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APPLICATIONS ON FOURTH–ORDER DIFFERENTIAL SUBORDINATION FOR p-VALENT MEROMORPHIC FUNCTIONS

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Abstract. In this current study, we aim to give some applications on fourth-order differential subordination for p-valent meromorphic functions in the region $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U\backslash\{0\}$, where $U = \{z \in \mathbb{C} : |z| < 1\}$, involving the linear operator $\mathcal{L}_p^* f$. By making use of basic concepts in theory of the fourth-order, we find new outcomes.

1. Introduction

Indicate by $\mathcal{H} = \mathcal{H}(U)$ the family of analytic functions in U that have the form

$$\mathcal{H}[\xi, n] = \left\{\mathfrak{f} \in \mathcal{H}(U) : \mathfrak{f}(z) = \xi + \xi_n z^n + \xi_{n+1} z^{n+1} + \cdots\right\}$$

$$(\xi \in \mathbb{C}, n \in \mathbb{N} = \{1, 2, 3, \ldots\}),$$

and let $\mathcal{A}_n$ be the collection of the form

$$\mathcal{A}_n = \left\{\mathfrak{f} \in \mathcal{H}(U) : \mathfrak{f}(z) = z + \xi_{n+1} z^{n+1} + \cdots\right\},$$

where $\mathcal{A}_1 = \mathcal{A}$ of normalized analytic functions in U. Further, indicate by $\Sigma(p)$ $(p \in \mathbb{N})$ the subfamily of $\mathcal{H}(\mathcal{U})$ of the form

$$\mathfrak{f}(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \xi_{n+p} z^{n+p},$$

which are meromorphic in U^*. For analytic functions f_1 and f_2, it is said that the function f_1 is subordinate to f_2, if

$$f_1(z) = f_2(\vartheta(z)) \quad (z \in U),$$

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where $\vartheta(z)$ is analytic and $\vartheta(0) = 0,\ |\vartheta(z)| < 1$. This subordination is indicated by $f_1(z) \prec f_2(z)$. Albehbah and Darus [1] defined the linear operator $\mathcal{L}_p^*(\xi, \delta) : \Sigma(p) \to \Sigma(p)$ which is expressed by

$$\mathcal{L}_p^*(\lambda, \delta)\mathfrak{f}(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \left| \frac{(\lambda)_{n+2}}{(\delta)_{n+2}} \right| \xi_{n+p} z^{n+p}, \tag{1}$$

where

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \lambda(\lambda + 1) \dots (\lambda + n - 1),$$
$$(\delta \neq 0, -1, -2, \dots, \quad \lambda \in \mathbb{C}\backslash\{0\})$$

is the Pochhammer symbol. Later, Liu [5] derived from (1) the relation

$$z\left(\mathcal{L}_p^*(\lambda, \delta)\mathfrak{f}(z)\right)' = \lambda\mathcal{L}_p^*(\lambda + 1, \delta)\mathfrak{f}(z) - (\lambda + p)\mathcal{L}_p^*(\lambda, \delta)\mathfrak{f}(z), \tag{2}$$

where

$$\mathcal{L}_p^*(\lambda + 1, \delta)\mathfrak{f}(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \left| \frac{(\lambda + 1)_{n+2}}{(\delta)_{n+2}} \right| \xi_{n+p} z^{n+p}.$$

2. Problem Formulation

In recent times, various subcollection of analytic and univalent functions, which are connected to differential subordination and superordination in the open unit disk U, have been initiated from many interesting outcomes and perspective (see [6], [7], [8], [9], [10], [11]).

In order to demonstrate the outcomes, we shall give the basic concepts in theory of the fourth-order below.

Definition 2.1. *(See [2]) Indicate by Q the collection of all analytic functions $\mathfrak{q}$ on $\bar{U}\backslash E(\mathfrak{q})$, where*

$$E(\mathfrak{q}) = \left\{\kappa : \kappa \in \partial U \text{ and } \lim_{z \to \kappa} \mathfrak{q}(z) = \infty\right\},$$

and are such that

$$\min |\mathfrak{q}'(\kappa)| = \rho > 0 \qquad (\kappa \in \partial U\backslash E(\mathfrak{q})).$$

Also, indicate by $Q(\beta)$ the collection of functions $\mathfrak{q}$ for which $\mathfrak{q}(0) = \beta$. Note that $Q_1 = Q(1) = \{\mathfrak{q}(z) \in Q : q(0) = 1\}$.

Lemma 2.2. *(See [3]) Let $z_0 \in U$ with $r_0 = |z_0|$. For $n \geq 1$, assume that*

$$\mathfrak{f}(z) = \xi_n z^n + \xi_{n+1} z^{n+1} + \xi_{n+2} z^{n+2} + \cdots \quad (\mathfrak{f}(z) \neq 0)$$

is continuous on $\bar{U}_{r_0}$ and analytic on $U_{r_0} \cup \{z_0\}$. If

$$|\mathfrak{f}(z_0)| = \max\left\{|\mathfrak{f}(z)| : z \in \bar{U}_{r_0}\right\},$$

there exists an $m \geq n$ such that

$$\frac{z_0\mathfrak{f}'(z_0)}{\mathfrak{f}(z_0)} = m, \quad \Re\left\{\frac{z_0\mathfrak{f}''(z_0)}{\mathfrak{f}'(z_0)} + 1\right\} \geq m,$$

$$\Re\left\{\frac{z_0\mathfrak{f}'(z_0)+3z_0^2\mathfrak{f}''(z_0)+z_0^3\mathfrak{f}'''(z_0)}{z_0\mathfrak{f}'(z_0)}\right\}\geq m^2.$$

Then

$$\Re\left\{\frac{z_0\mathfrak{f}'(z_0)+7z_0^2\mathfrak{f}''(z_0)+6z_0^3\mathfrak{f}'''(z_0)+z_0^4\mathfrak{f}^{(4)}(z_0)}{z_0\mathfrak{f}'(z_0)}\right\}\geq m^3.$$

Lemma 2.3. *(See* [3]*) Let $p\in\mathcal{H}[\xi,n]$ and $\mathfrak{q}\in Q$ with $\mathfrak{q}(0)=\kappa$. Assume that*

$$\kappa=\mathfrak{q}^{-1}[p(z)]=\mathfrak{f}(z)\qquad\left(z\in\bar{U}_{r_0}\right).$$

If $z_0\in U$ and $\kappa_0\in\partial U\backslash E(\mathfrak{q})$ fulfills

$$p(z_0)=\mathfrak{q}(\kappa_0),$$
$$p(\bar{U}_{r_0})\subset\mathfrak{q}(U)$$

and

$$\Re\left\{\frac{\kappa_0\mathfrak{q}''(\kappa_0)}{\mathfrak{q}'(\kappa_0)}\right\}\geq 0,\qquad\left|\frac{zp'(z)}{\mathfrak{q}'(\kappa)}\right|\leq k$$
$$\Re\left\{\frac{\kappa_0^2\mathfrak{q}'''(\kappa_0)}{\mathfrak{q}'(\kappa_0)}\right\}\geq 0,\qquad\left|\frac{z^2p''(z)}{\mathfrak{q}'(\kappa)}\right|\leq k^2\qquad(k=1,2,\ldots,n-1),$$

then there exists an $m\geq n\geq 1$ such that

$$z_0p'(z_0)=m\kappa_0\mathfrak{q}'(\kappa_0),$$
$$\Re\left\{1+\frac{z_0p''(z_0)}{p'(z_0)}\right\}\geq m\Re\left\{1+\frac{\kappa_0\mathfrak{q}''(\kappa_0)}{\mathfrak{q}'(\kappa_0)}\right\},$$
$$\Re\left\{\frac{z_0p'(z_0)+3z_0^2p''(z_0)+z_0^3p'''(z_0)}{z_0p'(z_0)}\right\}\geq m^2\Re\left\{\frac{\kappa_0\mathfrak{q}'(\kappa_0)+3\kappa_0^2\mathfrak{q}''(\kappa_0)+\kappa_0^3\mathfrak{q}'''(\kappa_0)}{\kappa_0\mathfrak{q}'(\kappa_0)}\right\}.$$

Then

$$\Re\left\{\frac{z_0p'(z_0)+7z_0^2p''(z_0)+6z_0^3p'''(z_0)+z_0^4p^{(4)}(z_0)}{z_0p'(z_0)}\right\}$$
$$\geq m^2\Re\left\{\frac{\kappa_0\mathfrak{q}'(\kappa_0)+7\kappa_0^2\mathfrak{q}''(\kappa_0)+6\kappa_0^3\mathfrak{q}'''(\kappa_0)+\kappa_0^4\mathfrak{q}^{(4)}(\kappa_0)}{\kappa_0\mathfrak{q}'(\kappa_0)}\right\}$$

or,

$$\Re\left\{\frac{z_0^3p^{(4)}(z_0)}{p'(z_0)}\right\}\geq k^3\Re\left\{\frac{\kappa_0^3\mathfrak{q}^{(4)}(\kappa_0)}{\kappa_0\mathfrak{q}'(\kappa_0)}\right\}.$$

Definition 2.4. *(See* [4]*) Assume h is univalent in U and $F:\mathbb{C}^5\times U\to\mathbb{C}$. If the analytic function p fulfills the fourth-order differential subordination*

$$F\left(p(z),zp'(z),z^2p''(z),z^3p'''(z),z^4p^{(4)}(z);z\right)\prec h(z),\tag{3}$$

then the function p is named a solution of the differential subordination. A univalent function $\mathfrak{q}$ is named a dominant of the solution of the differential subordination if $p(z)\prec\mathfrak{q}(z)$ for all p satisfying (3). A dominant $\tilde{\mathfrak{q}}(z)$ that fulfills $\tilde{\mathfrak{q}}(z)\prec\mathfrak{q}(z)$ for all dominants $\mathfrak{q}$ of (3) is named the best dominant.

Definition 2.5. *(See [3]) Assume Π is a set in $\mathbb{C}$ and $\mathfrak{q} \in Q$. The admissible functions class $\Phi_n[\Pi, \mathfrak{q}]$ $(n \in \mathbb{N}\backslash\{2\})$ consists of those functions $\digamma : \mathbb{C}^5 \times U \to \mathbb{C}$ that fulfill the following admissibility condition*

$$\digamma(r, s, t, w, b; z) \notin \Pi,$$

whenever

$$r = \mathfrak{q}(\kappa),\ s = k\kappa\mathfrak{q}'(\kappa),\ \Re\left\{\frac{t}{s} + 1\right\} \geq k\Re\left\{1 + \frac{\kappa\mathfrak{q}''(\kappa)}{\mathfrak{q}'(\kappa)}\right\},$$

$$\Re\left\{\frac{w}{s}\right\} \geq k^2\Re\left\{\frac{\kappa^2\mathfrak{q}'''(\kappa)}{\mathfrak{q}'(\kappa)}\right\},\ \Re\left\{\frac{b}{s}\right\} \geq k^3\Re\left\{\frac{\kappa^3\mathfrak{q}^{(4)}(\kappa)}{\mathfrak{q}'(\kappa)}\right\},$$
$$(z \in U,\ \kappa \in \partial U\backslash E(\mathfrak{q}),\ k \geq n).$$

Theorem 2.6. *(See [3]) Let $p \in \mathcal{H}[\xi, n]$ $(n \in \mathbb{N}\backslash\{2\})$. Also, let $q \in Q$ and fulfill $\mathfrak{q}$ fulfill the conditions*

$$\Re\left\{\frac{\kappa^2\mathfrak{q}'''(\kappa)}{\mathfrak{q}'(\kappa)}\right\} \geq 0,\ \left|\frac{z^2p'(\kappa)}{\mathfrak{q}'(\kappa)}\right| \leq m^2, \qquad (z \in U,\ \kappa \in \partial U\backslash E(\mathfrak{q}),\ k \geq n).$$

If $\digamma \in \Phi_n[\Pi, \mathfrak{q}]$ and

$$\digamma\left(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p^{(4)}(z); z\right) \in \Pi,$$

then

$$p(z) \prec \mathfrak{q}(z).$$

In this current study, we aim to give some applications on fourth-order differential subordination for p-valent meromorphic functions in the region U^* involving the linear operator $\mathcal{L}_p^* f$. Afterwards, new outcomes of differential subordination will be noted.

3. Subordination properties

We will establish the collection of admissible functions, which is necessary to discuss subordination properties.

Definition 3.1. *Assume $\mathfrak{q} \in Q_0 \cap \mathcal{H}$ and Π is a set in $\mathbb{C}$. The admissible functions class $\Phi_{\mathcal{L}}[\Pi, \mathfrak{q}]$ consists of functions $\digamma : \mathbb{C}^5 \times U \to \mathbb{C}$ fulfilling the admissibility case*

$$\digamma(u, v, x, y, g; z) \notin \Pi,$$

whenever

$$u = \mathfrak{q}(\kappa),\ v = \frac{k\kappa\mathfrak{q}'(\kappa) + (\lambda + p)\mathfrak{q}(\kappa)}{\lambda},$$

$$\Re\left\{\frac{\lambda(\lambda+1)x - (\lambda+p)(\lambda+p+1)u}{\lambda v - (\lambda+p)u} - 2(\lambda+p) - 1\right\} \geq k\Re\left\{\frac{\kappa\mathfrak{q}''(\kappa)}{\mathfrak{q}'(\kappa)} + 1\right\},$$

$$\Re\left\{\frac{\lambda(\lambda+1)(\lambda+2)y - 3\lambda(\lambda+1)(\lambda+p+2)x + 4(\lambda+p)(\lambda+p+1)(\lambda+p+2)u}{\lambda v - (\lambda+p)u} + 3(\lambda+p+1)(\lambda+p+2)\right\}$$

$$\geq k^2\Re\left\{\frac{\kappa^2\mathfrak{q}'''(\kappa)}{\mathfrak{q}'(\kappa)}\right\}$$

$$\Re\left\{\frac{\lambda(\lambda+1)(\lambda+2)(\lambda+3)g+9(\lambda+p)(\lambda+p+1)(\lambda+p+2)(\lambda+p+3)u-4\lambda(\lambda+1)(\lambda+2)(\lambda+p+3)y+6\lambda(\lambda+1)(\lambda+p+2)(\lambda+p+3)x}{\lambda v-(\lambda+p)u}\right.$$

$$\left.+4(\lambda+p+1)(\lambda+p+2)(\lambda+p+3)\right\} \geq k^3\Re\left\{\frac{\kappa^3\mathfrak{q}^{(4)}(\kappa)}{\mathfrak{q}'(\kappa)}\right\},$$

where $z \in U$, $\kappa \in \partial U\backslash E(\mathfrak{q})$ *and* $k \in \mathbb{N}\backslash\{1\}$.

Theorem 3.2. *Let* $\mu \in \Phi_{\mathcal{L}}[\Pi, \mathfrak{q}]$. *If* $\mathfrak{f} \in \Sigma(p)$ *and* $\mathfrak{q} \in Q_1$ *fulfills*

$$\Re\left\{\frac{\kappa^2\mathfrak{q}'''(\kappa)}{\mathfrak{q}'(\kappa)}\right\} \geq 0, \quad \left|\frac{\mathcal{L}_p^*(\lambda+2,c)\mathfrak{f}(z)}{\mathfrak{q}'(\kappa)}\right| \leq k \tag{4}$$

and

$$\left\{F\left(\mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+1,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+2,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+3,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+4,\delta)\mathfrak{f}(z); z\right) : z \in U\right\} \subset \Pi, \tag{5}$$

then

$$\mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z) \prec \mathfrak{q}(z).$$

Proof. Put

$$\vartheta(z) = \mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z). \tag{6}$$

From differentiation of (6) and (2), we see that

$$\mathcal{L}_p^*(\lambda+1,\delta)\mathfrak{f}(z) = \frac{z\vartheta'(z)+(\lambda+p)\vartheta(z)}{\lambda}.$$

Further computations give

$$\mathcal{L}_p^*(\lambda+2,\delta)\mathfrak{f}(z) = \frac{z^2\vartheta''(z)+2(\lambda+p+1)z\vartheta'(z)+(\lambda+p)(\lambda+p+1)\vartheta(z)}{\lambda(\lambda+1)},$$

$$\mathcal{L}_p^*(\lambda+3,\delta)\mathfrak{f}(z) = \frac{z^3\vartheta'''(z)+3(\lambda+p+2)z^2\vartheta''(z)+3(\lambda+p+2)(\lambda+p+1)z\vartheta'(z)+(\lambda+p)(\lambda+p+1)(\lambda+p+2)\vartheta(z)}{\lambda(\lambda+1)(\lambda+2)},$$

and

$$\mathcal{L}_p^*(\lambda+4,\delta)\mathfrak{f}(z) = \frac{z^4\vartheta^{(4)}(z)+4(\lambda+p+3)z^3\vartheta'''(z)+6(\lambda+p+3)(\lambda+p+2)z^2\vartheta''(z)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}$$

$$+\frac{4(\lambda+p+3)(\lambda+p+2)(\lambda+p+1)z\vartheta'(z)+(\lambda+p)(\lambda+p+1)(\lambda+p+2)(\lambda+p+3)\vartheta(z)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}.$$

Now, we will establish a transformation from $\mathbb{C}^5$ to $\mathbb{C}$ by

$$u(r,s,t,w,b) = r, \quad v(r,s,t,w,b) = \frac{s+(\lambda+p)r}{\lambda}, \quad x(r,s,t,w,b) = \frac{t+2(\lambda+p+1)s+(\lambda+p)(\lambda+p+1)r}{\lambda(\lambda+1)}, \tag{7}$$

$$y(r,s,t,w,b) = \frac{w+3(\lambda+p+2)t+3(\lambda+p+2)(\lambda+p+1)s+(\lambda+p)(\lambda+p+1)(\lambda+p+2)r}{\lambda(\lambda+1)(\lambda+2)} \tag{8}$$

and

(9)

$$g(r,s,t,w,b) = \frac{b+4(\lambda+p+3)w+6(\lambda+p+3)(\lambda+p+2)t}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)} + \frac{4(\lambda+p+3)(\lambda+p+2)(\lambda+p+1)s+(\lambda+p)(\lambda+p+1)(\lambda+p+2)(\lambda+p+3)r}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}.$$

Next, suppose that

(10)

$$\mu(r,s,t,w,b;z) = F(u,v,x,y,g;z)$$

$$= F\Big(r, \tfrac{s+(\lambda+p)r}{\lambda}, \tfrac{t+2(\lambda+p+1)s+(\lambda+p)(\lambda+p+1)r}{\lambda(\lambda+1)}, \tfrac{w+3(\lambda+p+2)t+3(\lambda+p+2)(\lambda+p+1)s+(\lambda+p)(\lambda+p+1)(\lambda+p+2)r}{\lambda(\lambda+1)(\lambda+2)},$$

$$\tfrac{b+4(\lambda+p+3)w+6(\lambda+p+3)(\lambda+p+2)t+4(\lambda+p+3)(\lambda+p+2)(\lambda+p+1)s+(\lambda+p)(\lambda+p+1)(\lambda+p+2)(\lambda+p+3)r}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}\Big).$$

It follows from (10) and Theorem 2.6 that

(11)

$$\mu\left(\vartheta(z), z\vartheta'(z), z^2\vartheta''(z), z^3\vartheta'''(z), z^4\vartheta^{(4)}(z); z\right) =$$

$$F\left(\mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+1,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+2,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+3,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+4,\delta)\mathfrak{f}(z); z\right).$$

Hence, the inclusion (5) leads to

$$\mu\Big(\vartheta(z), z\vartheta'(z), z^2\vartheta''(z), z^3\vartheta'''(z), z^4\vartheta^{(4)}(z); z\Big) \in \Pi.$$

Moreover, by implementing (7) and (8), we get

$$\frac{t}{s}+1 = \frac{\lambda(\lambda+1)x-(\lambda+p)(\lambda+p+1)u}{\lambda v-(\lambda+p)u} - 2(a+p)-1,$$

$$\frac{w}{s} = \frac{\lambda(\lambda+1)(\lambda+2)y - 3\lambda(\lambda+1)(\lambda+p+2)x + 4(\lambda+p)(\lambda+p+1)(\lambda+p+2)u}{\lambda v-(\lambda+p)u}$$

$$+3(\lambda+p+1)(\lambda+p+2)$$

and

$$\frac{b}{s} = \tfrac{\lambda(\lambda+1)(\lambda+2)(\lambda+3)g+9(\lambda+p)(\lambda+p+1)(\lambda+p+2)(\lambda+p+3)u-4\lambda(\lambda+1)(\lambda+2)(\lambda+p+3)y+6\lambda(\lambda+1)(\lambda+p+2)(\lambda+p+3)x}{\lambda v-(\lambda+p)u}$$

$$+4(\lambda+p+1)(\lambda+p+2)(\lambda+p+3).$$

Therefore, the admissibility condition in Definition 3.1 for $\mu \in \Phi_{\mathcal{L}}[\Pi, \mathfrak{q}]$ is equivalent to Definition 2.5 for $n = 3$. Hence, by making use of (4) and applying Theorem 2.6, we see that

$$\mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z) \prec \mathfrak{q}(z).$$

□

The next corollaries are extensions of Theorem 3.2 to the case where the behaviour of $\mathfrak{q}$ on ∂U is not known.

Corollary 3.3. *Let* $\Pi \subset \mathbb{C}$ *and suppose that* $\mathfrak{q}$ *is univalent in* U *with* $\mathfrak{q}(0) = 1$, $F \in \Phi_{\mathcal{L}}[\Pi, \mathfrak{q}_\rho]$ *for some* $\rho \in (0,1)$, *where* $\mathfrak{q}_\rho(z) = \mathfrak{q}(\rho z)$. *If* $\mathfrak{f} \in \Sigma(p)$ *and* $\mathfrak{q}_\rho \in Q_0$ *fulfills*

$$\Re\left\{\frac{\kappa^2 \mathfrak{q}_\rho'''(\kappa)}{\mathfrak{q}_\rho'(\kappa)}\right\} \geq 0, \quad \left|\frac{\mathcal{L}_p^*(a+2,c)\mathfrak{f}(z)}{\mathfrak{q}_\rho'(\kappa)}\right| \leq k^2 \quad (k \geq n)$$

and

$$\left\{F\left(\mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+1,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+2,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+3,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+4,\delta)\mathfrak{f}(z); z\right) : z \in U\right\} \subset \Omega,$$

then

$$\mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z) \prec \mathfrak{q}(z).$$

Proof. By using Theorem 3.2, we get

$$\mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z) \prec \mathfrak{q}_\rho(z).$$

Then, we deduce the outcome from

$$\mathfrak{q}_\rho(z) \prec \mathfrak{q}(z).$$

□

Corollary 3.4. *Let* $\Pi \subset \mathbb{C}$ *and suppose that* $\mathfrak{q}$ *is univalent in* U *with* $\mathfrak{q}(0) = 1$, $F \in \Phi_{\mathcal{L}}[\Pi, \mathfrak{q}_\rho]$ *for some* $\rho \in (0,1)$, *where* $\mathfrak{q}_\rho(z) = \mathfrak{q}(\rho z)$. *If* $\mathfrak{f} \in \Sigma(p)$ *and* $\mathfrak{q}_\rho \in Q_0$ *fulfills the condition (4) and*

$$F\left(\mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+1,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+2,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+3,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+4,\delta)\mathfrak{f}(z); z\right) \prec h(z),$$

then

$$\mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z) \prec \mathfrak{q}(z).$$

Corollary 3.5. *Let* $F \in \Phi_{\mathcal{L}}[\Pi, \mathfrak{q}]$. *If* $\mathfrak{f} \in \Sigma(p)$ *and* $\mathfrak{q} \in Q_0$ *fulfills the condition (4) and*

(12)
$$F\left(\mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+1,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+2,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+3,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+4,\delta)\mathfrak{f}(z); z\right) \prec h(z),$$

then

$$\mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z) \prec \mathfrak{q}(z).$$

Theorem 3.6. *Assume* $\digamma : \mathbb{C}^5 \times U \to \mathbb{C}$*, h is univalent in U and the differential equality*

(13)

$$\digamma\left(\mathfrak{q}(z), \frac{z\mathfrak{q}'(z) + (\lambda+p)\mathfrak{q}(z)}{\lambda}, \frac{z^2\mathfrak{q}''(z) + 2(\lambda+p+1)z\mathfrak{q}'(z) + (\lambda+p)(\lambda+p+1)\mathfrak{q}(z)}{\lambda(\lambda+1)},\right.$$

$$\frac{z^3\mathfrak{q}'''(z) + 3(\lambda+p+2)z^2\mathfrak{q}''(z) + 3(\lambda+p+2)(\lambda+p+1)z\mathfrak{q}'(z) + (\lambda+p)(\lambda+p+1)(\lambda+p+2)\mathfrak{q}(z)}{\lambda(\lambda+1)(\lambda+2)},$$

$$\frac{z^4\mathfrak{q}^{(4)}(z)+4(\lambda+p+3)z^3\mathfrak{q}'''(z)+6(\lambda+p+3)(\lambda+p+2)z^2\mathfrak{q}''(z)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}$$

$$\left.+\frac{4(\lambda+p+3)(\lambda+p+2)(\lambda+p+1)z\mathfrak{q}'(z)+(\lambda+p)(\lambda+p+1)(\lambda+p+2)(\lambda+p+3)\mathfrak{q}(z)}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}\right)$$

$$= h(z)$$

has a solution $\mathfrak{q} \in Q_0 \cap \mathcal{H}$ *and fulfills the condition (4). If the function* $\mathfrak{f} \in \Sigma(p)$ *fulfills (12) and*

$$\digamma\left(\mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+1,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+2,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+3,\delta)\mathfrak{f}(z), \mathcal{L}_p^*(\lambda+4,\delta)\mathfrak{f}(z); z\right)$$

is analytic in U*, then (12) gives that*

$$\mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z) \prec \mathfrak{q}(z)$$

and $\mathfrak{q}$ *is the best dominant.*

Proof. Theorem 3.2 gives that $\mathfrak{q}(z)$ is a dominant of (12). Indeed, $\mathfrak{q}(z)$ is a solution of (12) and fulfills (13), it is dominated by all dominants. Therefore, $\mathfrak{q}(z)$ is best dominant. This completes the proof of Theorem 3.6. $\square$

For the case $\mathfrak{q}(z) = Mz, (M > 0)$, the admissible functions class $\Phi_{\mathcal{L}}[\Pi, \mathfrak{q}] = \Phi_{\mathcal{L}}[\Pi, M]$ is expressed below.

Definition 3.7. *Let U be a set in $\mathbb{C}$ and $M > 0$. The admissible functions class $\Phi_{\mathcal{L}}[\Pi, M]$ including the functions $\digamma : \mathbb{C}^5 \times U \to \mathbb{C}$ such that*

$$\digamma\left(Me^{i\theta}, \frac{k+(\lambda+p)Me^{i\theta}}{\lambda}, \frac{L+2[(\lambda+p+1)k+(\lambda+p)(\lambda+p+1)]Me^{i\theta}}{\lambda(\lambda+1)},\right.$$

$$\frac{N+3(\lambda+p+2)L+[3(\lambda+p+2)(\lambda+p+1)K+(\lambda+p)(\lambda+p+1)(\lambda+p+2)]Me^{i\theta}}{\lambda(\lambda+1)(\lambda+2)},$$

$$\left.\frac{A+4(\lambda+p+3)N+6(\lambda+p+3)(\lambda+p+2)L+[4(\lambda+p+3)(\lambda+p+2)(\lambda+p+1)k+(\lambda+p)(\lambda+p+1)(\lambda+p+2)(\lambda+p+3)]Me^{i\theta}}{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}; z\right) \notin \Pi,$$

where $z \in U$, $\Re(Le^{i\theta}) \geq (k-1)kM \quad (k \geq 3)$, $\Re(Ne^{i\theta}) \geq 0$ *and* $\Re(Ae^{i\theta}) \geq 0 \ (\theta \in \mathbb{R})$.

Corollary 3.8. *Let* $\digamma \in \Phi_{\mathcal{L}}[\Pi, M]$*. If the function* $\mathfrak{f} \in \Sigma(p)$ *fulfills*

$$\left|\mathcal{L}_p^*(\lambda+2,\delta)\mathfrak{f}(z)\right| \leq k^2 M \qquad (k \geq 3,\ M > 0)$$

and

$$F\left(\mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z),\mathcal{L}_p^*(\lambda+1,\delta)\mathfrak{f}(z),\mathcal{L}_p^*(\lambda+2,\delta)\mathfrak{f}(z),\mathcal{L}_p^*(\lambda+3,\delta)\mathfrak{f}(z),\mathcal{L}_p^*(\lambda+4,\delta)\mathfrak{f}(z);z\right)\in\Pi,$$

then, as a consequence,

$$\left|\mathcal{L}_p^*(\lambda,\delta)\mathfrak{f}(z)\right| < M.$$

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