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A BIFURCATION PHENOMENON FOR ONE-DIMENSIONAL MINKOWSKI-CURVATURE EQUATION

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Abstract. In this paper, applying the bifurcation method and topological analysis, we investigate the global structures of solutions for one-dimensional Minkowski-curvature problems under certain behavior of nonlinear term near zero.

1. Introduction

In this paper, we are concerned with the global structures of nodal solutions for the following one-dimensional problem

$$(P_\lambda) \qquad \begin{cases} -\Big(\phi(u'(t))\Big)' = \lambda m(t) f(u(t)), & t \in (0,T), \\ u(0) = 0 = u(T), \end{cases}$$

where $\phi(y) = \frac{y}{\sqrt{1-|y|^2}}$, $y \in (-1,1)$, λ is a positive real parameter, $m : (0,T) \to [0,\infty)$ satisfies $m \not\equiv 0$ in any compact subinterval of $[0,T]$, $f : (-a,a) \to \mathbb{R}$ is a continuous function with $0 < a \leq \infty$ and $f(s)s > 0$ for $s \neq 0$. Denote $f_0 \triangleq \lim\limits_{s\to 0} \frac{f(s)}{s}$ and certain category of weight functions $\mathcal{A}$ can be defined as

$$\mathcal{A} \triangleq \{m \in L^1_{loc}(0,T) : \int_0^T \tau(T-\tau)m(\tau)d\tau < \infty\}.$$

In differential geometry and the theory of classical relativity, it plays a critical role in the study of determining existence and regularity properties of maximal and constant mean curvature hypersurfaces, see [1, 2, 3] and the references therein.

We say u a solution of problem (P_λ) if $u \in C[0,T] \cap C^1(0,T)$, $|u'(t)| < 1$ for $t \in (0,T)$, $\phi(u')$ is absolutely continuous in any compact subinterval of $(0,T)$, and u satisfies the equation and the boundary conditions in problem (P_λ). In

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[26], the authors classified the solutions by introducing "non $\frac{\pi}{4}$-tangential solution" defined as $u \in C^1[0,T]$ and $|u'(t)| < 1$ for $t \in [0,T]$ and "$\frac{\pi}{4}$-tangential solution" defined as $u \in C^1[0,T]$ and either $|u'(0)| = 1$ or $|u'(1)| = 1$. Non $\frac{\pi}{4}$-tangential solution has better topological properties for problem (P_λ) characterized by the second order mean curvature operator, for instance, [4, 5] used Leray-Schauder degree type arguments to study the nonexistence, existence, and multiplicity of radial solutions involving mean curvature operator in a bounded domain, which correspond to non $\frac{\pi}{4}$-tangential solutions.

In Theorem 2.2 of [7], Coelho-Corsato-Obersnel-Omari studied positive solutions of the following one-dimensional problem by global bifurcation technique

$$\begin{cases} -\Big(\phi(u'(t))\Big)' = \lambda f\big(t, u(t)\big), \ t \in (0,T), \\ u(0) = 0 = u(T). \end{cases} \tag{1}$$

Under the assumptions

(A_1) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the L^∞-Carath*é*odory conditions,

(A_2) $\lim\limits_{s\to 0^+} \frac{f(t,s)}{s} = m(t)$ uniformly almost everywhere in $[0,T]$,

(A_3) $m \in L^\infty(0,T)$ satisfies $\max\{m,0\} > 0$,

they proved that there exists $\lambda_* \in (0, \lambda_1(m)]$ such that for all $\lambda \in (0, \lambda_*)$, problem (1) has no positive solution, and for all $\lambda \in (\lambda_1(m), \infty)$, it has at least one positive solution, where $\lambda_1(m)$ is the positive principal eigenvalue of problem (2) given below.

It is necessary to point out that the positive solution in [7] means non $\frac{\pi}{4}$-tangential solution. It is interesting to note that results on nodal solutions for the Dirichlet problem of the one-dimensional Minkowski-curvature equation, such as problem (P_λ), have not been introduced yet. This motivates us to investigate bifurcations and asymptotic behaviors of solutions curves of problem (P_λ) under several behaviors of nonlinear term f near zero, *i.e.*, linear, superlinear, sublinear, respectively.

In this paper, we consider a bifurcation phenomenon of nodal solutions for the case that the nonlinear term is linear near zero, *i.e.* $0 < f_0 < \infty$. To state our main result, we define the subspace $E := \{u \in C^1[0,T] : u(0) = u(T) = 0\}$ with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$. Let N_k^+ $(k \in \mathbb{N})$ denote the set of $u \in E$ such that u has exactly $k-1$ simple interior zeros in $(0,T)$, $u'(0^+) > 0$ and all zeros of u on $[0,T]$ are simple. Set $N_k^- = -N_k^+$ and $N_k = N_k^- \cup N_k^+$. Denote the closure of the set of nontrivial solution pairs of problem (P_λ) by $\mathcal{S}$, that is,

$$\mathcal{S} = \overline{\{(\lambda, u) \in \mathbb{R} \times E : u \ is \ a \ nontrivial \ solution \ of \ problem \ (D) \ with \ \lambda > 0\}}.$$

Let $\nu \in \{+,-\}$ and $\lambda_k(m)$ be the k-th eigenvalue of the following problem

$$\begin{cases} -u'' = \lambda m(t) u, \qquad t \in (0,T), \\ u(0) = u(T) = 0. \end{cases} \tag{2}$$

The main result of this paper is the following.

Theorem 1.1. *Assume $m \in \mathcal{A}$ and $0 < f_0 < \infty$. Then, for each $k \in \mathbb{N}$, there exist two unbounded continua $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ of $\mathcal{S}$ bifurcating from $(\frac{\lambda_k(m)}{f_0}, 0)$, satisfying*

(a) $\mathcal{C}_k^\nu \subset (((0,\infty) \times \{u \in N_k^\nu : \|u'\|_\infty < 1\}) \cup \{(\frac{\lambda_k(m)}{f_0}, 0)\})$;

(b) $Proj_{\mathbb{R}} \mathcal{C}_k^\nu = [\lambda_*, \infty) \subset (0, \infty)$, *for some* $\lambda_* \in (0, \frac{\lambda_k(m)}{f_0}]$;

(c) $\lim\limits_{\lambda \to \infty} \|u'\|_\infty = 1$, *for* $(\lambda, u) \in \mathcal{C}_k^+$ *or* $\mathcal{C}_k^-$.

2. Proof of Theorem 1.1

In this section, under conditions $m \in \mathcal{A}$ and $0 < f_0 < \infty$, we prove the existence of unbounded continuum $\mathcal{C}_k$ of problem (P_λ) using bifurcation theory and then show some properties of solutions in $\mathcal{C}_1$.

To get a continuous function on $\mathbb{R}$, we define $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ as

$$\widetilde{f}(s) = \begin{cases} f(s), & s \in [-\frac{T}{2}, \frac{T}{2}], \\ \text{linear}, & s \in (-T, -\frac{T}{2}) \cup (\frac{T}{2}, T), \\ 0, & s \in (-\infty, -T] \cup [T, \infty). \end{cases}$$

We see that problem (P_λ) is equivalent to the same type problem with f replaced with $\widetilde{f}$. So we replace f with $\widetilde{f}$ and for simplicity, we still denote $\widetilde{f}$ by f. Define function

$$h(s) = \begin{cases} \left(\sqrt{1 - s^2}\right)^3, & |s| \le 1, \\ 0, & |s| > 1. \end{cases}$$

Then we transform the problem (P_λ) to the following form

$$(S_\lambda) \qquad \begin{cases} -u'' = \lambda m(t) f(u) h(u'), \ t \in (0, T), \\ u(0) = 0 = u(T). \end{cases}$$

The following lemma shows that problem (S_λ) is equivalent to problem (P_λ). The proof is similar to Lemma 3.1 in [19].

Lemma 2.1. *A function $u \in E$ is a non $\frac{\pi}{4}$-tangential solution of problem (P_λ) if and only if it is a solution of problem (S_λ).*

Proof. It is clear that every solution $u \in E$ of problem (P_λ) is a solution of problem (S_λ). Now we show that every solution $u \in E$ of problem (S_λ) is also a solution of problem (P_λ). For this, we need to prove $\|u'\|_\infty < 1$. We prove it by contradiction. Suppose that $\|u'\|_\infty = 1$. It is known that there exists $t^* \in (0, T)$ such that $u'(t^*) = 0$, so $u'(t^*) = 0$. Since u' is continuous and $\|u'\|_\infty = 1$, without loss of generality, we choose $t_{\max} \in (0, T)$ satisfying $|u'(t_{\max})| = 1$. Thus, $0 < t^* < t_{\max} \le T$ or $0 \le t_{\max} < t^* < T$. We only consider the former case. The other case can be proved similarly. It satisfies

$u'(t^*) = 0$, $|u'(t)| < 1$ in $(t^*, t_{\max})$, and $|u'(t_{\max})| = 1$. It is easy to see that u satisfies the equation

$$-\phi(u'(t))' = \lambda m(t) f\big(u(t)\big),\ t \in [t^*, t_{\max}).$$

Integrating both sides of the above equation over $[t^*, t)$ for $t \in [t^*, t_{\max})$, by using the fact $mf(u) \in L^1(0,T)$, we get

$$\phi(u'(t)) = -\lambda \int_{t^*}^{t} m(\tau) f\big(u(\tau)\big) d\tau,$$

and then

$$\begin{aligned} |u'(t)| &= \left| \phi^{-1}\left(-\lambda \int_{t^*}^{t} m(\tau) f\big(u(\tau)\big) d\tau \right) \right| \\ &= \phi^{-1}\left(\lambda \left| \int_{t^*}^{t} m(\tau) f\big(u(\tau)\big) d\tau \right| \right),\ t \in [t^*, t_{\max}), \end{aligned}$$

and

$$\lim_{t \to t_{\max}} |u'(t)| = \phi^{-1}\left(\lambda \left| \int_{t^*}^{t_{\max}} m(\tau) f\big(u(\tau)\big) d\tau \right| \right).$$

Since $\lambda \left| \int_{t^*}^{t_{\max}} m(\tau) f\big(u(\tau)\big) d\tau \right| < \infty$, we get $|u'(t_{\max})| < 1$. This is a contradiction. □

Existence results and properties of eigenvalues for the weighted eigenvalue problem (2) are studied by Asakawa [16] as follows.

Lemma 2.2. *Assume $m \in \mathcal{A}$. Then the set of all nonnegative eigenvalues of problem (3.1) is a countable set $\{\lambda_n(m) : n \in \mathbb{N}\}$ satisfying $0 < \lambda_1(m) < \cdots < \lambda_n(m) < \cdots \to \infty$. Moreover, the algebraic multiplicity of $\lambda_n(m)$ is 1. Let u_n be a corresponding characteristic function to $\lambda_n(m)$, then the number of interior simple zeros of u_n in $(0,T)$ is $n-1$.*

Note that $N_k \cap N_j = \emptyset$ if $k \neq j$ and $N_k^{\pm}$ and N_k are open in E. By the condition $0 < f_0 < \infty$, we define a continuous function $\xi : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(s) = (f_0 + \xi(s))\, s \quad \text{and} \quad \lim_{s \to 0} \xi(s) = 0.$$

From now on, we consider the following problem as a bifurcation problem

$$(W) \qquad \begin{cases} -u'' = \lambda\,(f_0 + \xi(u))\, m(t) u h(u'),\ t \in (0,T), \\ u(0) = 0 = u(T). \end{cases}$$

The pair $(\lambda, u) \in \mathbb{R} \times E$ is a solution of problem (W) if and only if it is a solution of the equation

$$(D) \qquad u = \lambda \mathcal{L} u + \mathcal{H}(\lambda, u),$$

where the operator $\mathcal{L} : E \to E$ is defined as

$$\mathcal{L}u(t) = f_0 \int_0^T G(t,s) m(s) u(s) ds,$$

and operator $\mathcal{H} : \mathbb{R} \times E \to E$ is defined as

$$\mathcal{H}(\lambda, u(t)) = \int_0^T G(t,s) \left\{\lambda f_0 m(s)u(s) \left[h(u'(s)) - 1\right] + \lambda\xi(u(s))m(s)u(s)h(u'(s))\right\} ds,$$

with $G(t,s)$ given by

$$G(t,s) = \begin{cases} \frac{1}{T}(T-t)s, \ 0 \le s \le t \le T, \\ \frac{1}{T}(T-s)t, \ 0 \le t \le s \le T. \end{cases}$$

It is not difficult to check that $\mathcal{L}$ is compact linear in E, $\mathcal{H}$ is completely continuous in $\mathbb{R} \times E$ and $\mathcal{H} = o(\|u\|)$ near $u = 0$ uniformly on bounded λ intervals. It is either not difficult to check that problem (W) does not have a nontrivial solution if $\lambda \le 0$. Denote by $\mathcal{S}$ the closure in $\mathbb{R} \times E$ of the set of all nontrivial solution pairs of problem (D) with $\lambda > 0$, that is,

$$\mathcal{S} = \overline{\{(\lambda, u) \in \mathbb{R} \times E : u \text{ is a nontrivial solution of problem } (D) \text{ with } \lambda > 0\}}.$$

Similar to Lemma 3.5 and Lemma 3.7 in [20] with $\beta \equiv 1$, the following Lemma 2.3, Lemma 2.4, and Lemma 2.5 can be proved respectively.

Lemma 2.3. *Assume* $(\lambda, u) \in \mathcal{C}_k$ *and* $u \in \partial N_k^+$. *Also assume that there exists a sequence* $\{(\lambda_n, u_n)\} \subset \mathcal{S} \cap ([0, \infty) \times N_k^+)$ *converging to* (λ, u) *in* $\mathbb{R} \times E$. *Then,* $(\lambda, u) = (\frac{\lambda_k(m)}{f_0}, 0)$.

Lemma 2.4. *Assume* $(\lambda, u) \in \mathcal{C}_k$ *and* $u \in \partial N_k^-$. *Also assume that there exists a sequence* $\{(\lambda_n, u_n)\} \subset \mathcal{S} \cap ([0, \infty) \times N_k^-)$ *converging to* (λ, u) *in* $\mathbb{R} \times E$. *Then,* $(\lambda, u) = (\frac{\lambda_k(m)}{f_0}, 0)$.

Lemma 2.5. $\mathcal{C}_k \subset ([0, \infty) \times N_k^+) \cup ([0, \infty) \times N_k^-) \cup (\frac{\lambda_k(m)}{f_0}, 0)$.

We show the asymptotic behavior of $\|u'\|_\infty$. The following two lemmas are inspired by Lemma 4.1 and Theorem 1.3 in [20]. we also give their proofs for readers' convenience.

Lemma 2.6. *Assume that* $(\lambda, u) \in \mathcal{C}_k^\nu \setminus \{(\frac{\lambda_k(m)}{f_0}, 0)\}$. *Then there exists a positive constant* b_0 *such that* $\|u'\|_\infty \ge b_0$ *as* $\lambda \to \infty$.

Proof. Suppose on the contrary that there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathcal{C}_k^+ \setminus \{(\frac{\lambda_k(m)}{f_0}, 0)\}$ satisfying $\|u_n'\|_\infty \to 0$ as $\lambda_n \to \infty$. Together with the fact that $|u_n(t)| \le \frac{T}{2}\|u_n'\|_\infty$ for $t \in [0, T]$, it follows that

$$\lim_{n\to\infty} (\lambda_n, u_n) = (\infty, 0).$$

On the other hand, $\{(\lambda_n, u_n)\}$ satisfies problem (D) and then by Lemma 2.3, we can deduce that $\lim_{n\to\infty} \lambda_n = \frac{\lambda_k(m)}{f_0}$, which contradicts $\lim_{n\to\infty} \lambda_n = \infty$. □

Lemma 2.7. $\lim_{\lambda\to\infty} \|u'\|_\infty = 1$, *for* $(\lambda, u) \in \mathcal{C}_k^\nu$.

Proof. To show the asymptotic behavior of $\|u'\|_\infty$ as $\lambda \to \infty$ for $(\lambda, u) \in \mathcal{C}_k^\pm \setminus \{(\frac{\lambda_k(m)}{f_0}, 0)\}$, without loss of generality, we choose a sequence $(\lambda_n, u_n) \in \mathcal{C}_k^+ \setminus \{(\frac{\lambda_k(m)}{f_0}, 0)\}$ satisfying $\lambda_n \to \infty$ as $n \to \infty$. It follows from Lemma 2.6 that there exist two constants $\delta > 0$ and $N_0 \in \mathbb{N}^+$ such that $\|u_n'\|_\infty \geq \delta$ for all $n \geq N_0$. Hence, we will consider the subsequence $(\lambda_n, u_n) \in \mathcal{C}_k^+ \setminus \{(\frac{\lambda_k(m)}{f_0}, 0)\}$ for all $n \geq N_0$ from now. By the definition of N_k^+, $u_n(t)$ has $k-1$ simple zeros in $(0, T)$. Then we denote the zeros on $[0, T]$ by $0 = t_n^0 < t_n^1 < \cdots < t_n^{k-1} < t_n^k = T$ and $t_n^{**} \in [0, T]$ satisfying $|u_n'(t_n^{**})| = \|u_n'\|_\infty \geq \delta$. Then there exists $j \in \{0, 1 \cdots, k-1\}$ such that $t_n^{**} \in [t_n^j, t_n^{j+1}]$.

Claim 1. $t_n^{**} = t_n^j$ or t_n^{j+1}.
Assume that $u_n(t)$ is positive in (t_n^j, t_n^{j+1}) (similar argument for the negative case). Obviously, u_n' has at least one zero in (t_n^j, t_n^{j+1}). Let $t_n^* = \min\{t \in (t_n^j, t_n^{j+1}) : u_n'(t) = 0\}$. We integrate the first equation in problem (P_λ) from t_n^* to t for $t \in [t_n^j, t_n^{j+1}]$ to get

$$u_n'(t) = -\phi^{-1}\left(\int_{t_n^*}^t \lambda_n m(\tau) f(u_n(\tau)) d\tau\right).$$

It can be seen that $u_n'(t)$ is decreasing on $[t_n^*, t_n^{j+1}]$ and $u_n'(t) < 0$ in $(t_n^*, t_n^{j+1}]$, $u_n'(t)$ is decreasing on $[t_n^j, t_n^*]$ and $u_n'(t) > 0$ in $[t_n^j, t_n^*)$. Thus, t_n^* is the unique zero of $u_n'(t)$, $u'(t) \leq u'(t_n^j)$ on $[t_n^j, t_n^*]$ and $|u'(t)| \leq |u'(t_n^{j+1})|$ on $[t_n^*, t_n^{j+1}]$. It follows that $t_n^{**} = t_n^j$ or t_n^{j+1}. Without loss of generality, we only consider case $t_n^{**} = t_n^j$, then we can get case (ii) after some suitable modifications. Define

$$F_n(s) = \max_{t \in [s, t_n^*]} u_n'(t), \ s \in [t_n^j, t_n^*], \text{ for all } n \geq N_0.$$

Then $F_n(s)$ is nonincreasing on $[t_n^j, t_n^*]$. Furthermore, there exists $\delta_0 > 0$ such that

$$F_n(t_n^j) = u_n'(t_n^j) = \|u_n'\|_\infty \geq \delta_0, \text{ for all } n \geq N_0.$$

Continuity of u_n' provides the existence of $\rho_n \in (t_n^j, t_n^*)$ with $F_n(\rho_n) = \frac{\delta_0}{2}$. Set

$$\rho_* = \liminf_{n \to \infty} \rho_n,$$

$$t_\infty^* = \liminf_{n \to \infty} t_n^*,$$

$$t_\infty^j = \liminf_{n \to \infty} t_n^j,$$

$$F(s) = \limsup_{n \to \infty} F_n(s).$$

It can be obtained directly that $\rho_* \leq t_\infty^*$. Choosing a subsequence, we have

$$F(\rho_*) = \lim_{n \to \infty} F_n(\rho_n) = \frac{\delta_0}{2} \ \text{ and } \ F(t_\infty^*) = \lim_{n \to \infty} F_n(t_n^*).$$

If $\rho_* = t^*_\infty$, choosing a subsequence, we have

$$\frac{\delta_0}{2} = F(t^*_\infty) = \lim_{n\to\infty} F_n(t^*_n) = 0,$$

and this is a contradiction. Thus, $\rho_* < t^*_\infty$. Similarly, it has $\rho_* > t^j_\infty$ since if $\rho_* = t^j_\infty$, we have $\frac{\delta_0}{2} = F(t^j_\infty) = \lim\limits_{n\to\infty} F_n(t^j_n) \geq \delta_0$ and we get a contradiction.

Claim 2. For any given $\widetilde{\rho} \in (0, t^*_\infty - \rho_*)$, there exists a constant σ_0 such that $u_n(t^*_\infty - \widetilde{\rho}) \geq \sigma_0$ for n large enough.
Suppose on the contrary that $u_n(t^*_\infty - \widetilde{\rho}) \to 0$ as $n \to \infty$. For any sufficiently small $\overline{\rho} \in (0, \rho_* - t^j_\infty)$ and n large enough, we have $[t^j_\infty + \overline{\rho}, t^*_\infty - \widetilde{\rho}] \subset [t^j_n, t^*_n]$.

Integrating the first equation in (P_λ) from t^*_n to t for $t \in [t^j_n, t^{j+1}_n]$, we get

$$u'_n(t) = -\phi^{-1}\left(\int_{t^*_n}^{t} \lambda_n m(\tau) f(u_n(\tau)) d\tau\right),$$

it can be seen that $u_n(t)$ is increasing in (t^j_n, t^*_n) and decreasing in (t^*_n, t^{j+1}_n). By virtue of the monotonicity of u_n in (t^j_n, t^*_n), one has that $u_n(t) \to 0$ as $n \to \infty$, for all $t \in [t^j_\infty + \overline{\rho}, t^*_\infty - \widetilde{\rho}]$. It follows that

$$\int_{t^j_\infty + \overline{\rho}}^{t} u'_n(\tau) d\tau = u_n(t) - u_n(t^j_\infty + \overline{\rho}) \to 0,$$

as $n \to \infty$, for all $t \in [t^j_\infty + \overline{\rho}, t^*_\infty - \widetilde{\rho}]$.

From the Fatou Lemma, we obtain

$$0 \leq \int_{t^j_\infty + \overline{\rho}}^{t} \liminf_{n\to\infty} u'_n(\tau) d\tau \leq \liminf_{n\to\infty} \int_{t^{**}_\infty + \overline{\rho}}^{t} u'_n(\tau) d\tau = 0,$$

for all $t \in [t^j_\infty + \overline{\rho}, t^*_\infty - \widetilde{\rho}]$. It follows that

$$\int_{t^j_\infty + \overline{\rho}}^{t} \liminf_{n\to\infty} u'_n(\tau) d\tau = 0,$$

for all $t \in [t^j_\infty + \overline{\rho}, t^*_\infty - \widetilde{\rho}]$. In particular, one has that

$$\int_{t^j_\infty + \overline{\rho}}^{t^*_\infty - \widetilde{\rho}} \liminf_{n\to\infty} u'_n(\tau) d\tau = 0.$$

We deduce that $\liminf\limits_{n\to\infty} u'_n \equiv 0$ *a.e.* on $[t^j_\infty + \overline{\rho}, t^*_\infty - \widetilde{\rho}]$. Thus, there exists a subsequence of $\{u'_n\}$, say $\{u'_n\}$ again, such that $u'_n \to 0$ as $n \to \infty$ for all $t \in [t^j_\infty + \overline{\rho}, t^*_\infty - \widetilde{\rho}]$. And $u'_n(t^j_\infty + \overline{\rho}) \to 0$ as $n \to \infty$. It follows from the arbitrary of $\overline{\rho}$ that $u'_n(t^j_\infty) \to 0$ as $n \to \infty$. By choosing a subsequence, it gives

$$u'_n(t^j_n) \to 0 \text{ as } n \to \infty.$$

Consequently, for $\varepsilon = \frac{1}{2}\delta_0$, there exists a sufficiently large n such that $|u'_n(t^j_n)| < \varepsilon$. It follows that $\|u'_n\|_\infty = \left|u'_n(t^j_n)\right| < \frac{\delta_0}{2}$ for sufficiently large n, which contradicts the fact of $\|u'_n\|_\infty \geq \delta_0$, for all $n \geq N_0$.

Now we take n large enough such that $[t^j_\infty + \frac{\overline{\rho}}{2}, t^*_\infty - \frac{\widetilde{\rho}}{2}] \subset [t^j_n, t^*_n]$. It follows from (3.9) that, for all $t \in [t^j_\infty + \overline{\rho}, t^*_\infty - \widetilde{\rho}]$,

$$|u'_n(t)| = \phi^{-1}\left(\left|\lambda_n \int_{t^*_n}^{t} m(\tau) f(u_n(\tau)) d\tau\right|\right).$$

By setting $t = t^*_\infty - \widetilde{\rho}$ in the above equality and using the Claim, it gives

$$|u'_n(t^*_\infty - \widetilde{\rho})| = \phi^{-1}\left(\lambda_n \int_{t^*_\infty - \widetilde{\rho}}^{t^*_n} m(\tau) f(u_n(\tau)) d\tau\right),$$

and then

$$1 \geq \|u'_n\|_\infty \geq |u'_n(t^*_\infty - \widetilde{\rho})| \geq \phi^{-1}\left(\lambda_n \int_{t^*_\infty - \widetilde{\rho}}^{t^*_\infty - \frac{\widetilde{\rho}}{2}} m(\tau) f(u_n(\tau)) d\tau\right) \to 1,$$

as $n \to \infty$. Therefore, $\lim\limits_{n\to\infty} \|u'_n\|_\infty = 1$. □

Proof of Theorem 1.1. By applying the Rabinowitz type global bifurcation theory (see Theorem 1.3 in [15]) and Lemma 2.2, we conclude that, for each $k \in \mathbb{N}^+$, there exists a continuum $\mathcal{C}_k$ in $\mathcal{S}$ bifurcating from $\mathbb{R} \times \{0\}$ at $\left(\frac{\lambda_k(m)}{f_0}, 0\right)$ which either is unbounded or contains a pair $\left(\frac{\widehat{\lambda}(m)}{f_0}, 0\right)$ for characteristic value $\frac{\widehat{\lambda}(m)}{f_0}$ of $\mathcal{L}$ with $\widehat{\lambda}(m) \neq \lambda_k(m)$. It follows from Lemma 2.5 that the second alternative cannot occur. Consequently, the first alternative is the only possibility and then $\mathcal{C}_k$ is unbounded. Furthermore, using Theorem 2 in [25], we can decompose $\mathcal{C}_k$ into two unbounded subcontinua $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ such that either $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ are both unbounded or $\mathcal{C}_k^+ \cap \mathcal{C}_k^- \neq \{(\frac{\lambda_k(m)}{f_0}, 0)\}$. By Lemma 2.5 and the fact $\|u'\|_\infty < 1$ for all $(\lambda, u) \in \mathcal{C}_k$, we can set that $\mathcal{C}_k^+ \subset ([0,\infty) \times \{u \in N_k^+ : \|u'\|_\infty < 1\}) \cup \{(\frac{\lambda_k(m)}{f_0}, 0)\}$ and $\mathcal{C}_k^- \subset ([0,\infty) \times \{u \in N_k^- : \|u'\|_\infty < 1\}) \cup \{(\frac{\lambda_k(m)}{f_0}, 0)\}$. Claim that $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ are both unbounded in λ-direction. Suppose the contrary, then there exists $(\lambda_0, u_0) \neq (\frac{\lambda_k(m)}{f_0}, 0)$ such that $(\lambda_0, u_0) \in \mathcal{C}_k^+ \cap \mathcal{C}_k^-$ and $u_0 \in N_k^+ \cap N_k^-$. This is a contradiction. Together with Lemma 2.1, (a) is attained.

Now, we prove (b). Denote $L_{f_0} = \max\limits_{s \in [-\frac{T}{2}, \frac{T}{2}]} \frac{f(s)}{s}$. Obviously, $f_0 \leq L_{f_0} < \infty$. Let χ be an eigenfunction corresponding to the eigenvalue $\frac{\lambda_k(m)}{f_0}$ of problem (2). Let $\lambda \geq 0$ and $u \in E$ be a $(k-1)$-nodal solution of problem (P_λ). Let $\{t_i\}$ and $\{s_i\}$ be the simple zeros of u and χ with $0 = t_0 < t_1 < \cdots < t_k = 1$ and $0 = s_0 < s_1 < \cdots < s_k = 1$, respectively. Then there exist $i, j \in \mathbb{N}$ such that $(t_{i-1}, t_i) \subset (s_{j-1}, s_j)$. We rewrite (t_{i-1}, t_i) by (a, b). Taking $y = u$, $b_1(t) = \lambda m(t) f(u(t))/u(t)$, and $z = \chi$, $b_2(t) = \lambda_k(m) m(t)$ in Lemma 2.5 of [20] (set $\beta(t) \equiv 1$), we have

$$\int_a^b (b_1(t) - b_2(t))\, |u|^2 dt > 0.$$

Since $b_1(t) = \lambda m(t) f(u(t))/u(t) \leq \lambda L_{f_0} m(t)$, we get

$$\lambda L_{f_0} \int_a^b m(t)|u|^2 dt > \lambda_k(m) \int_a^b m(t)|u|^2 dt.$$

This implies that

$$\lambda > \frac{\lambda_k(m)}{L_{f_0}}.$$

Thus, there exists $\lambda_* > 0$ such that $\lambda \geq \lambda_*$ for all $(\lambda, u) \in \mathcal{C}_k^\nu$. Together with the unboundedness of $\mathcal{C}_k^\nu$ in λ-direction, we conclude $Proj_{\mathbb{R}} \mathcal{C}_k^\nu = [\lambda_*, \infty) \subset (0, \infty)$. Moreover, it can be obtained directly from Lemma 2.7 that $\lim\limits_{\lambda \to \infty} \|u'\|_\infty = 1$, for $(\lambda, u) \in \mathcal{C}_k^\nu \setminus \{(\frac{\lambda_k(m)}{f_0}, 0)\}$, *i.e.*, (c) is proved. The proof is completed. □

Remark 2.8. *If $L_{f_0} = f_0$ in the above proof, then $Proj_{\mathbb{R}} \mathcal{C}_k^\nu = [\frac{\lambda_k(m)}{f_0}, \infty)$.*

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