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# COMPLETE LIFTS OF PROJECTABLE LINEAR CONNECTION TO SEMI-TANGENT BUNDLE

Murat Polat* and Furkan Yildirim

**Abstract.** We study the complete lifts of projectable linear connection for semi-tangent bundle. The aim of this study is to establish relations between these and complete lift already known. In addition, the relations between infinitesimal linear transformations and projectable linear connections are studied. We also have a new example for good square in this work.

## 1. Introduction

Let $M_n$ be a $C^\infty-$manifold of finite dimension $n$, and let $(M_n, \pi_1, B_m)$ be a differentiable bundle over $B_m$. We use the notation $(x^i) = (x^a, x^\alpha)$, where the indices $i, j, ...$ run over the range $\{1, 2, ..., n\}$, the indices $a, b, ...$ run over the range $\{1, 2, ..., n-m\}$ and the indices $\alpha, \beta, ...$ run over the range $\{n-m+1, ..., n\}$, $x^\alpha$ are coordinates in $B_m$, $x^a$ are fiber coordinates of the bundle

$$\pi_1 : M_n \to B_m.$$

Let now $(T(B_m), \widetilde{\pi}, B_m)$ be a tangent bundle [16] over base space $B_m$, and let $M_n$ be differentiable bundle determined by a submersion (natural projection) $\pi_1 : M_n \to B_m$. The semi-tangent bundle (pull-back) of the tangent bundle $(T(B_m), \widetilde{\pi}, B_m)$ is the bundle $(t(B_m), \pi_2, M_n)$ over differentiable bundle $M_n$ with a total space

$$\begin{aligned} t(B_m) &= \left\{ \left( (x^a, x^\alpha), x^{\overline{\alpha}} \right) \in M_n \times T_x(B_m) : \pi_1 (x^a, x^\alpha) = \widetilde{\pi} \left( x^\alpha, x^{\overline{\alpha}} \right) = (x^\alpha) \right\} \\ &\subset M_n \times T_x(B_m) \end{aligned}$$

and with the projection map $\pi_2 : t(B_m) \to M_n$ defined by $\pi_2(x^a, x^\alpha, x^{\overline{\alpha}}) = (x^a, x^\alpha)$, where $T_x(B_m) \left( x = \pi_1 (\widetilde{x}), \widetilde{x} = (x^a, x^\alpha) \in M_n \right)$ is the tangent space

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*Corresponding

at a point $x$ of $B_m$, where $x^{\overline{\alpha}} = y^{\alpha}$ $\left(\overline{\alpha}, \overline{\beta}, ... = n+1, ..., 2n\right)$ are fiber coordinates of the tangent bundle $T(B_m)$ (see, for pull-back bundle [4],[5],[10],[11],[17],[18],[13],[14],[3]).

Where the semi-tangent bundle $t(B_m)$ of the differentiable bundle $M_n$ also has the natural bundle structure over $B_m$, its bundle projection $\pi : t(B_m) \to B_m$ being defined by $\pi : (x^a, x^\alpha, x^{\overline{\alpha}}) \to (x^\alpha)$, and hence $\pi = \pi_1 \circ \pi_2$.

Thus $(t(B_m)$,$\pi_1 \circ \pi_2)$ is the step-like bundle [7] or composite bundle [[9], p.9]. As a result, we notice the semi-tangent bundle $(t(B_m)$,$\pi_2)$ is a pull-back (Pontryagin [8]) bundle of the tangent bundle over $B_m$ by $\pi_1$ [10].

If $(x^{i'}) = (x^{a'}, x^{\alpha'})$ is another local adapted coordinates in $M_n$, then we have

$$\text{(1)} \qquad \begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right). \end{cases}$$

The Jacobian of (1) has the components

$$\left(A_j^{i'}\right) = \left(\frac{\partial x^{i'}}{\partial x^j}\right) = \begin{pmatrix} A_b^{a'} & A_\beta^{a'} \\ 0 & A_\beta^{\alpha'} \end{pmatrix},$$

where $A_b^{a'} = \frac{\partial x^{a'}}{\partial x^b}$, $A_\beta^{a'} = \frac{\partial x^{a'}}{\partial x^\beta}$, $A_\beta^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}$ [10].

To a transformation (1) of local coordinates of differentiable bundle $M_n$, there corresponds on $t(B_m)$ the change of coordinate

$$\text{(2)} \qquad \begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right), \\ x^{\overline{\alpha}'} = \frac{\partial x^{\alpha'}}{\partial x^\beta} y^\beta. \end{cases}$$

The Jacobian of (2) is:

$$\text{(3)} \qquad \bar{A} = \left(A_J^{I'}\right) = \begin{pmatrix} A_b^{a'} & A_\beta^{a'} & 0 \\ 0 & A_\beta^{\alpha'} & 0 \\ 0 & A_{\beta\varepsilon}^{\alpha'} y^\varepsilon & A_\beta^{\alpha'} \end{pmatrix},$$

where $A_{\beta\varepsilon}^{\alpha'} = \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\varepsilon}$; $I = (a, \alpha, \overline{\alpha})$, $J = (b, \beta, \overline{\beta})$, $I, J, .... = 1, ..., 2n$ [10]. Writing the inverse of (2) as

$$\text{(4)} \qquad \begin{cases} x^a = x^a(x^{b'}, x^{\beta'}), \\ x^\alpha = x^\alpha\left(x^{\beta'}\right), \\ x^{\overline{\alpha}} = \frac{\partial x^\alpha}{\partial x^{\beta'}} y^{\beta'}, \end{cases}$$

we have

$$\text{(5)} \qquad \left(A_{J'}^I\right) = \begin{pmatrix} A_{b'}^a & A_{\beta'}^a & 0 \\ 0 & A_{\beta'}^\alpha & 0 \\ 0 & A_{\beta'\varepsilon'}^\alpha y^{\varepsilon'} & A_{\beta'}^\alpha \end{pmatrix}.$$

Now, consider a diagram as

$$\begin{array}{ccc} A & \overset{\tau_3}{\rightarrow} & B \\ {}^{\tau_1}\downarrow & & \downarrow^{\tau_2} \\ C & \underset{\tau_4}{\rightarrow} & D \end{array}$$

A good square of vector bundles satisfies the following conditions:
(a) $\tau_1$ and $\tau_2$ are fiber bundles, (but not necessarily vector bundles);
(b) $\tau_3$ and $\tau_4$ are vector bundles;
(c) $\tau_4 \circ \tau_1 = \tau_2 \circ \tau_3$, the square (diagram) is commutative;
(d) the local expression

$$\begin{array}{ccc} A & \overset{\tau_3}{\rightarrow} & B \\ {}^{\tau_1}\downarrow & & \downarrow^{\tau_2} \\ C & \underset{\tau_4}{\rightarrow} & D \end{array} \quad \begin{array}{ccc} U^n \times R^r \times M^s \times R^t & \rightarrow & U^n \times M^s \\ \downarrow & & \downarrow \\ U^n \times R^r & \rightarrow & U^n \end{array} \quad \begin{array}{ccc} (x^i, a^a, g^\lambda, b^\sigma) & \rightarrow & (x^i, g^\lambda) \\ \downarrow & & \downarrow \\ (x^i, a^a) & \rightarrow & (x^i) \end{array}$$

where superindices denote the dimension of the manifolds and $M$ is a manifold [2].

Hence we have

**Theorem 1.1.** *Let now $\pi : t(B_m) \rightarrow B_m$ be a semi-tangent bundle and $\pi_1 : M_n \rightarrow B_m$ be a fiber bundle. So that*

$$\begin{array}{ccc} t(B_m) & \overset{\pi_2}{\rightarrow} & M_n \\ {}^{id}\downarrow & & \downarrow^{\pi_1} \\ t(B_m) & \underset{\pi}{\rightarrow} & B_m \end{array} \quad \begin{array}{ccc} M_n \times T_x(B_m) & \overset{\pi_2}{\rightarrow} & M_n \\ {}^{id}\downarrow & & \downarrow^{\pi_1} \\ M_n \times T_x(B_m) & \underset{\pi}{\rightarrow} & B_m \end{array} \quad \begin{array}{ccc} (x^a, x^\alpha, x^{\overline{\alpha}}) & \overset{\pi_2}{\rightarrow} & (x^a, x^\alpha) \\ {}^{id}\downarrow & & \downarrow^{\pi_1} \\ (x^a, x^\alpha, x^{\overline{\alpha}}) & \underset{\pi}{\rightarrow} & (x^\alpha) \end{array}$$

*is a good square and the diagram commutes ($\pi = \pi_1 \circ \pi_2$).*

We note that projectable linear connections in $t(B_m)$ and their some properties were investigated in [[13], [14]]. In this paper, we continue to study the complete lifts of projectable linear connection from differentiable manifold $B_m$ to semi-tangent (pull-back) bundle $(t(B_m), \pi_2)$ initiated by V. V. Vishnevskii [13].

We denote by $\Im_q^p(M_n)$ the module over $F(M_n)$ of all $C^\infty-$tensor fields of type $(p, q)$ on $M_n$, i.e., of contravariant degree $p$ and covariant degree $q$, where $F(M_n)$ is the algebra of $C^\infty-$functions on $M_n$. We now put $\Im(M_n) = \sum_{p,q=0}^{\infty} \Im_q^p(M_n)$, which is the set of all tensor fields on $M_n$. Smilarly, we denote by $\Im_q^p(B_m)$ and $\Im(B_m)$ respectively the corresponding sets of tensor fields in the base space $B_m$.

## 2. Complete Lifts of Projectable Linear Connection

If $f$ is a function on $B_m$, we write ${}^{vv}f$ for the function on $t(B_m)$ obtained by forming the composition of $\pi : t(B_m) \rightarrow B_m$ and ${}^{v}f = f \circ \pi_1$, so that

$${}^{vv}f = {}^{v}f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi.$$

Thus, the vertical lift ${}^{vv}f$ of the function $f$ to $t(B_m)$ satisfies

$$ {}^{vv}f(x^a, x^\alpha, x^{\overline{\alpha}}) = f(x^\alpha). \tag{6} $$

We note here that value ${}^{vv}f$ is constant along each fibre of $\pi : t(B_m) \to B_m$.

On the other hand, if $f = f(x^a, x^\alpha)$ is a function in $M_n$, we write${}^{cc}f$ for the function in $t(B_m)$ defined by

$$ {}^{cc}f = \imath(df) = x^{\overline{\beta}}\partial_\beta f = y^\beta \partial_\beta f \tag{7} $$

and call of the complete lift ${}^{cc}f$ of the function $f$ [10].

Let $X \in \Im_0^1(B_m)$, i.e. $X = X^\alpha \partial_\alpha$. On putting

$$ {}^{vv}X = \left({}^{vv}X^I\right) = \begin{pmatrix} 0 \\ 0 \\ X^\alpha \end{pmatrix}, \tag{8} $$

from (3), we prove that ${}^{vv}X' = \bar{A}({}^{vv}X)$. ${}^{vv}X$ is a vector field which is called the vertical lift of $X$ to $t(B_m)$.

For $F \in \Im_1^1(B_m)$, we can define a vector field $\gamma F \in \Im_0^1(B_m)$:

$$ \gamma F = \left(\gamma F^I\right) = \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon F_\varepsilon^\alpha \end{pmatrix}. \tag{9} $$

From (3), we easily see that $\gamma F' = \bar{A}(\gamma F)$.

Let $\widetilde{X} \in \Im_0^1(M_n)$ be a projectable vector field [13] with projection $X = X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$. The complete lift ${}^{cc}\widetilde{X}$ of $\widetilde{X} \in \Im_0^1(M_n)$ to semi-tangent bundle $t(B_m)$ has components [14]

$$ {}^{cc}\widetilde{X} = \left({}^{cc}\widetilde{X}^I\right) = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix} \tag{10} $$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ in $t(B_m)$.

**Theorem 2.1.** *Let $\widetilde{X}$ and $\widetilde{Y}$ be projectable vector fields on $M_n$ with projections $X$ and $Y$ on $B_m$, respectively. For the Lie product, we have* [19]*:*

$$ [{}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}] = {}^{cc}[\widetilde{X}, \widetilde{Y}] (i.e. L_{{}^{cc}\widetilde{X}}({}^{cc}\widetilde{Y}) = {}^{cc}\left(L_{\widetilde{X}}\widetilde{Y}\right)). $$

**Theorem 2.2.** *Let $\widetilde{X} \in \Im_0^1(M_n)$. For the Lie product, we have*

$$ [{}^{cc}\widetilde{X}, \gamma F] = \gamma\left(L_X F\right) $$

*for any $F \in \Im_1^1(B_m)$, where $L_X$ the operator of Lie derivation with respect to $X$.*

Let $p : Y \to M$ be a fibred manifold. A classical connection $\nabla$ on $Y$ is called projectable (with respect to $p$) if there exists a (unique) classical linear connection $\underline{\nabla}$ on $M$ such that $\nabla$ is $p-$related to $\underline{\nabla}$ [[15], [1]]. If $T(B_m)$ is the tangent bundle of $B_m$, a linear connection $\underline{\nabla}$ is a classical linear connection on $B_m$ [6]. The last condition means that if $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$ and $X, Y \in \Im_0^1(B_m)$ are such that $Tp \circ \widetilde{X} = X \circ p$ and $Tp \circ \widetilde{Y} = Y \circ p$ then $Tp \circ \nabla_{\widetilde{X}} \widetilde{Y} = (\underline{\nabla}_X Y) \circ p$.

Where $T$ is given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for any $X, Y \in \Im_0^1(B_m)$.

From the above defination it follows that the $\nabla$ is a projectable (with respect to $p := \pi_1 : M_n \to B_m$) linear connection on $B_m$. Then there exists a unique projectable linear connection ${}^{cc}\nabla$ in semi-tangent bundle $t(B_m)$ ([13], [14]) which satisfies

$$ {}^{cc}\nabla_{{}^{cc}\widetilde{X}}({}^{cc}\widetilde{Y}) = {}^{cc}(\underline{\nabla}_X Y) \tag{11}$$

for any projectable vector fields $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$. This assertion may be verified by a simple calculation using connection components. Let $\Gamma_{\alpha\gamma}^{\beta}$ be components of $\underline{\nabla}$ with respect to local coordinates $(x^\alpha)$ in $B_m$ and ${}^{cc}\Gamma_{I\,K}^{J}$ components of ${}^{cc}\nabla$ with respect to the induced coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ in $t(B_m)$. Let $\widetilde{X} \in \Im_0^1(M_n)$ and $\widetilde{Y} \in \Im_0^1(M_n)$ be projectable vector fields with components $\widetilde{X}^I$ and $\widetilde{Y}^J$, respectively, with respect to the local coordinates $(x^a, x^\alpha)$ in $M_n$.

Then ${}^{cc}\widetilde{X}$ and ${}^{cc}\widetilde{Y}$ have, respectively, components

$$ {}^{cc}\widetilde{X} : \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix}, {}^{cc}\widetilde{Y} : \begin{pmatrix} \widetilde{Y}^a \\ Y^\alpha \\ y^\varepsilon \partial_\varepsilon Y^\alpha \end{pmatrix}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ in $t(B_m)$. If $\begin{pmatrix} {}^{cc}\widetilde{X}^{I\,cc}\nabla_I({}^{cc}\widetilde{Y}^b) \\ {}^{cc}\widetilde{X}^{I\,cc}\nabla_I({}^{cc}\widetilde{Y}^\beta) \\ {}^{cc}\widetilde{X}^{I\,cc}\nabla_I({}^{cc}\widetilde{Y}^{\overline{\beta}}) \end{pmatrix} =$ $\begin{pmatrix} X^a \underline{\nabla}_a Y^b \\ X^\alpha \underline{\nabla}_\alpha Y^\beta \\ y^\sigma \partial_\sigma (X^\alpha \underline{\nabla}_\alpha Y^\beta) \end{pmatrix}$ are the components of equalization ${}^{cc}\widetilde{X}^I \nabla_I({}^{cc}\widetilde{Y}^J) =$ $X^I \underline{\nabla}_I Y^J$ with respect to the induced coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t(B_m)$, then

we have by (11):
(12)

$$\begin{cases} {}^{cc}\Gamma^{b}_{a c} = {}^{cc}\Gamma^{b}_{a \gamma} = {}^{cc}\Gamma^{b}_{a \overline{\gamma}} = {}^{cc}\Gamma^{b}_{\alpha c} = {}^{cc}\Gamma^{b}_{\alpha \overline{\gamma}} = {}^{cc}\Gamma^{b}_{\overline{\alpha} c} = {}^{cc}\Gamma^{b}_{\overline{\alpha} \gamma} = {}^{cc}\Gamma^{b}_{\overline{\alpha} \overline{\gamma}} = 0, \\ {}^{cc}\Gamma^{b}_{\alpha \gamma} = \Gamma^{b}_{\alpha \gamma}, \\ {}^{cc}\Gamma^{\beta}_{a c} = {}^{cc}\Gamma^{\beta}_{a \gamma} = {}^{cc}\Gamma^{\beta}_{a \overline{\gamma}} = {}^{cc}\Gamma^{\beta}_{\alpha c} = {}^{cc}\Gamma^{\beta}_{\alpha \overline{\gamma}} = {}^{cc}\Gamma^{\beta}_{\overline{\alpha} c} = {}^{cc}\Gamma^{\beta}_{\overline{\alpha} \gamma} = {}^{cc}\Gamma^{\beta}_{\overline{\alpha} \overline{\gamma}} = 0, \\ {}^{cc}\Gamma^{\beta}_{\alpha \gamma} = \Gamma^{\beta}_{\alpha \gamma}, \\ {}^{cc}\Gamma^{\overline{\beta}}_{a c} = {}^{cc}\Gamma^{\overline{\beta}}_{a \gamma} = {}^{cc}\Gamma^{\overline{\beta}}_{a \overline{\gamma}} = {}^{cc}\Gamma^{\overline{\beta}}_{\alpha c} = {}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha} c} = {}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha} \overline{\gamma}} = 0, \\ {}^{cc}\Gamma^{\overline{\beta}}_{\alpha \overline{\gamma}} = \Gamma^{\beta}_{\alpha \gamma}, \\ {}^{cc}\Gamma^{\overline{\beta}}_{\alpha \gamma} = y^{\varepsilon} \partial_{\varepsilon} \Gamma^{\beta}_{\alpha \gamma}, \\ {}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha} \gamma} = \Gamma^{\beta}_{\alpha \gamma}. \end{cases}$$

By (3), (5), (12), we can easily prove that

$${}^{cc}\Gamma^{J'}_{I' K'} = A^{J'}_{J} A^{I}_{I'} A^{K}_{K'} {}^{cc}\Gamma^{J}_{I K} + A^{J'}_{J} A^{J}_{L'} {}^{cc}\Gamma^{L'}_{I' K'},$$

where $I = (a, \alpha, \overline{\alpha})$, $J = (b, \beta, \overline{\beta})$, $K = (c, \gamma, \overline{\gamma})$, $L = (d, \varphi, \overline{\varphi})$.

We can easily verify by means (3) and (5) that the $\Gamma^{J}_{I K}$ defined by (12) determine globally in $t(B_m)$ a projectable linear connection. This projectable linear connection is called the complete lift of the projectable linear connection $\nabla$ to $t(B_m)$ and denoted by ${}^{cc}\nabla$.

**Theorem 2.3.** *If $R$ and $T$ are respectively the curvature and the torsion tensors of $\nabla$, then ${}^{cc}R$ and ${}^{cc}T$ are respectively the curvature and the torsion tensors of ${}^{cc}\nabla$.*

*Proof.* Using Theorem 2.1, we obtain the following formulas:

$$\begin{aligned} {}^{cc}T\left({}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}\right) &= {}^{cc}\left(T\left(\widetilde{X}, \widetilde{Y}\right)\right) \\ &= {}^{cc}\left(\nabla_{\widetilde{X}}\widetilde{Y} - \nabla_{\widetilde{Y}}\widetilde{X} - \left[\widetilde{X}, \widetilde{Y}\right]\right) \\ &= {}^{cc}\nabla_{{}^{cc}\widetilde{X}}({}^{cc}\widetilde{Y}) - {}^{cc}\nabla_{{}^{cc}\widetilde{Y}}({}^{cc}\widetilde{X}) - \left[{}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}\right], \end{aligned}$$

$$\begin{aligned} &{}^{cc}R\left({}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}\right){}^{cc}\widetilde{Z} \\ &= {}^{cc}\left(R\left(\widetilde{X}, \widetilde{Y}\right)\widetilde{Z}\right) \\ &= {}^{cc}\left(\nabla_{\widetilde{X}}\nabla_{\widetilde{Y}}\widetilde{Z} - \nabla_{\widetilde{Y}}\nabla_{\widetilde{X}}\widetilde{Z} - \nabla_{\left[\widetilde{X}, \widetilde{Y}\right]}\widetilde{Z}\right) \\ &= {}^{cc}\nabla_{{}^{cc}\widetilde{X}} {}^{cc}\nabla_{{}^{cc}\widetilde{Y}} {}^{cc}\widetilde{Z} - {}^{cc}\nabla_{{}^{cc}\widetilde{Y}} {}^{cc}\nabla_{{}^{cc}\widetilde{X}} {}^{cc}\widetilde{Z} - {}^{cc}\nabla_{\left[{}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}\right]} {}^{cc}\widetilde{Z} \end{aligned}$$

for any projectable vector fields $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Im^{1}_{0}(M_n)$. Thus, Theorem 2.3 is proved. □

We shall now obtain the components of ${}^{cc}R$ and ${}^{cc}T$. The components $R_{\alpha\gamma}{}^{\beta}_{\varphi}$ of $R$ and $T_{\alpha}{}^{\beta}_{\gamma}$ of $T$ are respectively given by

$$R_{\alpha\gamma}{}^{\beta}_{\varphi} = \partial_{\alpha}\Gamma^{\beta}_{\gamma\varphi} - \partial_{\gamma}\Gamma^{\beta}_{\alpha\varphi} + \Gamma^{\beta}_{\alpha\phi}\Gamma^{\phi}_{\gamma\varphi} - \Gamma^{\beta}_{\gamma\phi}\Gamma^{\phi}_{\alpha\varphi},$$

$$T_{\alpha\gamma}^{\ \ \beta} = \Gamma_{\alpha\gamma}^{\beta} - \Gamma_{\gamma\alpha}^{\beta}.$$

Thus, the components $\widetilde{R}_{IK\ L}^{\ \ \ J}$of ${}^{cc}R$ are given by

$$\begin{cases} \widetilde{R}_{\alpha\gamma\ \varphi}^{\ \ \ b} = R_{\alpha\gamma\ \varphi}^{\ \ \ b}, \\ \widetilde{R}_{\alpha\gamma\ \underline{\varphi}}^{\ \ \ \beta} = R_{\alpha\gamma\ \varphi}^{\ \ \ \beta}, \\ \widetilde{R}_{\alpha\gamma\ \underline{\varphi}}^{\ \ \ \overline{\beta}} = y^{\varepsilon}\partial_{\varepsilon} R_{\alpha\gamma\ \varphi}^{\ \ \ \beta}, \\ \widetilde{R}_{\alpha\overline{\gamma}\ \underline{\varphi}}^{\ \ \ \overline{\beta}} = R_{\alpha\gamma\ \varphi}^{\ \ \ \beta}, \\ \widetilde{R}_{\alpha\gamma\ \overline{\varphi}}^{\ \ \ \overline{\beta}} = R_{\alpha\gamma\ \varphi}^{\ \ \ \beta}, \\ \widetilde{R}_{\overline{\alpha}\gamma\ \varphi}^{\ \ \ \overline{\beta}} = R_{\alpha\gamma\ \varphi}^{\ \ \ \beta}, \end{cases}$$

all the others being zero, and the components $\widetilde{T}_{I\ K}^{\ J}$ of ${}^{cc}T$ are given by

$$\begin{cases} \widetilde{T}_{\alpha\gamma}^{\ \ b} = T_{\alpha\gamma}^{\ \ b}, \\ \widetilde{T}_{\alpha\underline{\gamma}}^{\ \ \beta} = T_{\alpha\gamma}^{\ \ \beta}, \\ \widetilde{T}_{\alpha\overline{\gamma}}^{\ \ \overline{\beta}} = T_{\alpha\gamma}^{\ \ \beta}, \\ \widetilde{T}_{\alpha\underline{\gamma}}^{\ \ \overline{\beta}} = y^{\varepsilon}\partial_{\varepsilon} T_{\alpha\gamma}^{\ \ \beta}, \\ \widetilde{T}_{\overline{\alpha}\gamma}^{\ \ \overline{\beta}} = T_{\alpha\gamma}^{\ \ \beta}, \end{cases} \tag{13}$$

all the others being zero, with respect to the induced coordinates in $t(B_m)$. Where $I = (a, \alpha, \overline{\alpha})$, $J = (b, \beta, \overline{\beta})$, $K = (c, \gamma, \overline{\gamma})$, $L = (d, \varphi, \overline{\varphi})$.

## 3. Horizontal Lifts of vector fields

Firstly, we will give some preliminary definitions. For any $F \in \Im_1^1(B_m)$, if we take account of (5), we can easily prove that $(\gamma F)' = \bar{A}(\gamma F)$, where $\gamma F$ is a vector field defined by

$$\gamma F = (\gamma F^I) = \begin{pmatrix} 0 \\ 0 \\ y^{\varepsilon} F_{\varepsilon}^{\alpha} \end{pmatrix} \tag{14}$$

with respect to the coordinates $(x^a, x^{\alpha}, x^{\overline{\alpha}})$.

Let now $\widetilde{X} \in \Im_0^1(M_n)$ be a projectable vector field on $M_n$ with projection $X \in \Im_0^1(B_m)$ [9]. Then we define the horizontal lift ${}^{HH}\widetilde{X}$ of $\widetilde{X}$ by

$${}^{HH}\widetilde{X} = {}^{cc}\widetilde{X} - \gamma(\nabla\widetilde{X}) \tag{15}$$

on $t(M_n)$. Where $\nabla$ is a projectable symmetric linear connection in a differentiable manifold $B_m$. Then, remembering that ${}^{cc}\widetilde{X}$ and $\gamma(\nabla\widetilde{X})$ have, respectively, local componenets

$${}^{cc}\widetilde{X} = \left({}^{cc}\widetilde{X}^I\right) = \begin{pmatrix} \widetilde{X}^a \\ X^{\alpha} \\ y^{\varepsilon}\partial_{\varepsilon} X^{\alpha} \end{pmatrix}, \gamma(\nabla\widetilde{X}) = \left(\gamma(\nabla\widetilde{X})^I\right) = \begin{pmatrix} 0 \\ 0 \\ y^{\varepsilon}\nabla_{\varepsilon} X^{\alpha} \end{pmatrix}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t(B_m)$. $\nabla_\alpha X^\varepsilon$ being the covariant derivative of $X^\varepsilon$, i.e.,

$$(\nabla_\alpha X^\varepsilon) = \partial_\alpha X^\varepsilon + X^\beta \Gamma^\varepsilon_{\beta\alpha}.$$

We find that the horizontal lift ${}^{HH}\widetilde{X}$ of $\widetilde{X} \in \Im^1_0(M_n)$ to semi-tangent bundle $t(B_m)$ has components

$$ {}^{HH}X = \left({}^{HH}X^I\right) = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ -\Gamma^\alpha_\beta X^\beta \end{pmatrix} \tag{16}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ in $t(B_m)$. Where

$$\Gamma^\alpha_\beta = y^\varepsilon \Gamma^\alpha_{\varepsilon\,\beta}.$$

**Theorem 3.1.** *Let $X \in \Im^1_0(B_m)$. If $f \in \Im^0_0(B_m)$, then*

$(i)$ ${}^{cc}\nabla_{{}^{vv}X}({}^{vv}f) = 0,$
$(ii)$ ${}^{cc}\nabla_{{}^{vv}X}({}^{cc}f) = {}^{vv}(\nabla_X f).$

*Proof.* $(i)$ If $f \in \Im^0_0(B_m)$ and $X \in \Im^1_0(B_m)$, then we obtain by (6) and (8),

$$\begin{aligned} {}^{cc}\nabla_{{}^{vv}X}({}^{vv}f) &= {}^{vv}X^{I\,cc}\nabla_I({}^{vv}f) \\ &= {}^{vv}X^{a\,cc}\nabla_a({}^{vv}f) + {}^{vv}X^{\alpha\,cc}\nabla_\alpha({}^{vv}f) + {}^{vv}X^{\overline{\alpha}\,cc}\nabla_{\overline{\alpha}}({}^{vv}f) \\ &= \underbrace{{}^{vv}X^a}_{0}\partial_a f + \underbrace{{}^{vv}X^\alpha}_{0}\partial_\alpha f + {}^{vv}X^{\overline{\alpha}}\underbrace{\partial_{\overline{\alpha}} f}_{0} \\ &= 0. \end{aligned}$$

$(ii)$ If $f \in \Im^0_0(B_m)$ and $X \in \Im^1_0(B_m)$, then we have by (7) and (8),

$$\begin{aligned} {}^{cc}\nabla_{{}^{vv}X}({}^{cc}f) &= {}^{vv}X\,({}^{cc}f) = {}^{vv}X^I\partial_I({}^{cc}f) \\ &= {}^{vv}X^a\partial_a({}^{cc}f) + {}^{vv}X^\alpha\partial_\alpha({}^{cc}f) + {}^{vv}X^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{cc}f) \\ &= \underbrace{{}^{vv}X^a}_{0}\partial_a({}^{cc}f) + \underbrace{{}^{vv}X^\alpha}_{0}\partial_\alpha({}^{cc}f) + \underbrace{{}^{vv}X^{\overline{\alpha}}}_{X^\alpha}\partial_{\overline{\alpha}}({}^{cc}f) \\ &= X^\alpha \partial_{\overline{\alpha}} y^\beta \partial_\beta f = X^\alpha \delta^\beta_\alpha \partial_\beta f = X^\alpha \partial_\alpha f \\ &= {}^{vv}(Xf) = {}^{vv}(\nabla_X f). \end{aligned}$$

□

**Theorem 3.2.** *Let $\widetilde{X}$ be a projectable vector field on $M_n$. If $f \in \Im^0_0(B_m)$, then*

$(i)$ ${}^{cc}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}f) = {}^{vv}\left(\nabla_{\widetilde{X}} f\right),$
$(ii)$ ${}^{cc}\nabla_{{}^{cc}\widetilde{X}}({}^{cc}f) = {}^{cc}\left(\nabla_{\widetilde{X}} f\right).$

*Proof.* (*i*) If $f \in \Im_0^0(B_m)$ and $\widetilde{X}$ is a projectable vector field on $M_n$ with projection $X \in \Im_0^1(B_m)$, then, by (6) and (10), we have

$$\begin{aligned}
{}^{cc}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}f) &= {}^{cc}\widetilde{X}\,({}^{vv}f) = {}^{cc}\,\widetilde{X}^I\partial_I({}^{vv}f) \\
&= {}^{cc}\widetilde{X}^a\partial_a({}^{vv}f) + {}^{cc}\,\widetilde{X}^\alpha\partial_\alpha({}^{vv}f) + {}^{cc}\,\widetilde{X}^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{vv}f) \\
&= \widetilde{X}^a\,\underbrace{\partial_a f}_{0} + X^\alpha\partial_\alpha f + y^\sigma\partial_\sigma X^\alpha\,\underbrace{\partial_{\overline{\alpha}} f}_{0} \\
&= X^\alpha\partial_\alpha f = {}^{vv}\left(\widetilde{X}f\right) = {}^{vv}\left(\nabla_{\widetilde{X}}f\right).
\end{aligned}$$

(*ii*) If $f \in \Im_0^0(B_m)$ and $\widetilde{X}$ is a projectable vector field on $M_n$ with projection $X \in \Im_0^1(B_m)$, then we have by (7) and (10),

$$\begin{aligned}
{}^{cc}\nabla_{{}^{cc}\widetilde{X}}({}^{cc}f) &= {}^{cc}\widetilde{X}\,({}^{cc}f) \\
&= {}^{cc}\widetilde{X}^I\partial_I({}^{cc}f) \\
&= {}^{cc}\widetilde{X}^a\partial_a({}^{cc}f) + {}^{cc}\,\widetilde{X}^\alpha\partial_\alpha({}^{cc}f) + {}^{cc}\,\widetilde{X}^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{cc}f) \\
&= \widetilde{X}^a\partial_a\left(y^\beta\partial_\beta f\right) + X^\alpha\partial_\alpha\left(y^\beta\partial_\beta f\right) + y^\beta\partial_\beta X^\alpha\partial_{\overline{\alpha}}\left(y^\sigma\partial_\sigma f\right) \\
&= \widetilde{X}^a\partial_a y^\beta\partial_\beta f + X^\alpha\partial_\alpha\left(y^\beta\partial_\beta f\right) + y^\beta\partial_\beta X^\alpha\partial_{\overline{\alpha}}\left(y^\sigma\partial_\sigma f\right) \\
&= y^\beta(\widetilde{X}^a\partial_a + X^\alpha\partial_\alpha)\partial_\beta f + y^\beta\partial_\beta X^\alpha\delta_\alpha^\sigma\partial_\sigma f \\
&= y^\beta\partial_\beta(\widetilde{X}^a\partial_a + X^\alpha\partial_\alpha)f + \left(y^\beta\partial_\beta X^\alpha\right)\partial_\alpha f \\
&= y^\beta\partial_\beta(\widetilde{X}^\alpha\partial_\alpha f) = {}^{cc}\left(\widetilde{X}f\right) \\
&= {}^{cc}\left(\nabla_{\widetilde{X}}f\right).
\end{aligned}$$

□

**Theorem 3.3.** *Let* $X, Y \in \Im_0^1(B_m)$. *If* $f \in \Im_0^0(B_m)$, *then*

$${}^{cc}\nabla_{{}^{vv}X}({}^{vv}Y) = 0.$$

*Proof.* If $X,Y\in\Im_0^1(B_m)$, from (8) and (12), we have

$$\begin{aligned}
&{}^{cc}\nabla_{{}^{vv}X}({}^{vv}Y)\\
&=\begin{pmatrix} {}^{vv}X^{I\,cc}\nabla_I({}^{vv}Y^b)\\ {}^{vv}X^{I\,cc}\nabla_I({}^{vv}Y^\beta)\\ {}^{vv}X^{I\,cc}\nabla_I({}^{vv}Y^{\overline{\beta}})\end{pmatrix}\\
&=\begin{pmatrix} 0\\ 0\\ \underbrace{{}^{vv}X^a}_{0}\,{}^{cc}\nabla_a({}^{vv}Y^{\overline{\beta}})+\underbrace{{}^{vv}X^\alpha}_{0}\,{}^{cc}\nabla_\alpha({}^{vv}Y^{\overline{\beta}})+{}^{vv}X^{\overline{\alpha}\,cc}\nabla_{\overline{\alpha}}({}^{vv}Y^{\overline{\beta}})\end{pmatrix}\\
&=\begin{pmatrix} 0\\ 0\\ X^\alpha(\partial_{\overline{\alpha}}({}^{vv}Y^{\overline{\beta}})+{}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}K}({}^{vv}Y^K)\end{pmatrix}\\
&=\begin{pmatrix} 0\\ 0\\ X^\alpha(\underbrace{\partial_{\overline{\alpha}}Y^\beta}_{0}+{}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}c}\underbrace{{}^{vv}Y^c}_{0}+{}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}\gamma}\underbrace{{}^{vv}Y^\gamma}_{0}+\underbrace{{}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}\overline{\gamma}}})({}^{vv}Y^{\overline{\gamma}})\end{pmatrix}\\
&=\begin{pmatrix}0\\0\\0\end{pmatrix},
\end{aligned}$$

which prove Theorem 3.3. □

**Theorem 3.4.** *Let $\widetilde{X}$ be a projectable vector field on $M_n$ with projections $X$ on $B_m$. If $Y\in\Im_0^1(B_m)$, then*

$$ {}^{cc}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}Y)={}^{vv}(\nabla_X Y).$$

*Proof.* If $Y \in \Im_0^1(B_m)$, and $\widetilde{X}$ is a projectable vector field on $M_n$, then using (8), (10) and (12) we can find

$$
\begin{aligned}
&{}^{cc}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}Y)\\
&= \begin{pmatrix} {}^{cc}X^{I\,cc}\nabla_I({}^{vv}Y^b) \\ {}^{cc}X^{I\,cc}\nabla_I({}^{vv}Y^{\beta}) \\ {}^{cc}X^{I\,cc}\nabla_I({}^{vv}Y^{\overline{\beta}}) \end{pmatrix}\\
&= \begin{pmatrix} 0 \\ 0 \\ {}^{cc}X^{a\,cc}\nabla_a({}^{vv}Y^{\overline{\beta}}) +^{cc} X^{\alpha\,cc}\nabla_\alpha({}^{vv}Y^{\overline{\beta}}) +^{cc} X^{\overline{\alpha}\,cc}\nabla_{\overline{\alpha}}({}^{vv}Y^{\overline{\beta}}) \end{pmatrix}\\
&= \begin{pmatrix} 0 \\ 0 \\ X^{\alpha}(\partial_\alpha({}^{vv}Y^{\overline{\beta}})) + (y^{\varepsilon}\partial_{\varepsilon}X^{\alpha})\,(\partial_{\overline{\alpha}}({}^{vv}Y^{\beta}) +^{cc}\Gamma^{\beta}_{\overline{\alpha}K}({}^{vv}Y^{K})) \end{pmatrix}\\
&= \begin{pmatrix} 0 \\ 0 \\ X^{\alpha}(\partial_\alpha({}^{vv}Y^{\overline{\beta}}) + \Gamma^{\beta}_{\alpha\gamma}({}^{vv}Y^{\gamma})) \end{pmatrix}\\
&= \begin{pmatrix} 0 \\ 0 \\ (\nabla_X Y)^{\beta} \end{pmatrix}\\
&=^{vv}(\nabla_X Y)\,.
\end{aligned}
$$

Thus, we have ${}^{cc}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}Y) =^{vv} (\nabla_X Y)$. Where $K = (c, \gamma, \overline{\gamma})$. □

Let there be given a projectable linear connection $\nabla$ and a projectable vector field on $M_n$ with projection $X \in \Im_0^1(B_m)$. Then the Lie derivative $L_{\widetilde{X}}\nabla$ with respect to $\widetilde{X}$ is, by definition, an element of $\Im_2^1(B_m)$ such that

$$(L_{\widetilde{X}}\nabla)(\widetilde{Y}, \widetilde{Z}) = L_{\widetilde{X}}(\nabla_{\widetilde{Y}}\widetilde{Z}) - \nabla_{\widetilde{Y}}(L_{\widetilde{X}}\widetilde{Z}) - \nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z} \tag{17}$$

for any projectable vector fields $\widetilde{Y}, \widetilde{Z} \in \Im_0^1(M_n)$.

A projectable vector field $\widetilde{X} \in \Im_0^1(M_n)$ [13] with components $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$ is said to be an infinitesimal linear (resp. affine) transformation ([[16], p.67], [12]) in an $m-$dimensional manifold $B_m$ with projectable linear connection $\nabla$, if $L_{\widetilde{X}}\nabla = 0$ (see (17)).

**Theorem 3.5.** *Let $\nabla$ be a projectable linear connection on $B_m$. Then,*

$$(L_{{}^{cc}\widetilde{X}}{}^{cc}\nabla)({}^{cc}\widetilde{Y}, {}^{cc}\widetilde{Z}) =^{cc} \left((L_{\widetilde{X}}\nabla)(\widetilde{Y}, \widetilde{Z})\right)$$

*for any projectable vector fields $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Im_0^1(M_n)$.*

*Proof.* Substituting (11) and Theorem 2.1 in (17), we have

$$\begin{aligned}(L_{\widetilde{X}}\nabla)(\widetilde{Y},\widetilde{Z}) &= L_{\widetilde{X}}(\nabla_{\widetilde{Y}}\widetilde{Z}) - \nabla_{\widetilde{Y}}(L_{\widetilde{X}}\widetilde{Z}) - \nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}\\ (L_{^{cc}\widetilde{X}}{}^{cc}\nabla)(^{cc}\widetilde{Y},^{cc}\widetilde{Z}) &= L_{^{cc}\widetilde{X}}(^{cc}\nabla_{^{cc}\widetilde{Y}}{}^{cc}\widetilde{Z}) -^{cc}\nabla_{^{cc}\widetilde{Y}}(L_{^{cc}\widetilde{X}}{}^{cc}\widetilde{Z}) -^{cc}\nabla_{[^{cc}\widetilde{X},^{cc}\widetilde{Y}]}{}^{cc}\widetilde{Z}\\ &= L_{^{cc}\widetilde{X}}(^{cc}(\nabla_{\widetilde{Y}}\widetilde{Z})) -^{cc}\nabla_{^{cc}\widetilde{Y}}\left(^{cc}(L_{\widetilde{X}}\widetilde{Z})\right) -^{cc}\nabla_{^{cc}[\widetilde{X},\widetilde{Y}]}{}^{cc}\widetilde{Z}\\ &= {}^{cc}\left(L_{\widetilde{X}}(\nabla_{\widetilde{Y}}\widetilde{Z})\right) -^{cc}\left(\nabla_{\widetilde{Y}}(L_{\widetilde{X}}\widetilde{Z})\right) -^{cc}\left(\nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}\right)\\ &= {}^{cc}\left(L_{\widetilde{X}}(\nabla_{\widetilde{Y}}\widetilde{Z}) - \nabla_{\widetilde{Y}}(L_{\widetilde{X}}\widetilde{Z}) - \nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}\right)\\ &= {}^{cc}\left((L_{\widetilde{X}}\nabla)(\widetilde{Y},\widetilde{Z})\right),\end{aligned}$$

which is the proof of Theorem 3.5. □

Thus we have

**Theorem 3.6.** *If $\widetilde{X}$ is an infinitesimal transformation of $t(B_m)$ with projectable linear connection $\nabla$, then $^{cc}\widetilde{X}$ is an infinitesimal transformation of $t(B_m)$ with projectable linear connection $^{cc}\nabla$.*

**Theorem 3.7.** *Let $\widetilde{X}$ and $\widetilde{Y}$ be projectable vector fields on $M_n$ with projection $X \in \Im^1_0(B_m)$ and $Y \in \Im^1_0(B_m)$. We have:*

(*i*) $^{cc}\nabla_{^{vv}X}(^{HH}\widetilde{Y}) = 0$,

(*ii*) $^{cc}\nabla_{^{HH}\widetilde{X}}(^{vv}Y) =^{vv}(\nabla_X Y)$.

*Proof.* (*i*) If $\widetilde{X},\widetilde{Y} \in \Im^1_0(M_n)$ and $\begin{pmatrix}\left(^{cc}\nabla_{^{vv}X}(^{HH}\widetilde{Y})\right)^b\\ \left(^{cc}\nabla_{^{vv}X}(^{HH}\widetilde{Y})\right)^\beta\\ \left(^{cc}\nabla_{^{vv}X}(^{HH}\widetilde{Y})\right)^{\overline{\beta}}\end{pmatrix}$ are the components of $^{cc}\nabla_{^{vv}X}(^{HH}\widetilde{Y})$ with respect to the coordinates $(x^b,x^\beta,x^{\overline{\beta}})$ on $t(B_m)$, then we have

$$\left(^{cc}\nabla_{^{vv}X}(^{HH}\widetilde{Y})\right)^J =^{vv} X^{a\,cc}\nabla_a(^{HH}\widetilde{Y}^J)+^{vv}X^{\alpha\,cc}\nabla_\alpha(^{HH}\widetilde{Y}^J)+^{vv}X^{\overline{\alpha}\,cc}\nabla_{\overline{\alpha}}(^{HH}\widetilde{Y}^J).$$

Firstly, for $J=b$, we have

$$\begin{aligned}\left(^{cc}\nabla_{^{vv}X}(^{HH}\widetilde{Y})\right)^b &= \underbrace{^{vv}X^a}_{0}\,{}^{cc}\nabla_a(^{HH}\widetilde{Y}^b) + \underbrace{^{vv}X^\alpha}_{0}\,{}^{cc}\nabla_\alpha(^{HH}\widetilde{Y}^b) +^{vv} X^{\overline{\alpha}\,cc}\nabla_{\overline{\alpha}}(^{HH}\widetilde{Y}^b)\\ &= X^a(\underbrace{\partial_{\overline{\alpha}}Y^b}_{0} + \underbrace{^{cc}\Gamma^b_{\overline{\alpha}c}}_{0}(^{HH}\widetilde{Y}^c)\\ &\quad + \underbrace{^{cc}\Gamma^b_{\overline{\alpha}\gamma}}_{0}(^{HH}\widetilde{Y}^\gamma) + \underbrace{^{cc}\Gamma^b_{\overline{\alpha}\overline{\gamma}}}_{0}(^{HH}\widetilde{Y}^{\overline{\gamma}}))\\ &= 0\end{aligned}$$

by virtue of (8), (12) and (16). Secondly, for $J=\beta$, we have

$$\begin{aligned}\left({}^{cc}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y})\right)^{\beta} &= \underbrace{{}^{vv}X^{a}}_{0}\,{}^{cc}\nabla_{a}({}^{HH}\widetilde{Y}^{\beta})+\underbrace{{}^{vv}X^{\alpha}}_{0}\,{}^{cc}\nabla_{\alpha}({}^{HH}\widetilde{Y}^{\beta})+{}^{vv}X^{\overline{\alpha}}\,{}^{cc}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y}^{\beta})\\ &= X^{a}(\underbrace{\partial_{\overline{\alpha}}Y^{\beta}}_{0}+\underbrace{{}^{cc}\Gamma^{\beta}_{\overline{\alpha}c}}_{0}({}^{HH}\widetilde{Y}^{c})\\ &\quad+\underbrace{{}^{cc}\Gamma^{\beta}_{\overline{\alpha}\gamma}}_{0}({}^{HH}\widetilde{Y}^{\gamma})+\underbrace{{}^{cc}\Gamma^{\beta}_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{HH}\widetilde{Y}^{\overline{\gamma}}))\\ &= 0\end{aligned}$$

by virtue of (8), (12) and (16). Thirdly, for $J=\overline{\beta}$, then we have

$$\begin{aligned}\left({}^{cc}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y})\right)^{\overline{\beta}} &= \underbrace{{}^{vv}X^{a}}_{0}\,{}^{cc}\nabla_{a}({}^{HH}\widetilde{Y}^{\overline{\beta}})+\underbrace{{}^{vv}X^{\alpha}}_{0}\,{}^{cc}\nabla_{\alpha}({}^{HH}\widetilde{Y}^{\overline{\beta}})+{}^{vv}X^{\overline{\alpha}}\,{}^{cc}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y}^{\overline{\beta}})\\ &= X^{a}(\partial_{\overline{\alpha}}({}^{HH}Y^{\overline{\beta}})+\underbrace{{}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}c}}_{0}({}^{HH}\widetilde{Y}^{c})\\ &\quad+\underbrace{{}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}\gamma}}_{\Gamma^{\beta}_{\alpha\gamma}}(\underbrace{{}^{HH}\widetilde{Y}^{\gamma}}_{Y^{\gamma}})+\underbrace{{}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{HH}\widetilde{Y}^{\overline{\gamma}}))\\ &= X^{a}\left(-\underbrace{\partial_{\overline{\alpha}}y^{\varepsilon}}_{\delta^{\varepsilon}_{\alpha}}\Gamma^{\beta}_{\varepsilon\gamma}Y^{\gamma}+\Gamma^{\beta}_{\alpha\gamma}Y^{\gamma}\right)\\ &= X^{a}\left(-\Gamma^{\beta}_{\alpha\gamma}Y^{\gamma}+\Gamma^{\beta}_{\alpha\gamma}Y^{\gamma}\right)\\ &= 0\end{aligned}$$

by virtue of (8), (12) and (16). Thus $(i)$ of Theorem 3.7 is proved.

$(ii)$ If $\widetilde{X},\widetilde{Y}\in\Im^{1}_{0}(M_n)$ and $\begin{pmatrix}\left({}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)^{b}\\ \left({}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)^{\beta}\\ \left({}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)^{\overline{\beta}}\end{pmatrix}$ are the components of $\left({}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)$ with respect to the coordinates $(x^{b},x^{\beta},x^{\overline{\beta}})$ on $t(B_m)$, then we have

$$\left({}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)^{J}={}^{HH}\widetilde{X}^{a}\,{}^{cc}\nabla_{a}({}^{vv}Y^{J})+{}^{HH}\widetilde{X}^{\alpha}\,{}^{cc}\nabla_{\alpha}({}^{vv}Y^{J})+{}^{HH}\widetilde{X}^{\overline{\alpha}}\,{}^{cc}\nabla_{\overline{\alpha}}({}^{vv}Y^{J}).$$

Firstly, for $J = b$, we have

$$\begin{aligned}
\left({}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)^{b} &= {}^{HH}\widetilde{X}^{a\,cc}\nabla_{a}({}^{vv}Y^{b}) +{}^{HH}\widetilde{X}^{\alpha\,cc}\nabla_{\alpha}({}^{vv}Y^{b}) +{}^{HH}\widetilde{X}^{\overline{\alpha}\,cc}\nabla_{\overline{\alpha}}({}^{vv}Y^{b}) \\
&= X^{a}(\partial_{a}(\underbrace{{}^{vv}Y^{b}}_{0}) +{}^{cc}\Gamma^{b}_{ac}(\underbrace{{}^{vv}Y^{c}}_{0}) \\
&\quad +{}^{cc}\Gamma^{b}_{a\gamma}(\underbrace{{}^{vv}Y^{\gamma}}_{0}) + \underbrace{{}^{cc}\Gamma^{b}_{a\overline{\gamma}}}_{0}({}^{vv}Y^{\overline{\gamma}})) \\
&\quad +X^{\alpha}(\partial_{\alpha}(\underbrace{{}^{vv}Y^{b}}_{0}) +{}^{cc}\Gamma^{b}_{\alpha c}(\underbrace{{}^{vv}Y^{c}}_{0}) \\
&\quad +{}^{cc}\Gamma^{b}_{\alpha\gamma}(\underbrace{{}^{vv}Y^{\gamma}}_{0}) + \underbrace{{}^{cc}\Gamma^{b}_{\alpha\overline{\gamma}}}_{0}({}^{vv}Y^{\overline{\gamma}})) \\
&\quad +{}^{HH}\widetilde{X}^{\overline{\alpha}}(\partial_{\overline{\alpha}}(\underbrace{{}^{vv}Y^{b}}_{0}) +{}^{cc}\Gamma^{b}_{\overline{\alpha}c}(\underbrace{{}^{vv}Y^{c}}_{0}) \\
&\quad +{}^{cc}\Gamma^{b}_{\overline{\alpha}\gamma}(\underbrace{{}^{vv}Y^{\gamma}}_{0}) + \underbrace{{}^{cc}\Gamma^{b}_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{vv}Y^{\overline{\gamma}})) \\
&= 0
\end{aligned}$$

by virtue of (8), (12) and (16). Secondly, for $J = \beta$, we have

$$\begin{aligned}
\left({}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)^{\beta} &= {}^{HH}\widetilde{X}^{a\,cc}\nabla_{a}(\underbrace{{}^{vv}Y^{\beta}}_{0}) +{}^{HH}\widetilde{X}^{\alpha\,cc}\nabla_{\alpha}(\underbrace{{}^{vv}Y^{\beta}}_{0}) +{}^{HH}\widetilde{X}^{\overline{\alpha}\,cc}\nabla_{\overline{\alpha}}(\underbrace{{}^{vv}Y^{\beta}}_{0}) \\
&= X^{a}(\partial_{a}(\underbrace{{}^{vv}Y^{\beta}}_{0}) +{}^{cc}\Gamma^{\beta}_{ac}(\underbrace{{}^{vv}Y^{c}}_{0}) \\
&\quad +{}^{cc}\Gamma^{\beta}_{a\gamma}(\underbrace{{}^{vv}Y^{\gamma}}_{0}) + \underbrace{{}^{cc}\Gamma^{\beta}_{a\overline{\gamma}}}_{0}({}^{vv}Y^{\overline{\gamma}})) \\
&\quad +X^{\alpha}(\partial_{\alpha}(\underbrace{{}^{vv}Y^{\beta}}_{0}) +{}^{cc}\Gamma^{\beta}_{\alpha c}(\underbrace{{}^{vv}Y^{c}}_{0}) \\
&\quad +{}^{cc}\Gamma^{\beta}_{\alpha\gamma}(\underbrace{{}^{vv}Y^{\gamma}}_{0}) + \underbrace{{}^{cc}\Gamma^{\beta}_{\alpha\overline{\gamma}}}_{0}({}^{vv}Y^{\overline{\gamma}}) \\
&\quad +{}^{HH}\widetilde{X}^{\overline{\alpha}}(\partial_{\overline{\alpha}}(\underbrace{{}^{vv}Y^{\beta}}_{0}) +{}^{cc}\Gamma^{\beta}_{\overline{\alpha}c}(\underbrace{{}^{vv}Y^{c}}_{0}) \\
&\quad +{}^{cc}\Gamma^{\beta}_{\overline{\alpha}\gamma}(\underbrace{{}^{vv}Y^{\gamma}}_{0}) + \underbrace{{}^{cc}\Gamma^{\beta}_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{vv}Y^{\overline{\gamma}})) \\
&= 0
\end{aligned}$$

by virtue of (8), (12) and (16). Thirdly, for $J = \overline{\beta}$, then we have

$$\begin{aligned}
\left({}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)^{\overline{\beta}} &= {}^{HH}\widetilde{X}^{a\,cc}\nabla_a({}^{vv}Y^{\overline{\beta}}) + {}^{HH}\widetilde{X}^{\alpha\,cc}\nabla_\alpha({}^{vv}Y^{\overline{\beta}}) + {}^{HH}\widetilde{X}^{\overline{\alpha}\,cc}\nabla_{\overline{\alpha}}({}^{vv}Y^{\overline{\beta}}) \\
&= X^a(\underbrace{\partial_a(Y^\beta)}_{0} + {}^{cc}\Gamma^{\overline{\beta}}_{a\,c}(\underbrace{{}^{vv}Y^c}_{0}) \\
&\quad + {}^{cc}\Gamma^{\overline{\beta}}_{a\,\gamma}(\underbrace{{}^{vv}Y^\gamma}_{0}) + \underbrace{{}^{cc}\Gamma^{\overline{\beta}}_{a\,\overline{\gamma}}}_{0}({}^{vv}Y^{\overline{\gamma}})) \\
&\quad + X^\alpha(\partial_\alpha(Y^\beta) + {}^{cc}\Gamma^{\overline{\beta}}_{\alpha c}(\underbrace{{}^{vv}Y^c}_{0}) \\
&\quad + {}^{cc}\Gamma^{\overline{\beta}}_{\alpha\gamma}(\underbrace{{}^{vv}Y^\gamma}_{0}) + \underbrace{{}^{cc}\Gamma^{\overline{\beta}}_{\alpha\overline{\gamma}}}_{\Gamma^\beta_{\alpha\gamma}}(\underbrace{{}^{vv}Y^{\overline{\gamma}}}_{Y^\gamma})) \\
&\quad + X^\alpha(\underbrace{\partial_{\overline{\alpha}}(Y^\beta)}_{0} + {}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}c}(\underbrace{{}^{vv}Y^c}_{0}) \\
&\quad + {}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}\gamma}(\underbrace{{}^{vv}Y^\gamma}_{0}) + \underbrace{{}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{vv}Y^{\overline{\gamma}})) \\
&= X^\alpha(\partial_\alpha Y^\beta + \Gamma^\beta_{\alpha\gamma}Y^\gamma) \\
&= (\nabla_X Y)^\beta
\end{aligned}$$

by virtue of (8), (12) and (16). On the other hand, we know that ${}^{vv}(\nabla_X Y)$ have the components

$$ {}^{vv}(\nabla_X Y) = \begin{pmatrix} 0 \\ 0 \\ (\nabla_X Y)^\beta \end{pmatrix}$$

with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t(B_m)$. Thus, we have

$$ {}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y) = {}^{vv}(\nabla_X Y)$$

in $t(B_m)$. □

Let there be given a projectable linear connection $\nabla$ in $B_m$. Taking account of the definition (15) of the horizontal lift, we have from (13):

$$ {}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y}) = {}^{HH}(\nabla_X Y) + \gamma(R(\ ,X)Y),$$

for any $\widetilde{X}, \widetilde{Y} \in \Im^1_0(M_n)$. Where $R(\ ,X)Y$ denotes a tensor field $F$ of type $(1,1)$ in $B_m$ such that $F(Z) = R(Z,X)Y$ for any $Z \in \Im^1_0(B_m)$.

*Proof.* If $\widetilde{X}, \widetilde{Y} \in \Im^1_0(M_n)$ and $\begin{pmatrix} {}^{HH}\widetilde{X}^I(\partial_I({}^{HH}\widetilde{Y}^b) + {}^{cc}\Gamma^b_{IK}({}^{HH}\widetilde{Y}^K)) \\ {}^{HH}\widetilde{X}^I(\partial_I({}^{HH}\widetilde{Y}^\beta) + {}^{cc}\Gamma^\beta_{IK}({}^{HH}\widetilde{Y}^K)) \\ {}^{HH}\widetilde{X}^I(\partial_I({}^{HH}\widetilde{Y}^{\overline{\beta}}) + {}^{cc}\Gamma^{\overline{\beta}}_{IK}({}^{HH}\widetilde{Y}^K)) \end{pmatrix}$ are the components of ${}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y})$ with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$

on $t(B_m)$, then we have

$$^{HH}(\nabla_X Y)^J = \left({}^{cc}\nabla_{^{HH}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^J.$$

Firstly, for $J = b$, we have

$$\begin{aligned}
^{HH}(\nabla_X Y)^b &= \left({}^{cc}\nabla_{^{HH}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^b \\
&= {}^{HH}\widetilde{X}^I(\partial_I({}^{HH}\widetilde{Y}^b) + {}^{cc}\Gamma^b_{IK}({}^{HH}\widetilde{Y}^K)) \\
&= {}^{HH}\widetilde{X}^a(\partial_a({}^{HH}\widetilde{Y}^b) + {}^{cc}\Gamma^b_{aK}({}^{HH}\widetilde{Y}^K)) + {}^{HH}\widetilde{X}^\alpha(\partial_\alpha({}^{HH}\widetilde{Y}^b) \\
&\quad + {}^{cc}\Gamma^b_{\alpha K}({}^{HH}\widetilde{Y}^K)) + {}^{HH}\widetilde{X}^{\overline{\alpha}}(\partial_{\overline{\alpha}}({}^{HH}\widetilde{Y}^b) + {}^{cc}\Gamma^b_{\overline{\alpha}K}({}^{HH}\widetilde{Y}^K)) \\
&= {}^{HH}\widetilde{X}^a(\underbrace{\partial_a(\widetilde{Y}^b)}_{0} + \underbrace{{}^{cc}\Gamma^b_{ac}}_{0}({}^{HH}\widetilde{Y}^c) + \underbrace{{}^{cc}\Gamma^b_{a\gamma}}_{0}({}^{HH}\widetilde{Y}^\gamma) + \underbrace{{}^{cc}\Gamma^b_{a\overline{\gamma}}}_{0}({}^{HH}\widetilde{Y}^{\overline{\gamma}})) \\
&\quad + {}^{HH}\widetilde{X}^\alpha(\partial_\alpha(\widetilde{Y}^b) + \underbrace{{}^{cc}\Gamma^b_{\alpha c}}_{0}({}^{HH}\widetilde{Y}^c) + {}^{cc}\Gamma^b_{\alpha\gamma}({}^{HH}\widetilde{Y}^\gamma) + \underbrace{{}^{cc}\Gamma^b_{\alpha\overline{\gamma}}}_{0}({}^{HH}\widetilde{Y}^{\overline{\gamma}})) \\
&\quad + {}^{HH}\widetilde{X}^{\overline{\alpha}}(\underbrace{\partial_{\overline{\alpha}}(\widetilde{Y}^b)}_{0} + \underbrace{{}^{cc}\Gamma^b_{\overline{\alpha}c}}_{0}({}^{HH}\widetilde{Y}^c) + \underbrace{{}^{cc}\Gamma^b_{\overline{\alpha}\gamma}}_{0}({}^{HH}\widetilde{Y}^\gamma) + \underbrace{{}^{cc}\Gamma^b_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{HH}\widetilde{Y}^{\overline{\gamma}})) \\
&= X^\alpha(\partial_\alpha\widetilde{Y}^b + \Gamma^b_{\alpha\gamma}Y^\gamma) \\
&= {}^{HH}(\nabla_X Y)^b
\end{aligned}$$

by virtue of (12) and (16). Secondly, for $J = \beta$, we have

$$\begin{aligned}
^{HH}(\nabla_X Y)^\beta &= \left({}^{cc}\nabla_{^{HH}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^\beta \\
&= {}^{HH}\widetilde{X}^I(\partial_I({}^{HH}\widetilde{Y}^\beta) + {}^{cc}\Gamma^\beta_{IK}({}^{HH}\widetilde{Y}^K)) \\
&= {}^{HH}\widetilde{X}^a(\partial_a({}^{HH}\widetilde{Y}^\beta) + {}^{cc}\Gamma^\beta_{aK}({}^{HH}\widetilde{Y}^K)) + {}^{HH}\widetilde{X}^\alpha(\partial_\alpha({}^{HH}\widetilde{Y}^\beta) \\
&\quad + {}^{cc}\Gamma^\beta_{\alpha K}({}^{HH}\widetilde{Y}^K)) + {}^{HH}\widetilde{X}^{\overline{\alpha}}(\partial_{\overline{\alpha}}({}^{HH}\widetilde{Y}^\beta) + {}^{cc}\Gamma^\beta_{\overline{\alpha}K}({}^{HH}\widetilde{Y}^K)) \\
&= {}^{HH}\widetilde{X}^a(\underbrace{\partial_a Y^\beta}_{0} + \underbrace{{}^{cc}\Gamma^\beta_{ac}}_{0}({}^{HH}\widetilde{Y}^c) + \underbrace{{}^{cc}\Gamma^\beta_{a\gamma}}_{0}({}^{HH}\widetilde{Y}^\gamma) + \underbrace{{}^{cc}\Gamma^\beta_{a\overline{\gamma}}}_{0}({}^{HH}\widetilde{Y}^{\overline{\gamma}})) \\
&\quad + {}^{HH}\widetilde{X}^\alpha(\partial_\alpha Y^\beta + \underbrace{{}^{cc}\Gamma^\beta_{\alpha c}}_{0}({}^{HH}\widetilde{Y}^c) + {}^{cc}\Gamma^\beta_{\alpha\gamma}({}^{HH}\widetilde{Y}^\gamma) + \underbrace{{}^{cc}\Gamma^\beta_{\alpha\overline{\gamma}}}_{0}({}^{HH}\widetilde{Y}^{\overline{\gamma}})) \\
&\quad + {}^{HH}\widetilde{X}^{\overline{\alpha}}(\underbrace{\partial_{\overline{\alpha}} Y^\beta}_{0} + \underbrace{{}^{cc}\Gamma^\beta_{\overline{\alpha}c}}_{0}({}^{HH}\widetilde{Y}^c) + \underbrace{{}^{cc}\Gamma^\beta_{\overline{\alpha}\gamma}}_{0}({}^{HH}\widetilde{Y}^\gamma) + \underbrace{{}^{cc}\Gamma^\beta_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{HH}\widetilde{Y}^{\overline{\gamma}})) \\
&= X^\alpha(\partial_\alpha Y^\beta + \Gamma^\beta_{\alpha\gamma}Y^\gamma) \\
&= {}^{HH}(\nabla_X Y)^\beta
\end{aligned}$$

by virtue of (12) and (16). Thirdly, for $J=\overline{\beta}$, then we have

$$
\begin{aligned}
{}^{HH}(\nabla_X Y)^{\overline{\beta}} &= \left({}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^{\overline{\beta}} \\
&= {}^{HH}\widetilde{X}^I(\partial_I({}^{HH}\widetilde{Y}^{\overline{\beta}}) + {}^{cc}\,\Gamma^{\overline{\beta}}_{I\,K}({}^{HH}\widetilde{Y}^K)) \\
&= {}^{HH}\widetilde{X}^a(\partial_a({}^{HH}\widetilde{Y}^{\overline{\beta}}) + {}^{cc}\,\Gamma^{\overline{\beta}}_{a\,K}({}^{HH}\widetilde{Y}^K)) \\
&\quad + {}^{HH}\widetilde{X}^\alpha(\partial_\alpha({}^{HH}\widetilde{Y}^{\overline{\beta}}) + {}^{cc}\,\Gamma^{\overline{\beta}}_{\alpha K}({}^{HH}\widetilde{Y}^K)) \\
&\quad + {}^{HH}\widetilde{X}^{\overline{\alpha}}(\partial_{\overline{\alpha}}({}^{HH}\widetilde{Y}^{\overline{\beta}}) + {}^{cc}\,\Gamma^{\overline{\beta}}_{\overline{\alpha}K}({}^{HH}\widetilde{Y}^K)) \\
&= {}^{HH}\widetilde{X}^a(\partial_a({}^{HH}\widetilde{Y}^{\overline{\beta}}) + \underbrace{{}^{cc}\Gamma^{\overline{\beta}}_{a\,c}}_{0}({}^{HH}\widetilde{Y}^c) \\
&\quad + \underbrace{{}^{cc}\Gamma^{\overline{\beta}}_{a\,\gamma}}_{0}({}^{HH}\widetilde{Y}^\gamma) + \underbrace{{}^{cc}\Gamma^{\overline{\beta}}_{a\,\overline{\gamma}}}_{0}({}^{HH}\widetilde{Y}^{\overline{\gamma}})) \\
&\quad + {}^{HH}\widetilde{X}^\alpha(\partial_\alpha({}^{HH}\widetilde{Y}^{\overline{\beta}}) + \underbrace{{}^{cc}\Gamma^{\overline{\beta}}_{\alpha c}}_{0}({}^{HH}\widetilde{Y}^c) \\
&\quad + {}^{cc}\Gamma^{\overline{\beta}}_{\alpha\gamma}({}^{HH}\widetilde{Y}^\gamma) + \underbrace{{}^{cc}\Gamma^{\overline{\beta}}_{\alpha\overline{\gamma}}}_{\Gamma^{\beta}_{\alpha\,\gamma}}({}^{HH}\widetilde{Y}^{\overline{\gamma}})) \\
&\quad + {}^{HH}\widetilde{X}^{\overline{\alpha}}(\partial_{\overline{\alpha}}(\widetilde{Y}^{\overline{\beta}}) + \underbrace{{}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}c}}_{0}({}^{HH}\widetilde{Y}^c) \\
&\quad + \underbrace{{}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}\gamma}}_{\Gamma^{\beta}_{\alpha\,\gamma}}({}^{HH}\widetilde{Y}^\gamma) + \underbrace{{}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{HH}\widetilde{Y}^{\overline{\gamma}})) \\
&= X^a(\partial_a(-y^\varepsilon\Gamma^\beta_{\varepsilon\,\phi}Y^\phi)) + X^\alpha(\partial_\alpha(-y^\varepsilon\Gamma^\beta_{\varepsilon\,\phi}Y^\phi) \\
&\quad + y^\varepsilon\partial_\varepsilon\Gamma^\beta_{\alpha\gamma}Y^\gamma)) + \Gamma^\beta_{\alpha\gamma}(-y^\varepsilon\Gamma^\gamma_{\varepsilon\,\beta}Y^\beta) \\
&\quad + (-y^\varepsilon\Gamma^\alpha_{\varepsilon\,\phi}Y^\phi)(\partial_{\overline{\alpha}}(-y^\varepsilon\Gamma^\beta_{\varepsilon\,\sigma}Y^\sigma) + \Gamma^\beta_{\alpha\sigma}Y^\sigma) \\
&= X^a((-\partial_a\Gamma^\beta_{\varepsilon\,\phi})y^\varepsilon Y^\phi) + X^\alpha(-\partial_\alpha\Gamma^\beta_{\varepsilon\,\phi})y^\varepsilon Y^\phi) \\
&\quad - y^\varepsilon X^a\Gamma^\beta_{\varepsilon\,\phi}(\partial_a Y^\phi) - y^\varepsilon\Gamma^\beta_{\varepsilon\,\phi}(\partial_\alpha Y^\phi) \\
&\quad + X^\alpha(\partial_\varepsilon\Gamma^\beta_{\alpha\gamma})y^\varepsilon Y^\gamma - X^\alpha\Gamma^\beta_{\alpha\gamma}\Gamma^\gamma_{\varepsilon\,\sigma}y^\varepsilon Y^\sigma) \\
&\quad + \Gamma^\alpha_{\varepsilon\,\phi}\Gamma^\beta_{\varepsilon\,\sigma}y^\varepsilon X^\phi Y^\sigma - \Gamma^\alpha_{\varepsilon\,\phi}\Gamma^\beta_{\alpha\sigma}y^\varepsilon X^\phi Y^\sigma \\
&= X^\alpha Y^\phi y^\varepsilon(-\partial_\alpha\Gamma^\beta_{\varepsilon\,\phi} + \partial_\varepsilon\Gamma^\beta_{\alpha\phi} - \Gamma^\beta_{\alpha\sigma}\Gamma^\sigma_{\varepsilon\,\phi} + \Gamma^\beta_{\varepsilon\,\sigma}\Gamma^\sigma_{\varepsilon\,\phi}) \\
&\quad - \Gamma^\beta_{\varepsilon\,\sigma}\Gamma^\sigma_{\alpha\phi}X^\alpha Y^\phi y^\varepsilon - \Gamma^\beta_{\varepsilon\,\phi}y^\varepsilon X^\alpha\partial_\alpha Y^\phi \\
&= y^\varepsilon R^\beta_{\varepsilon\alpha\phi}X^\alpha Y^\phi + {}^{HH}(\nabla_X Y)^{\overline{\beta}}
\end{aligned}
$$

by virtue of (12) and (16). On the other hand, we know that ${}^{HH}(\nabla_X Y) + \gamma(R(\ ,X)Y)$ have the components

$$^{HH}(\nabla_X Y) + \gamma(R(\ ,X)Y) = \begin{pmatrix} {}^{HH}(\nabla_X Y)^b \\ {}^{HH}(\nabla_X Y)^\beta \\ {}^{HH}(\nabla_X Y)^{\overline{\beta}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon R^\beta_{\varepsilon\alpha\phi} X^\alpha Y^\phi \end{pmatrix}$$

with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t(B_m)$. Thus, we have

$$^{cc}\nabla_{^{HH}\widetilde{X}}({}^{HH}\widetilde{Y}) = {}^{HH}(\nabla_X Y) + \gamma(R(\ ,X)Y)$$

in $t(B_m)$. □

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Murat Polat
Department of Mathematics, Faculty of Sci. Dicle University,
25240, Diyarbakır, Turkey
E-mail: murat.polat@dicle.edu.tr

Furkan Yildirim
Narman Vocational Training School, Ataturk University,
25530, Erzurum, Turkey
E-mail: furkan.yildirim@atauni.edu.tr