\alpha(t) = \big(-2+\sqrt{3}\cos(-\frac{\pi}{12}), t\big), \quad \sin(-\frac{\pi}{12}) \le t \le \sin(\frac{\pi}{12})$$

in the Poincaré disk with the above one.

```
f=lambda x:-2+((3)**(1/2))*np.cos(-pi/12)

g=lambda x:x

lengthpoincare(f,g,((3)**(1/2))*np.sin(-pi/12),((3)**(1/2))*np.sin(pi/12))
```

2.182699643561364 which is greater than the above length.

**Example 1** Consider a geodesic $\alpha(t) = (-10 + \sqrt{99}\cos(t), \sqrt{99}\sin(t))$ for $-\pi/10000 \le t \le \pi/10000$ which is the part of circle $(x+10)^2 + y^2 = 99$ in the Poincaré disk.

```
f=lambda x:-10+((99)**(1/2))*np.cos(x)
g=lambda x:((99)**(1/2))*np.sin(x)
lengthpoincare(f,g,-pi/10000,pi/10000)
```

0.012534916887604922

Let us compare the length of a curve (a line segment in the sense of Euclidean geometry)

$$\alpha(t) = \big(-10 + \sqrt{99}\cos(-\frac{\pi}{10000}), t\big), \quad \sin(-\frac{\pi}{10000}) \le t \le \sin(\frac{\pi}{10000})$$

in the Poincaré disk with the above one.

```
f=lambda x:-10+((99)**(1/2))*np.cos(-pi/10000)
g=lambda x:x
lengthpoincare(f,g,((99)**(1/2))*np.sin(-pi/10000),((99)**(1/2))*np.sin(pi/10000))
```

0.012534917094007081 which is greater than the above length.

**Example 2** Let us denote by $C_1$ the circle whose part in the poincaré disk is the shortest path connecting $P$ and $Q$. Let us denote by $C_2$ the circle passing through two points $P$ and $Q$ whose radius is greater than that of $C_1$. Let $O_1$, $O_2$ be the origin of the circle $C_1$, $C_2$, respectively. We see that the length of the line segment $\overline{O_1P}$ is $\sqrt{3}$ with $O_1 = (-2, 0)$. We can calculate the radius $c$ of $C_2$ with $O_2 = (-4, 0)$ and $\angle PO_1R = \pi/12$ by using the Pythagorean Theorem of $\triangle PO_2R$

$$r = c^2 = (-2 - (-4) + \sqrt{3}\cos(\pi/12))^2 + (\sqrt{3}\sin(\pi/12))^2.$$

Let us denote by $\theta$ the angel $\angle PO_2R = \angle QO_2R$. From $\sqrt{r}\sin\theta = \sqrt{3}\sin(\pi/12)$, it follows that

$$\theta = \arcsin(\sqrt{3/r}\sin(\pi/12)).$$

We can see that arc length connecting $P$ and $Q$ of $C_2$ becomes more greater than arc length connecting $P$ and $Q$ of $C_1$ as the radius of $C_2$ increases by the following Python calculations.

```
def makecircle(c):
    r= ((3**(1/2))*np.cos(pi/12)-(c-(-2)))**2 + ((3**(1/2))*np.sin(pi/12))**2
    return [c,r]
```

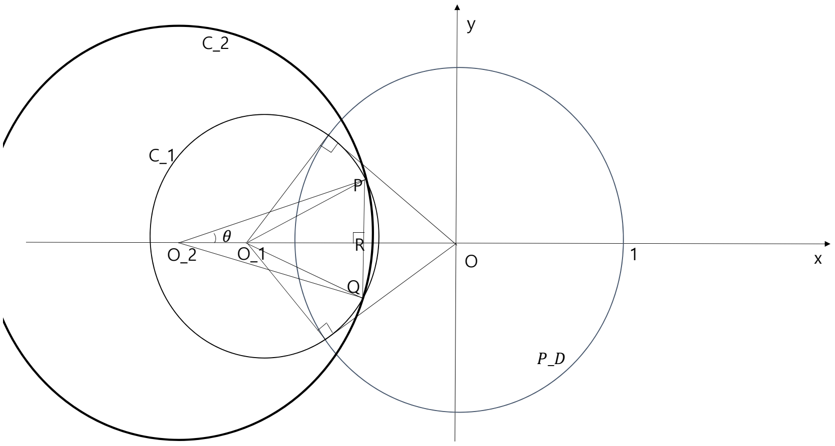


FIGURE 5. Example 2

```
radius,lengthpoincarePQ=[],[]

for k in range(0,4):
    c=makecircle(-(2)-k*0.1)[0]
    r=makecircle(-(2)-k*0.1)[1]
    d=-2+(3**(1/2))*np.cos(pi/12)
    e=(3**(1/2))*np.sin(pi/12)
    if (d-(c))**2+e**2==r:
      radius.append(r**(1/2))
    f=lambda x:c+((r)**(1/2))*np.cos(x)
    g=lambda x:((r)**(1/2))*np.sin(x)
    a=np.arcsin(((3/r)**(1/2))*np.sin(-pi/12))
    b=np.arcsin(((3/r)**(1/2))*np.sin(pi/12))
    h=lengthpoincare(f,g,a,b)
    lengthpoincarePQ.append(h)

radius

[1.7320508075688772, 1.8288265422109127, 1.92593173373052, 2.0233189477898357]

lengthpoincarePQ

[2.151268379413075, 2.1513591858797745, 2.1515959867425436, 2.151936496223713]
```

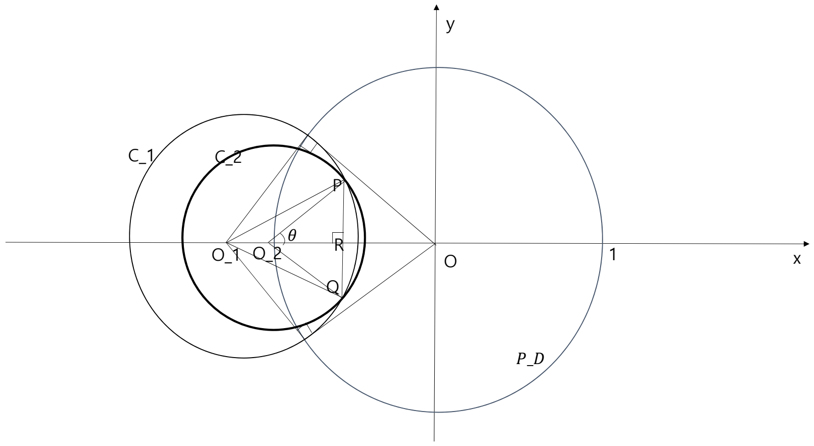


FIGURE 6. Example 3

**Example 3** Let us denote by $C_2$ the circle passing through two points $P$ and $Q$ whose radius is less than that of $C_1$ so that the origin of $C_2$ is on the right side of the origin of $C_1$ in the picture of Example 2. We can also see that arc length connecting $P$ and $Q$ of $C_2$ becomes more greater than arc length connecting $P$ and $Q$ of $C_1$ as the radius of $C_2$ decreases by the following Python calculations.

```
radius,lengthpoincarePQ=[],[]

for k in range(0,4):
    c=makecircle(-(2)+k*0.1)[0]
    r=makecircle(-(2)+k*0.1)[1]
    d=-2+(3**(1/2))*np.cos(pi/12)
    e=(3**(1/2))*np.sin(pi/12)
    if (d-(c))**2+e**2==r:
      radius.append(r**(1/2))
    f=lambda x:c+((r)**(1/2))*np.cos(x)
    g=lambda x:((r)**(1/2))*np.sin(x)
    a=np.arcsin(((3/r)**(1/2))*np.sin(-pi/12))
    b=np.arcsin(((3/r)**(1/2))*np.sin(pi/12))
    h=lengthpoincare(f,g,a,b)
    lengthpoincarePQ.append(h)
```

radius

[1.7320508075688772, 1.6356630088453048, 1.5397360023750024, 1.444361601370872]

lengthpoincarePQ

[2.151268379413075, 2.1513819028313836, 2.1517810421417263, 2.1525803914346207]

Indeed, the above two numerical calculations indicate that the arc length connecting $P$ and $Q$ of $C_1$ is the shortest.

f=lambda x:-2+((3)**(1/2))*np.cos(x)

g=lambda x:((3)**(1/2))*np.sin(x)

lengthpoincare(f,g,-pi/12,pi/12)

2.151268379413075

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