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NEW KINDS OF CONTINUITY IN FUZZY NORMED SPACES

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Abstract. We first define the notions of filter continuous, filter sequentially continuous and filter strongly continuous in the framework of fuzzy normed space (FNS), and then we introduce the notion of filter slowly oscillating sequences in the setting of FNS and shows that this notion is stronger than slowly oscillating sequences. Further, we define the concept of filter slowly oscillating continuous functions, filter Cesàro slowly oscillating sequences as well as some other related notions in the aforementioned space and investigate several related results.

1. Introduction

The notion of statistical convergence was first appeared under the name of almost convergence in [52]. Later, this notion was defined by Fast [22] and Steinhaus [47] in the same year (also see [24]), and many researchers were further studied on it (see [3, 2, 30, 32, 40, 41]). The natural density of subsets of $\mathbb{N}$ is the basic concept to define the statistical convergence. Let $E \subseteq \mathbb{N}$. The natural or asymptotic density of E is denoted by $\delta(E)$ and is defined by

$$\delta(E) = \lim_{k\to\infty} \frac{|E(k)|}{k},$$

where the limit exists, $E(k) = \{n \leq k : n \in E\}$, and $|E|$ denotes the cardinality of the set E. A sequence (v_m) of real numbers is said to be *statistically convergent* to v if for each $\epsilon > 0$, $\delta(\{m \in \mathbb{N} : |v_m - v| \geq \epsilon\}) = 0$, i.e.

$$\lim_{k\to\infty} \frac{1}{k} |\{m \leq k : |v_m - v| \geq \epsilon\}| = 0.$$

The ideal convergence is the dual (equivalent) of the notion of filter convergence initiated by Cartan [15] in 1937. The filter convergence is a generalized form of classical convergence of a sequence and it is an important tool in general topology and functional analysis. The ideal convergence was studied by

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Kostyrko, et al. [38], Nuray and Ruckle [44] independently, which is based on the structure of admissible ideal I of subsets of $\mathbb{N}$ (also see [27, 42, 43, 45]). An ideal I on $\mathbb{N}$ is a family of subsets of $\mathbb{N}$ which is closed under finite unions and subsets of its elements.

A real valued function is continuous if and only if it preserves Cauchy sequences. Based on the idea of compactness in terms of sequences and continuity of a real valued function, different types of continuities were introduced, for example, we refer to [5, 6, 8, 9, 10, 11, 12, 13, 21, 48, 49]. A real sequence (v_m) is called quasi-Cauchy if the sequence (Δv_m) converges to 0, where $\Delta v_m = v_{m+1} - v_m$. Burton and Coleman [4] named these sequences as "quasi-Cauchy" and Çakallı [7] named as "ward convergent to 0" sequences. Recall that a real sequence (v_m) is said to be I-convergent to ℓ if for every $\epsilon > 0$, the set $\{m \in \mathbb{N} : |v_m - \ell| \geq \epsilon\} \in I$. This can be written as I-$\lim v_m = \ell$.

In this paper, we define the notions of filter continuous, filter sequentially continuous and filter strongly continuous in the setting of fuzzy normed space and then introduce the concepts of filter slowly oscillating sequences and filter slowly oscillating continuous functions in the aforementioned space and investigate several related results.

2. Preliminaries and notations

A real sequence (v_m) is said to be slowly oscillating (**SO**, in short) if

$$\lim_{\gamma \to 1^+} \overline{\lim}_m \max_{m+1 \leq p \leq [\gamma m]} |v_p - v_m| = 0,$$

where $[\gamma m]$ is the integral part of γm. This is equivalent that if $(v_p - v_m) \to 0$ whenever $1 \leq \frac{p}{m} \to 1$, as $p, m \to \infty$. For any given $\epsilon > 0$, there exists $a = a(\epsilon) > 0$ and $\mathcal{N} = \mathcal{N}(\epsilon)$ such that $|v_p - v_m| < \epsilon$ if $m \geq \mathcal{N}(\epsilon)$ and $m \leq p \leq (1+a)m$ (see [8, 14, 26, 28]). Let $E \subset \mathbb{R}$. A function defined on E is called **SO** continuous if it preserves **SO** sequences.

The fuzzy set theory was developed by L. Zadeh [51] in 1965. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space. The fuzzy metric space with the help of continuous t-norms was studied by George and Veeramani [25]. Katsaras [34] was initiated the idea of fuzzy norm to developed the field of fuzzy functional analysis in 1984. The compactness in fuzzy minimal spaces introduced by Alimohammady and Roohi [1]. Felbin [23] put forward the concept of fuzzy norm on a linear space, which is based on the treatment of a fuzzy metric introduced by Kaleva and Seikkala [33]. From a different approach Cheng and Mordeson [17] defined another type of fuzzy norm on a linear space whose associated fuzzy metric is of Kramosil and Michalek type [39]. Some topological properties of fuzzy normed spaces were found in [50]. One can see [29, 31, 46] for more details on fuzzy normed spaces. Based on George and

Veeramani fuzzy metric space, Chugh and Rathi [16] introduced a new concept of fuzzy normed space as a fuzzy metric space in which if $\mu(x,y,t)=\nu(x-y,t)$ for all $x,y\in\mathcal{X}$ and $t>0$, where $\mathcal{X}$ is a fuzzy normed space. The notion of fuzziness are attracted many workers on sequence spaces and summability theory to introduce different types of sequence spaces and study their different properties. For further work on fuzzy, we refer to [18, 19, 20, 35, 36, 37].

We now recall the concept of the continuous t-norm, fuzzy metric space and fuzzy normed space. Throughout the paper, we denoted by $(0,1)$ the open interval $]0,1[$.

Definition 2.1. *A function* $*:[0,1]\times[0,1]\to[0,1]$ *is called a continuous triangular norm (in short a t-norm) if the following conditions hold:*

(i) $*$ *is commutative and associative;*
(ii) $*$ *is continuous;*
(iii) *for every* $\alpha,\beta,\gamma,\lambda\in[0,1]$ *if* $\alpha\le\gamma$ *and* $\beta\le\lambda$, *then* $\alpha*\beta\le\gamma*\lambda$ *(order-preserving in both variables);*
(iv) *for every* $\alpha\in[0,1],\alpha*1=\alpha$ *(natural element).*

The followings are examples of t-norm:

(a) the product t-norm: for every $\alpha,\beta\in[0,1],\alpha*\beta=\alpha\beta$.
(b) the Zadeh's t-norm or the minimum: for every $\alpha,\beta\in[0,1],\alpha*\beta=\min\{\alpha,\beta\}$.
(c) the Lukasiewicz t-norm: for every $\alpha,\beta\in[0,1],\alpha*\beta=\max\{\alpha+\beta-1,0\}$.

Let $\mathcal{X}$ be a nonempty set. A fuzzy subset A of $\mathcal{X}$ is characterized by its membership function $A:\mathcal{X}\to[0,1]$ and $A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for each $x\in\mathcal{X}$.

Definition 2.2. *A fuzzy metric space is an ordered triple* $(\mathcal{X},\mu,*)$ *such that* $\mathcal{X}$ *is an arbitrary (nonempty) set,* $*$ *is a continuous t-norm and* μ *is a fuzzy subset on* $\mathcal{X}^2\times]0,+\infty[$ *satisfying, for all* $x,y,z\in\mathcal{X}$, *and* $s,t>0$ *the following conditions:*

(i) $\mu(x,y,t)>0$;
(ii) $\mu(x,y,t)=1$ *if and only if* $x=y$;
(iii) $\mu(x,y,t)=\mu(y,x,t)$;
(iv) $\mu(x,y,t)*\mu(y,z,s)\le\mu(x,z,t+s)$;
(v) $\mu(x,y,.):]0,+\infty[\to[0,1]$ *is continuous.*

μ *is called a fuzzy metric on* $\mathcal{X}$. *The function* $\mu(x,y,t)$ *denote the degree of nearness between* x *and* y *with respect to* t.

Also the condition (ii) is equivalent to: $\mu(x,x,t)=1$ for all $x\in\mathcal{X}$ and $t>0$, and $\mu(x,y,t)<1$ for all $x\neq y$ and $t>0$.

Definition 2.3. *A triplet* $(\mathcal{X},\nu,*)$ *is called a fuzzy normed space (FNS, in short) if* $\mathcal{X}$ *is a linear (or real) vector space,* $*$ *is a continuous t-norm, and* ν *is a fuzzy subset on* $\mathcal{X}\times[0,+\infty[$ *satisfying the following conditions:*

(i) $\nu(x,0) = 0$ *for all* $x \in \mathcal{X}$;
(ii) $\nu(x,t) = 1$ *for all* $t > 0$ *if and only if* $x = 0$;
(iii) $\nu(\alpha x,t) = \nu(x, \frac{t}{|\alpha|})$ *for all* $\alpha(\neq 0) \in \mathbb{R}$;
(iv) $\nu(x,t) * \nu(y,s) \leq \nu(x+y, t+s)$ *for all* $x, y \in \mathcal{X}$ *and* $t, s \in \mathbb{R}^+$;
(v) $\nu(x,.) : [0,+\infty) \longrightarrow [0,1]$ *is left continuous for all* $x \in \mathcal{X}$;
(vi) $\lim_{t \to +\infty} \nu(x,t) = 1$ *for all* $x \in \mathcal{X}$ *and* $t \in \mathbb{R}^+$.

A FNS is a fuzzy metric space by setting $\mu(x,y,t) = \nu(x-y,t)$, which is also called the fuzzy metric induced by the fuzzy norm ν. $\nu(x,t)$ can be considered as the degree of nearness of norm of x with respect to t.

Definition 2.4. *Let* $(\mathcal{X}, \nu, *)$ *be a FNS. The open ball* $B(x,r,t)$ *with center at* $x \in \mathcal{X}$ *and radius* $r \in (0,1)$, $t > 0$ *is defined as*

$$B(x,r,t) = \{y \in \mathcal{X} : \nu(x-y,t) > 1-r\}.$$

Also a subset $A \subseteq \mathcal{X}$ is called open if for each $x \in A$ there exists $t > 0$ and $r \in (0,1)$ such that $B(x,r,t) \subseteq A$.

Definition 2.5. *A sequence* (v_m) *in a FNS* $(\mathcal{X}, \nu, *)$ *is said to be convergent to* ℓ *if for each* $r \in (0,1)$ *and each* $t > 0$ *there exists* $m_0 \in \mathbb{N}$ *such that* $\nu(v_m - \ell, t) > 1 - r$ *for all* $m \geq m_0$.

Definition 2.6. *A sequence* (v_m) *in a FNS* $(\mathcal{X}, \nu, *)$ *is said to be Cauchy if for each* $r \in (0,1)$ *and each* $t > 0$ *there exists* $m_0 \in \mathbb{N}$ *such that* $\nu(v_m - v_n, t) > 1 - r$ *for all* $m, n \geq m_0$.

Definition 2.7. *A sequence* (v_m) *in a FNS* $(\mathcal{X}, \nu, *)$ *converges to* ℓ *if and only if* $\nu(v_m - \ell, t) \to 1$, *as* $m \to \infty$ *i.e.* $\lim_{m \to \infty} \nu(v_m - \ell, t) = 1$.

Definition 2.8. *A FNS* $(\mathcal{X}, \nu, *)$ *is said to be complete if every Cauchy sequence in* X *is converges to an element in* $\mathcal{X}$.

Definition 2.9. *Let* $(\mathcal{X}, \nu, *)$ *and* $(\mathcal{Y}, \psi, *)$ *be two FNSs. A mapping* $h : \mathcal{X} \to \mathcal{Y}$ *is said to be continuous at* $x_0 \in \mathcal{X}$ *if for all* $x \in \mathcal{X}$, *for each* $\epsilon \in (0,1)$ *and each* $t > 0$, *there exists* $a \in (0,1)$ *and* $s > 0$ *such that* $\psi(h(x) - h(x_0), t) > 1 - \epsilon$ *whenever* $\nu(x - x_0, s) > 1 - a$. *The mapping* h *is continuous on* $\mathcal{X}$ *if it is continuous at every point in* $\mathcal{X}$.

In the following sections, we consider $\mathcal{X}$ for a FNS $(\mathcal{X}, \nu, *)$. Otherwise, we will be mentioned it.

3. Filter slowly oscillating sequences

The concepts of filter slowly oscillating sequence and filter slowly oscillating continuous function are introduced and some interesting results related to these notions are established.

Definition 3.1. *A nonempty family of sets $\mathcal{F} \subset P(\mathbb{N})$ is said to be a filter on $\mathbb{N}$ if the following three properties are satisfied:*

(i) $\emptyset \notin \mathcal{F}$

(ii) *for each $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$*

(iii) *for each $A \in \mathcal{F}$ and each $B \supset A$, we have $B \in \mathcal{F}$.*

Definition 3.2. *A sequence (v_m) in a FNS $\mathcal{X}$ is said to be filter convergent to ℓ if for each $r \in (0,1)$ and each $t > 0$ such that $\{m \in \mathbb{N} : \nu(v_m - \ell, t) > 1 - r\} \in \mathcal{F}$. We denote the set of all filter convergent sequence on $(\mathcal{X}, \nu, *)$ by* **FC**.

Definition 3.3. *Let $(\mathcal{X}, \nu, *)$ and $(\mathcal{Y}, \psi, *)$ be two FNSs. A mapping $h : \mathcal{X} \to \mathcal{Y}$ is said to be filter continuous at $x_0 \in \mathcal{X}$ if for all $x \in \mathcal{X}$, for each $\epsilon \in (0,1)$ and each $t > 0$ there exists $a \in (0,1)$ and $s > 0$ such that*

$$\{x \in \mathcal{X} : \psi(h(x) - h(x_0), t) > 1 - \epsilon\} \in \mathcal{F}$$

whenever

$$\{x \in \mathcal{X} : \nu(x - x_0, s) > 1 - a\} \in \mathcal{F}.$$

The mapping h is filter continuous on $\mathcal{X}$ if it is filter continuous at each point on $\mathcal{X}$.

Definition 3.4. *Let $(\mathcal{X}, \nu, *)$ and $(\mathcal{Y}, \psi, *)$ be two FNSs. A mapping $h : \mathcal{X} \to \mathcal{Y}$ is said to be filter sequentially continuous at $x_0 \in \mathcal{X}$ if for any sequence (v_m) in $\mathcal{X}$ such that $h(v_m) \to_{\mathcal{F}} h(x_0)$ whenever $v_m \to_{\mathcal{F}} x_0$. i.e. $\mathcal{F} - \lim_{m\to\infty} \nu(v_m - x_0, t) = 1 \Rightarrow \mathcal{F} - \lim_{m\to\infty} \psi(h(v_m) - h(x_0), t) = 1$ for all $t > 0$. If h is filter sequentially continuous at each point of $\mathcal{X}$ then h is filter sequentially continuous on $\mathcal{X}$.*

Definition 3.5. *Let $(\mathcal{X}, \nu, *)$ and $(\mathcal{Y}, \psi, *)$ be two FNSs. A mapping $h : \mathcal{X} \to \mathcal{Y}$ is said to be filter strongly continuous at $x_0 \in \mathcal{X}$ if for each $t > 0$ there exists $s > 0$ such that*

$$\{x \in \mathcal{X} : \psi(h(x) - h(x_0), t) \geq \nu(x - x_0, s)\} \in \mathcal{F}.$$

h is filter strongly continuous on $\mathcal{X}$ if it is filter strongly continuous at every point on $\mathcal{X}$.

Definition 3.6. *A sequence (v_m) of points in a FNS $\mathcal{X}$ is called quasi-Cauchy if for each $r \in (0,1)$ and each $t > 0$ such that $\nu(v_{m+1} - v_m, t) > 1 - r$.*

Now we introduce the notion of filter quasi-Cauchy and filter slowly oscillating sequences in FNS.

Definition 3.7. *A sequence (v_m) of points in a FNS $\mathcal{X}$ is called filter quasi-Cauchy if for each $r \in (0,1)$ and each $t > 0$ such that*

$$\{m \in \mathbb{N} : \nu(v_{m+1} - v_m, t) > 1 - r\} \in \mathcal{F}.$$

Example 3.8. *For $\mathcal{X} = \mathbb{R}$, $(v_m) = (\ln m)$, $(v_m) = (\ln \ln m)$ are filter quasi-Cauchy sequences.*

Definition 3.9. *A sequence (v_m) of points in a FNS $\mathcal{X}$ is said to be filter slowly oscillating (**FSO**, in short) if for each $r \in (0,1)$ and each $t > 0$, there exist $a > 0$, a positive integer $\mathcal{N}$ such that*

$$\{\mathcal{N} \leq m \leq p \leq (1+a)m \text{ and } p \in \mathbb{N} : \nu(v_p - v_m, t) > 1 - r\} \in \mathcal{F}.$$

It is obvious that a filter convergent sequence is **FSO**, because **FC** sequence is filter Cauchy sequence, but the converse need not to be true. See the following example.

For $\mathcal{X} = \mathbb{R}$, the sequence $(v_m) = \left(\sum_{i=1}^{m} \frac{1}{i}\right)$ is **FSO** but not **FC**. From definition it is clear that **FSO** sequences are not filter Cauchy in general. Also from definition, follows that a **SO** sequence is **FSO** but the converse is not true in general.

Example 3.10. *For $\mathcal{X} = \mathbb{R}$, we define a sequence (v_m) by*

$$v_m = \begin{cases} (-2)^m & , \text{if } m = i^2, m = i^2+1, i \in \mathbb{N}; \\ 0 & , \text{otherwise} \end{cases}$$

*Then the sequence (v_m) is **FSO**, but not **SO** because*

$$|v_{i^2+1} - v_{i^2}| = 3.2^{i^2} \nrightarrow 0 \text{ as } i \to \infty,$$

whenever $1 < \frac{(i^2+1)}{i^2} \to 1$ as $i \to \infty$.

We now introduce the definition of filter slowly oscillating continuous as follows:

Definition 3.11. *Let K be subset of a FNS $\mathcal{X}$. A function h defined on K is called filter slowly oscillating continuous (**FSOC**, in short) if it transforms **FSO** sequences to **FSO** sequences of points in K, that is, $(h(v_m))$ is **FSO** whenever (v_m) is **FSO** sequences of points in K.*

Proposition 3.12. *The set **FC** is a proper subset of the set of **FSO**.*

Proof. The proof of this result follows from the both definitions. □

Proposition 3.13. *The set **FSO** is a closed subalgebra of ℓ_∞.*

Proof. The proof of this result follows from fact that sum of two **FSO** sequences and product of **FSO** sequences is a **FSO** sequence. Also, **FSO** sequences are bounded. □

Theorem 3.14. *If h is **FSOC** on $K \subset \mathcal{X}$ then it is filter continuous on K.*

Proof. Suppose that h is **FSOC** on K. Let (v_m) be a **FC** sequence of points in K with $\mathcal{F} - \lim v_m = x_0$. Then the sequence

$$(w_m) = (v_1, v_0, v_2, v_0, ..., v_{m-1}, v_0, v_m, v_0, ...)$$

is also **FC** to x_0 and hence (w_m) is **FSO**. Since h is **FSOC**, therefore

$$(h(w_m)) = (h(v_1), h(v_0), h(v_2), h(v_0), ..., h(v_{m-1}), h(v_0), h(v_m), h(v_0), ...)$$

is also a **FSO** sequence. Hence $(h(w_m))$ is a filter quasi-Cauchy sequence. Now for $r \in (0,1)$ and for every $t > 0$, there is an integer $\mathcal{N} > 0$ such that

$$\nu(h(v_m) - h(x_0), t) > 1 - r \text{ for } n \geq \mathcal{N}.$$

i.e.

$$\{m \in \mathbb{N} : \nu(h(v_m) - h(x_0), t) > 1 - r\} \in \mathcal{F}.$$

It gives $\mathcal{F} - \lim h(v_m) = h(v_0)$. □

In general the converse is not true. For $\mathcal{X} = \mathbb{R}$, it follows from the function $h(x) = e^x$ and the sequence $(v_m) = (\ln m)$.

Corollary 3.15. *If h is* **FSOC***, then it is continuous in the ordinary sense.*

Theorem 3.16. *The sum of two* **FSOC** *functions is* **FSOC**.

Proof. Let h and g be **FSOC** functions on $K \subset \mathcal{X}$. To show that $h + g$ is **FSOC** on K. Let (v_m) be a **FSO** sequence in K. Then $(h(v_m))$ and $(g(v_m))$ are **FSO** sequences. Therefore for every $r \in (0,1)$ and every $t > 0$, there exist a positive integers $m_1 = m_1(r)$ and $m_2 = m_2(r)$ such that

$$\left\{m_1 \leq p \leq (1+a)m \text{ and } p \in \mathbb{N} : \nu\left(h(v_p) - h(v_m), \frac{t}{2}\right) > 1 - r\right\} \in \mathcal{F};$$

$$\left\{m_2 \leq p \leq (1+a)m \text{ and } p \in \mathbb{N} : \nu\left(g(v_p) - g(v_m), \frac{t}{2}\right) > 1 - r\right\} \in \mathcal{F}.$$

We choose $m_0 = \max\{m_1, m_2\}$, then we have

$$\begin{aligned} &\{m_0 \leq p \leq (1+a)m \text{ and } p \in \mathbb{N} : \nu((h+g)(v_p) - (h+g)(v_m), t) > 1 - r\} \\ &\supseteq \left\{m_1 \leq p \leq (1+a)m \text{ and } p \in \mathbb{N} : \nu\left(h(v_p) - h(v_m), \frac{t}{2}\right) > 1 - r\right\} \\ &\cap \left\{m_2 \leq p \leq (1+a)m \text{ and } p \in \mathbb{N} : \nu\left(g(v_p) - g(v_m), \frac{t}{2}\right) > 1 - r\right\}. \end{aligned}$$

Therefore,

$$\{m_0 \leq p \leq (1+a)m \text{ and } p \in \mathbb{N} : \nu((h+g)(v_p) - (h+g)(v_m), t) > 1 - r\} \in \mathcal{F}.$$

This completes the proof. □

Theorem 3.17. *If h is a uniformly continuous function on $K \subset \mathcal{X}$, then h is* **FSOC** *on K.*

Proof. Suppose h is a uniformly continuous function on K. Let (v_m) be a **FSO** sequence in K. Since h is uniformly continuous on K, then for $r \in (0,1)$ and every $t > 0$ there exists $s \in (0,1)$ and every $v > 0$ such that $\nu(h(x) - h(y), v) > 1 - s$ whenever $\nu(x - y, t) > 1 - r$ for every $x, y \in K$. Since (v_m) is **FSO**, for $r \in (0,1)$ and every $t > 0$, there is a $a > 0$ and a positive integer $\mathcal{N} = \mathcal{N}(r) = \mathcal{N}_1(s)$ such that

$$\{\mathcal{N} \leq m \leq p \leq (1+a)m \text{ and } p \in \mathbb{N} : \nu(v_p - v_m, t) > 1 - r\} \in \mathcal{F}.$$

Therefore, we have

$$\{\mathcal{N} \le m \le p \le (1+a)m \text{ and } p \in \mathbb{N} : \nu(h(v_p) - h(v_m), v) > 1 - s\} \in \mathcal{F}.$$

It follows that $(h(v_m))$ is **FSO**. Hence h is **FSOC** on K. □

Definition 3.18. *A sequence (v_m) in $\mathcal{X}$ is called filter Cesàro* **SO** *if (l_n) is* **FSO**, *where $l_m = \frac{1}{m}\sum_{p=1}^{m} v_p$, is the Cesàro means of the sequence (v_m). Also a function h on $K \subset \mathcal{X}$ is called filter Cesàro* **SO** *continuous if it preserves filter Cesàro* **SO** *sequences in K.*

By using the similar argument used in proof of Theorem 3.17, we immediately have the following result.

Theorem 3.19. *If h is a uniformly continuous on $K \subset \mathcal{X}$ and (v_m) is a filter* **SO** *sequence in K, then $(f(v_m))$ is filter Cesàro* **SO**.

Definition 3.20. *A sequence of functions (h_m) on $K \subset \mathcal{X}$ is said to be uniformly filter convergent to a function h if for $r \in (0,1)$ and for every $t > 0$ the set*

$$\{x \in K, m \in \mathbb{N} : \nu(h_m(x) - h(x), t) > 1 - r\} \in \mathcal{F}.$$

Theorem 3.21. *If (h_m) is a sequence of* **FSOC** *functions on $K \subset \mathcal{X}$ and (h_m) is uniformly filter convergent to a function h on K, then h is* **FSOC** *on K.*

Proof. Let (v_m) be any **FSO** sequence in K. Since (h_m) is uniformly filter convergence to h then we have for $r \in (0,1)$ and every $t > 0$

$$\left\{x \in K \text{ and } m \in \mathbb{N} : \nu\left(h_m(x) - h(x), \frac{t}{3}\right) > 1 - r\right\} \in \mathcal{F}.$$

Also since each h_m is **FSOC**, $\exists\, \mathcal{N} > 0$ and $a > 0$ such that

$$\left\{\mathcal{N} \le m \le k \le (1+a)m \text{ and } k \in \mathbb{N} : \nu\left(h_{\mathcal{N}}(v_k) - h_{\mathcal{N}}(v_m), \frac{t}{3}\right) > 1 - r\right\} \in \mathcal{F}.$$

Therefore, we have

$$\begin{aligned}
&\{\mathcal{N} \le m \le p \le (1+a)m \text{ and } p \in \mathbb{N} : \nu\left(h(v_p) - h(v_m), t\right) > 1 - r\} \\
&\supseteq \left\{\mathcal{N} \le m \le p \le (1+a)m \text{ and } p \in \mathbb{N} : \nu\left(h(v_p) - h_{\mathcal{N}}(x_m), \frac{t}{3}\right) > 1 - r\right\} \\
&\cap \left\{\mathcal{N} \le m \le p \le (1+a)m \text{ and } p \in \mathbb{N} : \nu\left(h_{\mathcal{N}}(v_p) - h_{\mathcal{N}}(v_m), \frac{t}{3}\right) > 1 - r\right\} \\
&\cap \left\{\mathcal{N} \le m \le p \le (1+a)m \text{ and } p \in \mathbb{N} : \nu\left(h_{\mathcal{N}}(v_p) - h(v_m), \frac{t}{3}\right) > 1 - r\right\}.
\end{aligned}$$

It implies that

$$\{\mathcal{N} \le m \le p \le (1+a)m \text{ and } p \in \mathbb{N} : \nu(h(v_p) - h(v_m), t) > 1 - r\} \in \mathcal{F}.$$

This gives that $(h(v_m))$ is a **FSO** sequences in K. □

Corollary 3.22. *If* (h_m) *is a sequence of* **FSOC** *functions on* $K \subset \mathcal{X}$ *and* (h_m) *is uniformly convergent to a function* h *on* K, *then* h *is* **SO** *continuous on* K.

Using the same techniques as in Theorem 3.21, one can obtained the following result.

Theorem 3.23. *If* (h_m) *is a sequence of filter Cesàro* **SO** *continuous functions on* $K \subset \mathcal{X}$ *and* (h_m) *is uniformly filter convergent to a function* h *on* K, *then* h *is filter Cesàro* **SO** *continuous on* K.

Theorem 3.24. *Let* $\mathcal{X}$ *be complete. The set of all* **FSOC** *functions on* $K \subset \mathcal{X}$ *is a closed subset of all continuous functions on* K, *that is* $\overline{\mathbf{FSCO}(K)} = \mathbf{FSOC}(K)$, *where* $\overline{\mathbf{FSOC}(K)}$ *denote the set of all cluster points of* $\mathbf{FSOC}(K)$.

Proof. Let h be an element of $\overline{\mathbf{FSOC}(K)}$. Then there is a sequence (h_m) in $\mathbf{FSOC}(K)$ such that $\lim_{m\to\infty} h_m = h$. To show that h is **FSOC** on K. Now let (v_m) be a **FSO** sequence in K. Since (h_m) converges to h, then for $r \in (0,1)$ and every $t > 0$, there exists a positive integer $\mathcal{N} = \mathcal{N}(r)$ such that for all $x \in K$ and for all $m \geq \mathcal{N}$,

$$\nu\left(h(x) - h_m(x), \frac{t}{3}\right) > 1 - r.$$

For all $x \in K$, we have

$$\left\{m \in \mathbb{N} : \nu\left(h(x) - h_m(x), \frac{t}{3}\right) > 1 - r\right\} \in \mathcal{F}.$$

Also since h_m is **FSOC** on K, then for $r \in (0,1)$ and for every $t > 0$ there exists a positive integer $\mathcal{N} = \mathcal{N}(r)$ such that

$$\left\{\mathcal{N} \leq m \leq p \leq (1+a)m \text{ and } p \in \mathbb{N} : \nu\left(h_{\mathcal{N}}(v_p) - h_{\mathcal{N}}(v_m), \frac{t}{3}\right) > 1 - r\right\} \in \mathcal{F}.$$

Also we have

$$\begin{aligned}
&\{\mathcal{N} \leq m \leq p \leq (1+a)m \text{ and } p \in \mathbb{N} : \nu(h(v_p) - h(v_m), t) > 1 - r\} \\
&\supseteq \left\{\mathcal{N} \leq m \leq p \leq (1+a)m \text{ and } p \in \mathbb{N} : \nu\left(h(v_p) - h_{\mathcal{N}}(v_m), \frac{t}{3}\right) > 1 - r\right\} \\
&\cap \left\{\mathcal{N} \leq m \leq p \leq (1+a)m \text{ and } p \in \mathbb{N} : \nu\left(h_{\mathcal{N}}(v_p) - h_{\mathcal{N}}(v_m), \frac{t}{3}\right) > 1 - r\right\} \\
&\cap \left\{\mathcal{N} \leq m \leq p \leq (1+a)m \text{ and } p \in \mathbb{N} : \nu\left(h_{\mathcal{N}}(v_p) - h(v_m), \frac{t}{3}\right) > 1 - r\right\}.
\end{aligned}$$

By the finite intersection property of $\mathcal{F}$, gives that

$$\{\mathcal{N} \leq m \leq p \leq (1+a)m \text{ and } p \in \mathbb{N} : \nu(h(v_p) - h(v_m), t) > 1 - r\} \in \mathcal{F}.$$

Thus h is **FSOC** on K. □

Corollary 3.25. *Let $\mathcal{X}$ be complete. The set of all* **FSOC** *functions on $K \subset \mathcal{X}$ is a complete subspace of the space of all continuous functions on K.*

Definition 3.26. *An element $x_0 \in \mathcal{X}$ is called a filter limit point of $K \subset \mathcal{X}$ if there is an K-valued sequence of points with filter limit x_0. It follows that the set of all filter limit points of K is equal to the set of all limit points of K in the ordinary sense. An element x_0 in $\mathcal{X}$ is called a filter accumulation point of a subset K if it is a filter limit point of the set $K \setminus \{x_0\}$. The set of all filter accumulation points of K is equal to the set of all accumulation points of K in the ordinary sense.*

Definition 3.27. *A function h on $\mathcal{X}$ is said to have a filter sequential limit at a point x_0 of $\mathcal{X}$ if the image sequence $(h(v_m))$ is filter convergent to x_0 for any* **FC** *sequence (v_m) with filter limit x_0 and a function h is to be filter sequentially continuous at a point x_0 of X if the sequence $(h(v_m))$ is* **FC** *to $h(x_0)$ for a* **FC** *sequence (v_m) with filter limit x_0. Then h is filter sequentially continuous on $\mathcal{X}$, if h is filter sequentially continuous at every point in $\mathcal{X}$.*

Lemma 3.28. *A function h on $\mathcal{X}$ has a filter sequential limit at a point x_0 of $\mathcal{X}$ if and only if it has a filter limit at a point x_0 of $\mathcal{X}$ in ordinary sense.*

Proof. The proof follows immediately, because any **FC** sequence has a convergent subsequence (also see [12]). □

Theorem 3.29. *A function h is filter sequentially continuous on $\mathcal{X}$ if and only if it is continuous in ordinary sense.*

Proof. The proof follows from the fact that any **FC** sequence has a convergent subsequence and Lemma 3.28. □

Theorem 3.30. *Let $h : \mathcal{X} \to \mathcal{X}$ be any function and (v_m) be a sequence in $\mathcal{X}$ such that $\mathcal{F} - \lim_{m\to\infty} v_m = x_0$ implies $\lim_{m\to\infty} h(v_m) = h(x_0)$, then it is a constant function.*

Proof. The proof follows form similar technique as used in [13, Theorem 3]. □

Theorem 3.31. *If a function is* **SO** *continuous on $K \subset \mathcal{X}$, then it is filter sequentially continuous on K.*

Proof. Let h be any **SO** continuous on K. Then h is continuous on K (see [13, Theorem 3.1]. Also from Theorem 3.29, we get h is filter sequentially continuous on K. □

Theorem 3.32. *If a function is δ-ward continuous on $K \subset \mathcal{X}$ then it is filter sequentially continuous on K.*

Proof. Given h is a δ-ward continuous function on K. Then h is continuous (see [9, Corollary 2]). Then h is filter sequentially continuous on K follows from Theorem 3.29. □

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