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SYMMETRIC TOEPLITZ DETERMINANTS ASSOCIATED WITH A LINEAR COMBINATION OF SOME GEOMETRIC EXPRESSIONS

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Abstract. Let f be the function defined on the open unit disk, with $f(0) = 0 = f'(0) - 1$, satisfying $\operatorname{Re}\big(\alpha f'(z) + (1-\alpha)zf'(z)/f(z)\big) > 0$ or $\operatorname{Re}\big(\alpha f'(z) + (1-\alpha)(1 + zf''(z)/f'(z)\big) > 0$ respectively, where $0 \leq \alpha \leq 1$. Estimates for the Toeplitz determinants have been obtained when the elements are the coefficients of the functions belonging to these two subclasses.

1. Introduction

Let $\mathcal{A}$ be the class of all normalized analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{S}$ be its subclass consisting of univalent functions in $\mathbb{D}$. Motivated by an open research problem raised by Hayman [6] in 1967, many linear combinations of the geometric expressions of the form $(1-\beta)F(z) + \beta G(z)$ for real or complex constant β were studied. In particular, attentions were devoted to the class of β-starlike (or β-convex) functions $f \in \mathcal{A}$ satisfying the condition

$$\operatorname{Re}\big((1-\beta)F(z) + \beta G(z)\big) > 0 \tag{2}$$

where $F(z) = zf'(z)/f(z)$ and $G(z) = 1 + zf''(z)/f'(z)$ for real or complex β and for all $z \in \mathbb{D}$; see for example (Mocanu [14], Miller, Mocanu and Reade 1973 [12] and Miller, Mocanu and Reade 1978 [13]).

In 1975, Al Amiri and Reade [3] introduced and studied properties of a class of functions f satisfying the condition (2), where $F(z) = f'(z)$ and $G(z) =$

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$1+zf''(z)/f'(z)$ for fixed β and for all $z \in \mathbb{D}$. More precisely, they studied the class

$$(3) \qquad \mathcal{Q}(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left((1-\beta)f'(z) + \beta\Big(1+\frac{zf''(z)}{f'(z)}\Big)\right) > 0, z \in \mathbb{D} \right\}$$

where β is a real number. They showed that $f \in \mathcal{Q}(\beta)$, $\beta \leq 0$, satisfies $\operatorname{Re} f'(z) > 0$ for all $z \in \mathbb{D}$. Therefore, by a criterion of Noshiro [15] and Warschawski [18], $f \in \mathcal{Q}(\beta)$, $\beta \leq 0$, must be univalent in $\mathbb{D}$. In 1987, Ahuja and Silverman [1] observed that the convex function f defined by $f(z) = z/(1-z)$ is not in the class $\mathcal{Q}(\beta)$ for any $\beta > 0$ and $\beta \neq 1$. Thus, a function $f \in \mathcal{Q}(\beta)$ for $\beta > 0$ and $\beta \neq 1$ need not be univalent in $\mathbb{D}$. Also see [17]. In addition to these properties, we observe that by dividing the inequality in (3) by β, $\beta \neq 0$, and letting $k = 1/\beta - 1$, we see that (3) can be written as

$$(4) \qquad \operatorname{Re}\Big(1+\frac{zf''(z)}{f'(z)} + kf'(z)\Big) > 0, z \in \mathbb{D},$$

where $k \to -1$ as $\beta \to \infty$.

For convenience, we write $\mathcal{Q}(1-\alpha)$ as $\mathcal{L}(\alpha)$ and define another class $\mathcal{M}(\alpha)$ by choosing functions $F(z)$ and $G(z)$ in (2) by the geometric expressions: $F(z) = f'(z)$, $G(z) = zf'(z)/f(z)$. More precisely, for any fixed real number α in $[0,1]$, we define

$$(5) \qquad \mathcal{L}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\alpha f'(z) + (1-\alpha)\Big(1+\frac{zf''(z)}{f'(z)}\Big)\right) > 0 \right\}.$$

and

$$(6) \qquad \mathcal{M}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\alpha f'(z) + (1-\alpha)\frac{zf'(z)}{f(z)}\right) > 0 \right\}.$$

Note that for $\beta = 1-\alpha$, $\mathcal{L}(\alpha) = \mathcal{Q}(\beta)$. In view of this fact, it follows that a function belonging to $\mathcal{L}(\alpha)$ must be univalent in $\mathbb{D}$ when $\alpha \geq 1$. On the other hand, using argument given in [1], it follows that a function f in $\mathcal{L}(\alpha)$ need not be univalent in $\mathbb{D}$ for $\alpha < 1$, and α not equal to zero.

Obviously, well known classes $\mathcal{R}$, $\mathcal{S}^*$ and $\mathcal{K}$ are given by

$$\mathcal{R} = \mathcal{L}(1) = \mathcal{M}(1) = \{f \in \mathcal{A} : \operatorname{Re} f'(z) > 0, z \in \mathbb{D}\},$$

$$\mathcal{K} = \mathcal{L}(0) = \left\{ f \in \mathcal{A} : \operatorname{Re}\Big(1+\frac{zf''(z)}{f'(z)}\Big) > 0, z \in \mathbb{D} \right\}$$

and

$$\mathcal{S}^* = \mathcal{M}(0) = \left\{ f \in \mathcal{A} : \operatorname{Re}\Big(\frac{zf'(z)}{f(z)}\Big) > 0, z \in \mathbb{D} \right\}.$$

It is also well-known that a function f in $\mathcal{R}$ and $\mathcal{K}$, respectively, is close-to-convex and convex in $\mathbb{D}$. Recall, that $\mathcal{S}^*$ is the well known class of starlike functions in $\mathbb{D}$.

We now recall some definitions and notations of Toeplitz determinants. For the history and applications of Toeplitz matrix and determinant to several areas of pure and applied mathematics, one may refer to a survey article by Ye and

Lim [19]. Also, see [9, 10]. Related Hankel determinants were also studied, in particular, we refer to [2, 7, 8, 11]. We recall that Toeplitz symmetric matrices have constant entries along the diagonal. In 2017, Thomas [4] initiated the study of symmetric Toeplitz determinant $T_q(n)$ given by

$$T_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix}.$$

for small values of q and n, where a_n's are the complex coefficients of analytic function f given by (1).

In this paper, we obtain sharp estimates of the Toeplitz determinant $T_q(n)$ for functions in the classes $\mathcal{L}(\alpha)$ and $\mathcal{M}(\alpha)$ for $q = 2, 3$ and $n = 1, 2, 3$. In particular, we compute the bounds for the following determinants

$$T_2(2) = |a_3^2 - a_2^2|,\ T_2(3) = |a_4^2 - a_3^2|,$$

and

$$T_3(1) = |1 + 2a_2^2(a_3 - 1) - a_3^2|,\ T_3(2) = |(a_2 - a_4)(a_2^2 - 2a_2^3 + a_2 a_4)|.$$

where the entries are the coefficients of the function f of the form (1) in class $\mathcal{L}(\alpha)$ or $\mathcal{M}(\alpha)$. We have considered the case when a_2 is real and it would be nice to get bounds when a_2 is not necessarily real.

2. The Class $\mathcal{L}(\alpha)$

The first theorem gives bound for $T_2(2)$ wherein the elements a_2 and a_3 of the determinant matrix are the coefficients of the function $f \in \mathcal{L}(\alpha)$.

Theorem 2.1. *For $f = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{L}(\alpha)$, with a_2 real, we have*

$$|T_2(2)| = |a_3^2 - a_2^2| \leq \frac{72 - 84\alpha + 25\alpha^2}{9(2-\alpha)^2}.$$

The class $\mathcal{P}$ of Caratheodory functions consists of analytic functions p defined on $\mathbb{D}$ with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{D}$. The function $p \in \mathcal{P}$ has Taylor series

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

We make use of the following lemma in order to compute the required bounds.

Lemma 2.2. [5] *If the function given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \tag{7}$$

is in $\mathcal{P}$, *then,*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$
$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)y,$$

for some x, y *with* $c_1 \geq 0$, $|x| \leq 1$ *and* $|y| \leq 1$.

Proof of Theorem 2.1. Since $f \in \mathcal{L}(\alpha)$, there is an analytic function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$ such that

$$\alpha f'(z) + (1-\alpha)\Big(1 + \frac{zf''(z)}{f'(z)}\Big) = p(z). \tag{8}$$

The Taylor series expansion of the function f gives

$$\alpha f'(z) + (1-\alpha)\Big(1 + \frac{zf''(z)}{f'(z)}\Big) \tag{9}$$

$$\begin{aligned} &= 1 + 2a_2 z + \big(3a_3(2-\alpha) - 4a_2^2(1-\alpha)\big)z^2 \\ &\quad + \big(8a_2^3(1-\alpha) - 18a_2a_3(1-\alpha) \\ &\quad + a_4(12 - 8\alpha)\big)z^3 + \cdots . \end{aligned} \tag{10}$$

Then using (8), (9) and the expansion for the function p, the coefficients a_2 and a_3 can be expressed as a function of the coefficients c_i of $p \in \mathcal{P}$:

$$a_2 = \frac{c_1}{2}, \tag{11}$$

and

$$a_3 = \frac{c_2 + c_1^2(1-\alpha)}{3(2-\alpha)}, \tag{12}$$

Using the values of a_2 and a_3 from equations (11) and (12) and a little simplification yields

$$|a_3^2 - a_2^2| = \frac{1}{36(2-\alpha)^2}\big|4c_1^4(1-\alpha)^2 + 4c_2^2 + 8c_1^2c_2(1-\alpha) - 9c_1^2(2-\alpha)^2\big|.$$

Substituting the value for c_2 from Lemma 2.2 in the previous equation, we have

$$\begin{aligned} |a_3^2 - a_2^2| = \frac{1}{36(2-\alpha)^2}\big|&c_1^4(3-2\alpha)^2 + (4-c_1^2)^2x^2 \\ &+ c_1^2\big((4-c_1^2)(6-4\alpha)x - 9(2-\alpha)^2\big)\big|. \end{aligned}$$

Using triangle inequality, choosing $c_1 = c \in [0,2]$ and replacing $|x|$ by μ in the above equation, we get

$$\begin{aligned} |a_3^2 - a_2^2| &= \frac{1}{36(2-\alpha)^2}\Big(c^4(3-2\alpha)^2 + (4-c^2)^2\mu^2 \\ &\qquad + 9c^2(2-\alpha)^2 + c^2(4-c^2)(6-4\alpha)\mu\Big) \\ &=: F(c,\mu). \end{aligned} \tag{13}$$

We shall now maximize the function $F(c,\mu)$ for $(c,\mu)\in[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu}=\frac{1}{36(2-\alpha)^2}\Big(c^2(4-c^2)(6-4\alpha)+2(4-c^2)^2\mu\Big).$$

For $0<\mu<1$, and for any fixed $c\in[0,2]$, we observe that $\partial F/\partial\mu>0$. Thus $F(c,\mu)$ is an increasing function of μ, and for $c\in[0,2]$, $F(c,\mu)$ has a maximum value at $\mu=1$. Thus, we have

$$\max_{\mu\in[0,1]} F(c,\mu)=F(c,1)=G(c). \tag{14}$$

Then equations (13) and (14) yield

$$G(c)=\frac{1}{36(2-\alpha)^2}\Big(c^4(3-2\alpha)^2+(4-c^2)^2 + 9c^2(2-\alpha)^2+c^2(4-c^2)(6-4\alpha)\Big).$$

Now, we need to find the maximum value of $G(c)$ for $c\in[0,2]$. Differentiating $G(c)$ with respect to c, we see that

$$G'(c)=\frac{1}{18(2-\alpha)^2}\Big(c\big(52-52\alpha+9\alpha^2+8c^2(1-\alpha)^2\big)\Big).$$

Clearly $G'(c)>0$ for $c\in[0,2]$, and therefore

$$T_2(2)\le\max_{c\in[0,2]}G(c)=G(2)=\frac{72-84\alpha+25\alpha^2}{9(2-\alpha)^2}. \tag{15}$$

Hence equations (13), (14) and (15) proves the result. □

Remark 2.3. 1. *When $\alpha=0$, the class $\mathcal{L}(\alpha)$ reduces to the class $\mathcal{K}$ and thus $|T_2(2)|\le 2$ for $f\in\mathcal{K}$ as in* [4].

2. *When $\alpha=1$, the class $\mathcal{L}(\alpha)$ reduces to the class $\mathcal{R}$ and thus $|T_2(2)|\le 13/9$ for $f\in\mathcal{R}$ as in* [4].

The next theorem gives bound for $T_3(1)$ wherein the elements a_2 and a_3 of the determinant matrix are the coefficients of the function $f\in\mathcal{L}(\alpha)$.

Theorem 2.4. *For $f\in\mathcal{L}(\alpha)$ with a_2 real,*

$$|T_3(1)|=|1+2a_2^2(a_3-1)-a_3^2|\le\frac{(12-5\alpha)(12-7\alpha)}{9(2-\alpha)^2}.$$

Proof. Expanding the expression for $T_3(1)$, and using the values of a_2 and a_3 from equations (11) and (12), it can be seen that

$$\begin{aligned}T_3(1)=&|1+2a_2^2(a_3-1)-a_3^2|\\ =&\frac{1}{18(2-\alpha)^2}\Big|18(2-\alpha)^2+c_1^4(4-\alpha)(1-\alpha)\\ &-2c_2^2+c_1^2c_2(2+\alpha)-9c_1^2(2-\alpha)^2\Big|.\end{aligned}$$

Substituting the values for c_2 from Lemma 2.2 in the above expression, we have with $M := 4 - c_1^2$

$$\begin{aligned}|1 + 2a_2^2(a_3 - 1) - a_3^2| =& \frac{1}{36(2-\alpha)^2}\Big|36(2-\alpha)^2 + c_1^4(3-\alpha)(3-2\alpha)\\ &- M^2x^2 - 18(2-\alpha)^2c_1^2 + \alpha M c_1^2 x\Big|.\end{aligned}$$

Using triangle inequality in the previous equation, choosing $c_1 = c \in [0,2]$ and replacing $|x|$ by μ, we get

$$\begin{aligned}T_3(1) \leq& \frac{1}{36(2-\alpha)^2}\Big(36(2-\alpha)^2 + c^4(3-\alpha)(3-2\alpha)\\ &+ M^2\mu^2 + 18(2-\alpha)^2c^2 + \alpha M \mu c^2\Big)\\ =&:F(c,\mu). \end{aligned} \tag{16}$$

where now $M := 4 - c_1^2$. We shall further maximize the function $F(c,\mu)$ for $(c,\mu) \in [0,2] \times [0,1]$. Substituting the value of M and differentiating $F(c,\mu)$ partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = \frac{1}{36(2-\alpha)^2}\big(\alpha c^2(4-c^2) + 2(4-c^2)^2\mu\big).$$

For $0 < \mu < 1$, and for any fixed $c \in [0,2]$, we observe that $\partial F/\partial\mu > 0$. Thus $F(c,\mu)$ is an increasing function of μ, and for $c \in [0,2]$, $F(c,\mu)$ has a maximum value at $\mu = 1$. Thus, we have

$$\max_{0\leq\mu\leq 1} F(c,\mu) = F(c,1) = G(c). \tag{17}$$

Then equations (16) and (17) yield

$$\begin{aligned}G(c) = & \frac{1}{36(2-\alpha)^2}\Big(36(2-\alpha)^2 + c^4(3-\alpha)(3-2\alpha)\\ &+ (4-c^2)^2 + 18(2-\alpha)^2c^2 + (4-c^2)\alpha c^2\Big).\end{aligned}$$

Now, we need to maximize $G(c)$ for $c \in [0,2]$. Differentiating $G(c)$ with respect to c, we get

$$G'(c) = \frac{1}{9(2-k)^2}\Big(c(2-\alpha)(16-9\alpha) + 2c^3(5 - 5\alpha + \alpha^2)\Big).$$

Clearly $G'(c) > 0$ for $c \in [0,2]$, and therefore

$$\max_{0\leq c\leq 2} G(c) = G(2) = \frac{27(2-\alpha)^2 + 4(3-2\alpha)(3-\alpha)}{9(2-\alpha)^2}. \tag{18}$$

Simplifying the equation (18) yields the required result. □

Remark 2.5. 1. *When $\alpha = 0$, the class $\mathcal{L}(\alpha)$ reduces to the class $\mathcal{K}$ and thus $|T_3(1)| \leq 4$ as in* [4].

2. *When* $\alpha = 1$*, the class* $\mathcal{L}(\alpha)$ *reduces to the class* $\mathcal{R}$ *and thus* $|T_3(1)| \leq 35/9$ *as in* [4].

3. The Class $\mathcal{M}(\alpha)$

Our first theorem in this section gives the bound for $T_2(2)$, wherein the elements a_2 and a_3 of the determinant matrix are the coefficients of the function $f \in \mathcal{M}(\alpha)$.

Theorem 3.1. *For* $f = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{M}(\alpha)$*, where* a_2 *is real, we have*

$$|T_2(2)| = |a_3^2 - a_2^2| \leq \frac{4\big((1+\alpha)^2(2+\alpha)^2 + (3+\alpha^2)^2\big)}{(1+\alpha)^4(2+\alpha)^2}.$$

Proof of Theorem 3.1. Since $f \in \mathcal{M}(\alpha)$, there is an analytic function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$ such that

$$\alpha f'(z) + (1-\alpha)\frac{zf'(z)}{f(z)} = p(z). \tag{19}$$

The Taylor series expansion of the function f yields

$$\begin{aligned} &\alpha f'(z) + (1-\alpha)\frac{zf'(z)}{f(z)} \\ &= 1 + a_2(1+\alpha)z + \big(a_3(2+\alpha) - a_2^2(1-\alpha)\big)z^2 \\ &\quad + \big(a_2^3(1-\alpha) - 3a_2a_3(1-k) + a_4(3+\alpha)\big)z^3 + \cdots . \end{aligned} \tag{20}$$

Then using (19), (20) and the expansion for the function p from (7), the coefficients a_2 and a_3 can be expressed as a function of the coefficients c_i of $p \in \mathcal{P}$:

$$a_2 = \frac{c_1}{1+\alpha}, \tag{21}$$

and

$$a_3 = \frac{c_1^2(1-\alpha) + c_2(1+\alpha)^2}{(1+\alpha)^2(2+\alpha)} \tag{22}$$

Using the values of a_3 and a_2, simplifying and collecting the coefficients of various powers of c_i's, we get

$$\begin{aligned} |a_3^2 - a_2^2| = \frac{1}{(1+\alpha)^4(2+\alpha)^2}\big|&c_1^4(1-\alpha)^2 + c_2^2(1+\alpha)^4 \\ &+ 2c_1^2c_2(1+\alpha)^2(1-\alpha) - c_1^2(1+\alpha)^2(2+\alpha)^2\big|. \end{aligned}$$

Substituting the value for c_2 from Lemma 2.2 in the previous equation, we have

$$\begin{aligned} |a_3^2 - a_2^2| = \frac{1}{4(1+\alpha)^4(2+\alpha)^2}|&c_1^4(3+\alpha^2)^2 + (4-c_1^2)^2(1+\alpha)^4x^2 \\ &+ 2c_1^2(1+\alpha)^2\big((4-c_1^2)(3+\alpha^2)x - 2(2+\alpha)^2\big)|. \end{aligned}$$

Using triangle inequality, choosing $c_1 = c \in [0,2]$ and replacing $|x|$ by μ in the above inequality, we get

$$\begin{aligned}|a_3^2 - a_2^2| \leq \frac{1}{4(1+\alpha)^4(2+\alpha)^2}\Big(&c^4(3+\alpha^2)^2 + (4-c^2)^2(1+\alpha)^4\mu^2 \\ &+ 2c^2(1+\alpha)^2\big((4-c^2)(3+\alpha^2)\mu + 2(2+\alpha)^2\big)\Big) \\ =:&\, F(c,\mu). \qquad (23)\end{aligned}$$

We shall now maximize the function $F(c,\mu)$ for $(c,\mu) \in [0,2]\times[0,1]$. Differentiating $F(c,\mu)$ partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = \frac{1}{4(1+\alpha)^4(2+\alpha)^2}\Big(2(1+\alpha)^2(3+\alpha^2)(4-c^2)c^2 + 2(1+\alpha)^4(4-c^2)^2\mu\Big).$$

For $0 < \mu < 1$, and for any fixed $c \in [0,2]$, we observe that $\partial F/\partial\mu > 0$. Thus $F(c,\mu)$ is an increasing function of μ, and for $c \in [0,2]$, $F(c,\mu)$ has a maximum value at $\mu = 1$. Thus, we have

$$\max_{\mu\in[0,1]} F(c,\mu) = F(c,1) = G(c). \qquad (24)$$

Then equations (23) and (24) yield

$$\begin{aligned}G(c) = \frac{1}{4(1+\alpha)^4(2+\alpha)^2}\Big(&c^4(3+\alpha^2)^2 + (4-c^2)^2(1+\alpha)^4 \\ &+ 2c^2(1+\alpha)^2\big((4-c^2)(3+\alpha^2) + 2(2+\alpha)^2\big)\Big).\end{aligned}$$

Now, we need to find the maximum value of $G(c)$ for $c \in [0,2]$. Differentiating $G(c)$ with respect to c, we see that

$$G'(c) = \frac{1}{4(1+\alpha)^4(2+\alpha)^2}\Big(8c\big((1+\alpha)^2(8+\alpha^2) + 2c^2(1-\alpha)^2\big)\Big).$$

Clearly $G'(c) > 0$ for $c \in [0,2]$, and therefore

$$\max_{c\in[0,2]} G(c) = G(2) = \frac{4\big((1+\alpha)^2(2+\alpha)^2 + (3+\alpha^2)^2\big)}{(1+\alpha)^4(2+\alpha)^2}. \qquad (25)$$

Thus, equations (23), (24) and (25) proves the result. □

Remark 3.2. 1. *When $\alpha = 0$, the class $\mathcal{M}(\alpha)$ reduces to the class $\mathcal{S}^*$ and thus $|T_2(2)| \leq 13$ for $f \in \mathcal{S}^*$ which is same as the bound obtained in* [4].
2. *When $\alpha = 1$, the class $\mathcal{M}(\alpha)$ reduces to the class $\mathcal{R}$ and thus $|T_2(2)| \leq 13/9$ for $f \in \mathcal{R}$ which is same as the bound in* [4].

The next theorem gives bound for $T_3(1)$ wherein the elements a_2 and a_3 of the determinant matrix are the coefficients of the function $f \in \mathcal{M}(\alpha)$.

Theorem 3.3. *For $f \in \mathcal{M}(\alpha)$, with a_2 real,*

$$\begin{aligned}|T_3(1)| &= |1+2a_2^2(a_3-1)-a_3^2| \\ &\le \frac{8(1+\alpha)(2+\alpha)^2+(1+\alpha)^3(2+\alpha)^2+4(5-\alpha)(3+\alpha^2)}{(1+\alpha)^3(2+\alpha)^2}.\end{aligned}$$

Proof. Expanding the expression for $T_3(1)$, and using the values of a_2 and a_3 from equations (21) and (22) it can be seen that

$$\begin{aligned}T_3(1) =& |1+2a_2^2(a_3-1)-a_3^2| \\ =& \frac{1}{(1+\alpha)^3(2+\alpha)^2}\Big|(1+\alpha)^3(2+\alpha)^2+3c_1^4(1-\alpha)-c_2^2(1+\alpha)^3 \\ &+2c_1^2c_2(1+\alpha)(1+2\alpha)-2c_1^2(1+\alpha)(2+\alpha)^2\Big|.\end{aligned}$$

Substituting the value of c_2 from Lemma 2.2, the above equation yields

$$\begin{aligned}T_3(1) =& \frac{1}{4(1+\alpha)^3(2+\alpha)^2}\Big|4(1+\alpha)^3(2+\alpha)^2+(5-\alpha)(3+\alpha^2)c_1^4 \\ &-(1+\alpha)^3M^2x^2-8c_1^2(1+\alpha)(2+\alpha)^2+2c_1^2(1+\alpha)(1+2\alpha-\alpha^2)Mx\Big|,\end{aligned}$$

where $M = 4-c_1^2$. Using triangle inequality in the previous equation, we get

$$\begin{aligned}T_3(1) \le& \frac{1}{4(1+\alpha)^3(2+\alpha)^2}\Big(4(1+\alpha)^3(2+\alpha)^2+c_1^4(5-\alpha)(3+\alpha^2) \\ &+(1+\alpha)^3M^2|x|^2+8c_1^2(1+\alpha)(2+\alpha)^2 \\ &+2c_1^2(1+\alpha)(1+2\alpha-\alpha^2)M|x|\Big). \qquad (26)\end{aligned}$$

Choose $c_1 = c \in [0,2]$ and replace $|x|$ by μ in the previous inequality to get

$$\begin{aligned}T_3(1) \le& \frac{1}{4(1+\alpha)^3(2+\alpha)^2}\Big(4(1+\alpha)^3(2+\alpha)^2+c^4(5-\alpha)(3+\alpha^2) \\ &+(1+\alpha)^3M^2\mu^2+8c^2(1+\alpha)(2+\alpha)^2+2c^2(1+\alpha)(1+2\alpha-\alpha^2)M\mu\Big) \\ =:& F(c,\mu). \qquad (27)\end{aligned}$$

We now maximize the function $F(c,\mu)$ for $(c,\mu) \in [0,2]\times[0,1]$. Differentiating $F(c,\mu)$ partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = \frac{1}{2(1+\alpha)^2(2+\alpha)^2}\Big((1+\alpha)^2M^2\mu+c^2(1+2\alpha-\alpha^2)M\Big).$$

For $0<\mu<1$, and for any fixed $c \in [0,2]$, we observe that $\partial F/\partial\mu > 0$ as $(1+2\alpha-\alpha^2) > 0$ for $0<\alpha\le 1$. Thus $F(c,\mu)$ is an increasing function of μ, and for $c \in [0,2]$, $F(c,\mu)$ has a maximum value at $\mu = 1$. Thus, we have

$$\max_{0\le\mu\le1} F(c,\mu) = F(c,1) = G(c). \qquad (28)$$

Now, using equations (27) and (28) we get

$$(29)\qquad \begin{aligned} G(c) =&\frac{1}{4(1+\alpha)^3(2+\alpha)^2}\Big(4(1+\alpha)^3(2+\alpha)^2 + c^4(5-\alpha)(3+\alpha^2)\\ &+(1+\alpha)^3M^2 + 8c^2(1+\alpha)(2+\alpha)^2\\ &+2c^2(1+\alpha)(1+2\alpha-\alpha^2)M\Big).\end{aligned}$$

On substituting $M = 4 - c^2$ in equation (29) and simplifying we obtain,

$$\begin{aligned} G(c) = &\frac{1}{2(1+\alpha)^3(2+\alpha)^2}\Big(c^4(7-3\alpha+3\alpha^2+\alpha^3)\\ &+4c^2(1+\alpha)(4+4\alpha-\alpha^2) + 2(1+\alpha)^3(8+4\alpha+\alpha^2)\Big).\end{aligned}$$

Now, we need to find the maximum value of $G(c)$ for $c \in [0, 2]$. Differentiating $G(c)$ with respect to c, we get

$$G'(c) = \frac{1}{(1+\alpha)^3(2+\alpha)^2}\Big(4c(1+\alpha)(4+4\alpha-\alpha^2) + 2c^3(7-3\alpha+3\alpha^2+\alpha^3)\Big).$$

Clearly $G'(c) > 0$ for $c \in [0, 2]$, and therefore

$$\begin{aligned} \max_{0\le c\le 2} G(c) = G(2) = &\frac{1}{(1+\alpha)^3(2+\alpha)^2}\Big(2(1+\alpha)^3(8+4\alpha+\alpha^2)\\ &+8(1+\alpha)(4+4\alpha-\alpha^2) + 8(7-3\alpha+3\alpha^2+\alpha^3)\Big).\end{aligned}$$

Simplifying the previous equation yields the required result. □

Remark 3.4. 1. *When $\alpha = 0$, the class $\mathcal{M}(\alpha)$ reduces to the class $\mathcal{S}^*$ and thus $|T_3(1)| \le 24$ as in* [4].

2. *When $\alpha = 1$, the class $\mathcal{M}(\alpha)$ reduces to the class $\mathcal{R}$ and thus $|T_3(1)| \le 35/9$ as in* [4].

The next theorem gives bound for $T_3(2)$ wherein the elements a_2, a_3 and a_4 of the determinant matrix are the coefficients of the function $f \in \mathcal{M}(\alpha)$.

Theorem 3.5. *For $f \in \mathcal{M}(\alpha)$ with a_2 real,*

$$\begin{aligned}|T_3(2)| &= |(a_2 - a_4)(a_2^2 - 2a_2^3 + a_2a_4)|\\ &\le \frac{32(3+\alpha^2)(7+10\alpha+7\alpha^2+\alpha^3)(9+5\alpha+18\alpha^2+3\alpha^3+\alpha^4)}{(1+\alpha)^7(2+\alpha)^3(3+\alpha)^2}.\end{aligned}$$

Proof. Expanding the expression for $T_3(2)$, it can be seen that

$$T_3(2) = |(a_2 - a_4)(a_2^2 - 2a_2^3 + a_2a_4)|.$$

First, we shall obtain an upper bound for $|a_2 - a_4|$. Using equations (19), (20) and the expansion for the function p from (7), the coefficient a_4 can be expressed as a function of the coefficients c_i of $p \in \mathcal{P}$:

$$a_4 = \frac{c_1^3(1-\alpha)(1-4\alpha) + 3c_1c_2(1-\alpha)(1+\alpha)^2 + c_3(1+\alpha)^3(2+\alpha)}{(1+\alpha)^3(2+\alpha)(3+\alpha)}. \tag{30}$$

Using the values of a_2 and a_4 from equations (21) and (30), we get

$$\begin{aligned}|a_2 - a_4| =& \frac{1}{(1+\alpha)^3(2+\alpha)(3+\alpha)}\Big| - c_3(1+\alpha)^3(2+\alpha) - c_1^3(1-\alpha)(1-4\alpha)\\ &- 3c_2c_1(1+\alpha)^2(1-\alpha) + c_1(1+\alpha)^2(2+\alpha)(3+\alpha)\Big|.\end{aligned}$$

Substituting the values for c_2 and c_3 from Lemma 2.2 in the above equation, we have with $M := 4 - c_1^2$ and $Z = 1 - x^2$,

$$\begin{aligned}|a_2 - a_4| =& \frac{1}{4(1+\alpha)^3(2+\alpha)(3+\alpha)}\Big|c_1^3(-12 + 7\alpha - 19\alpha^2 + \alpha^3 - \alpha^4)\\ &+ c_1(1+\alpha)^2\big(4(2+\alpha)(3+\alpha) - 2(5+\alpha^2)Mx\\ &+ (1+\alpha)(2+\alpha)Mx^2\big) - 2(1+\alpha)^3(2+\alpha)MyZ\Big|.\end{aligned}$$

Simplifying the above expression for $|a_2 - a_4|$ by substituting back $M := 4 - c_1^2$ and $Z := 1 - x^2$, and using triangle inequality and also making use of the fact that $|y| \le 1$, we get

$$\begin{aligned}|a_2 - a_4| \le& \frac{1}{4(1+\alpha)^3(2+\alpha)(3+\alpha)}\Big(2(1+\alpha)^3(2+\alpha)(4 - c_1^2)\\ &+ 4c_1(1+\alpha)^2(2+\alpha)(3+\alpha) + c_1^3(12 - 7\alpha + 19\alpha^2 - \alpha^3 + \alpha^4)\\ &+ 2c_1(4 - c_1^2)(1+\alpha)^2(5+\alpha^2)|x|\\ &+ (2 - c_1)(2 + c_1)^2(1+\alpha)^3(2+\alpha)|x|^2\Big)\\ =:& F(c_1, |x|) = F(c, \mu)(\text{say}).\end{aligned} \tag{31}$$

We shall further maximize the function $F(c, \mu)$ for $(c, \mu) \in [0, 2] \times [0, 1]$. Differentiating $F(c, \mu)$ partially with respect to μ, we get

$$\begin{aligned}\frac{\partial F}{\partial \mu} =& \frac{1}{2(1+\alpha)(2+\alpha)(3+\alpha)}\Big(c(4 - c^2)(5+\alpha^2)\\ &+ (2 - c)(2 + c)^2(1+\alpha)(2+\alpha)\mu\Big).\end{aligned}$$

For $0 < \mu < 1$, and for any fixed $c \in [0, 2]$, we observe that $\partial F/\partial \mu > 0$. Thus $F(c, \mu)$ is an increasing function of μ, and for $c \in [0, 2]$, $F(c, \mu)$ has a maximum value at $\mu = 1$. Thus, we have

$$\max_{0 \le \mu \le 1} F(c, \mu) = F(c, 1) = G(c). \tag{32}$$

Then equations (31) and (32) upon simplification yield

$$G(c) = \frac{1}{2(1+\alpha)^3(2+\alpha)(3+\alpha)}\Big(4c(1+\alpha)^2(9+4\alpha+2\alpha^2) - c^3\alpha(17+\alpha+5\alpha^2+\alpha^3)\Big).$$

Now, we need to maximize $G(c)$ for $c \in [0,2]$. Differentiating $G(c)$ with respect to c, we get

$$G'(c) = \frac{1}{2(1+\alpha)^3(2+\alpha)(3+\alpha)}\Big(4(1+\alpha)^2(9+4\alpha+2\alpha^2) - 3c^2\alpha(17+\alpha+5\alpha^2+\alpha^3)\Big).$$

Clearly $G'(c) > 0$ for $c \in [0,2]$, and therefore

$$\begin{aligned} |a_2 - a_4| &\le \max_{0\le c\le 2} G(c) = G(2) \\ &= \frac{2(1+\alpha)^2(9+4\alpha+2\alpha^2) - 6\alpha(17+\alpha+5\alpha^2+\alpha^3)}{(1+\alpha)^3(2+\alpha)(3+\alpha)}. \end{aligned} \tag{33}$$

We shall now obtain an upper bound for $|a_2^2 - 2a_3^2 + a_2a_4|$. Using the values of a_2, a_3 and a_4 from equations (21), (22) and (30), we get

$$\begin{aligned} a_2^2 - 2a_3^2 + a_2a_4 =& \frac{1}{(1+\alpha)^4(2+\alpha)^2(3+\alpha)}\Big(c_1c_3(1+\alpha)^3(2+\alpha)^2 \\ &- 2c_2^2(1+\alpha)^4(3+\alpha) + c_1^4(-4+\alpha+\alpha^2+2\alpha^3) \\ &+ c_1^2(1+\alpha)^2\big((2+\alpha)^2(3+\alpha) + c_2(-6+5\alpha+\alpha^2)\big)\Big). \end{aligned}$$

Substituting the values of c_2 and c_3 from Lemma 2.2 in the above expression, we have with $M := 4 - c_1^2$ and $Z = 1 - x^2$,

$$\begin{aligned} |a_2^2 - 2a_3^2 + a_2a_4| =& \frac{1}{4(1+\alpha)^4(2+\alpha)^2(3+\alpha)}\Big| - 2(1+\alpha)^4(3+\alpha)M^2x^2 \\ &- c_1^4(30+20\alpha+5\alpha^2-5\alpha^3+5\alpha^4+\alpha^5) \\ &+ c_1^2(1+\alpha)^2\big(4(2+\alpha)^2(3+\alpha) - 2(8+\alpha+4\alpha^2+\alpha^3)Mx \\ &- (1+\alpha)(2+\alpha)^2Mx^2\big) + 2c_1(1+\alpha)^3(2+\alpha)^2MyZ\Big|. \end{aligned}$$

Simplifying the above expression by using triangle inequality after substituting $M := 4 - c_1^2$ and $Z := 1 - x^2$, and using the fact that $|y| \leq 1$, we get

$$\begin{aligned}|a_2^2 - 2a_3^2 + a_2a_4| \leq &\frac{1}{4(1+\alpha)^4(2+\alpha)^2(3+\alpha)}\Big|2(4-c_1^2)^2(1+\alpha)^4(3+\alpha)|x|^2\\ &+ 2c_1(4-c_1^2)(1+\alpha)^3(2+\alpha)^2(1-|x|^2)\\ &+ c_1^2(1+\alpha)^2\big((4-c_1^2)(1+\alpha)(2+\alpha)^2|x|^2\\ &+ 2(4-c_1^2)(8+\alpha+4\alpha^2+\alpha^3)|x|\big) + 4c_1^2(1+\alpha)^2(2+\alpha)^2\\ &(3+\alpha) + c_1^4(30+20\alpha+5\alpha^2-5\alpha^3+5\alpha^4+\alpha^5)\Big|\\ &=: F(c,|x|) = F(c,\mu)(\text{say}).\end{aligned} \tag{34}$$

We shall further maximize the function $F(c,\mu)$ in (34) for $(c,\mu) \in [0,2]\times[0,1]$. Differentiating $F(c,\mu)$ partially with respect to μ, we get

$$\begin{aligned}\frac{\partial F}{\partial \mu} =&\frac{1}{4(1+\alpha)^4(2+\alpha)^2(3+\alpha)}\Big(4(4-c^2)^2(1+\alpha)^4(3+\alpha)\mu\\ &- 4c\mu(4-c^2)(1+\alpha)^3(2+\alpha)^2 + c^2(1+\alpha)^2\big(2(4-c^2)(8+\alpha+4\alpha^2+\alpha^3)\\ &+ 2(4-c^2)(1+\alpha)(2+\alpha)^2\mu\big)\Big).\end{aligned}$$

For $0 < \mu < 1$, and for any fixed $c \in [0,2]$, we observe that $\partial F/\partial\mu > 0$. Thus $F(c,\mu)$ is an increasing function of μ, and for $c \in [0,2]$, $F(c,\mu)$ has a maximum value at $\mu = 1$. Thus, we have

$$\max_{0\leq\mu\leq1} F(c,\mu) = F(c,1) = G(c). \tag{35}$$

Then equations (34) and (35) on simplifying yield

$$\begin{aligned}G(c) =&\frac{1}{(1+\alpha)^4(2+\alpha)^2(3+\alpha)}\Big(2c^2(10-\alpha)(1+\alpha)^2\\ &+ 8(1+\alpha)^4(3+\alpha) + c^4(1-\alpha)(4+3\alpha+2\alpha^2)\Big).\end{aligned}$$

Now, we need to maximize $G(c)$ for $c \in [0,2]$. Differentiating $G(c)$ with respect to c, we get

$$G'(c) = \frac{4c}{(1+\alpha)^4(2+\alpha)^2(3+\alpha)}\Big((10-\alpha)(1+\alpha)^2 + c^2(1-\alpha)(4+3\alpha+2\alpha^2)\Big).$$

Clearly $G'(c) > 0$ for $c \in [0,2]$, and therefore

$$\max_{0\leq c\leq2} G(c) = G(2) = \frac{8(3+\alpha^2)(7+10\alpha+7\alpha^2+\alpha^3)}{(1+\alpha)^4(2+\alpha)^2(3+\alpha)}. \tag{36}$$

Finally, from equation (33) and (36), we have

$$|T_3(2)| \leq \frac{32(3+\alpha^2)(7+10\alpha+7\alpha^2+\alpha^3)(9+5\alpha+18\alpha^2+3\alpha^3+\alpha^4)}{(1+\alpha)^7(2+\alpha)^3(3+\alpha)^2}.$$

□

Remark 3.6. 1. *When $\alpha = 0$, the class $\mathcal{M}(\alpha)$ reduces to the class $\mathcal{S}^*$ and thus $|T_3(2)| \leq 84$ as in* [4].
2. *When $\alpha = 1$, the class $\mathcal{M}(\alpha)$ reduces to the class $\mathcal{R}$ and thus $|T_3(2)| \leq 25/12$ as in* [4].

The next theorem gives bound for $T_2(3)$ wherein the elements a_3 and a_4 of the determinant matrix are the coefficients of the function $f \in \mathcal{M}(\alpha)$.

Theorem 3.7. *For $f \in \mathcal{M}(\alpha)$, $0 \leq \alpha \leq \alpha_0$,*

$$\begin{aligned}|T_2(3)| &= |a_4^2 - a_3^2| \\ &\leq \frac{4(1+\alpha)^2(3+\alpha)^2(3+\alpha^2)^2 + 4(12 - 7\alpha + 19\alpha^2 - \alpha^3 + \alpha^4)^2}{(1+\alpha)^6(2+\alpha)^2(3+\alpha)^2}.\end{aligned}$$

whereas, for $\alpha_0 < \alpha \leq 1$, $T_2(3) = |a_4^2 - a_3^2| \leq G(c_0)$, where $0 < c_0 < 2$ and $G(c_0)$ is a very complex quantity. Here α_0 is a root of $G(c_0)(1+\alpha)^6(2+\alpha)^2(3+\alpha)^2 = 4(1+\alpha)^2(3+\alpha)^2(3+\alpha^2)^2 + 4(12-7\alpha+19\alpha^2-\alpha^3+\alpha^4)^2$ and is equal to 0.359395.

Proof. Using the values of a_3 and a_4 from equations (22) and (30), we get

$$\begin{aligned}|a_4^2 - a_3^2| = &\frac{1}{(1+\alpha)^6(2+\alpha)^2(3+\alpha)^2}\Big|6c_1c_2c_3(1+\alpha)^5(1-\alpha)(2+\alpha) \\ &+ c_1^6(1-\alpha)^2(1-4\alpha)^2 + 2c_1^3c_3(1+\alpha)^3(1-\alpha)(2+\alpha)(1-4\alpha) \\ &+ c_1^2c_2(1-\alpha)(1+\alpha)^4\big(9c_2(1-\alpha) - 2(3+\alpha)^2\big) \\ &+ (1+\alpha)^6\big(c_3^2(2+\alpha)^2 - c_2^2(3+\alpha)^2\big) \\ &+ c_1^4(1-\alpha^2)^2\big(6c_2(1-4\alpha) - (3+\alpha)^2\big)\Big|.\end{aligned}$$

Substituting the values for c_2 and c_3 from Lemma 2.2 in the above equation, we have with $M := 4 - c_1^2$, $Z = 1 - x^2$ and $(12 - 7\alpha + 19\alpha^2 - \alpha^3 + \alpha^4) = N$

$$\begin{aligned}|a_4^2 - a_3^2| = &\frac{1}{16(1+\alpha)^6(2+\alpha)^2(3+\alpha)^2}\Big|c_1^6N^2 - 2c_1^4(1+\alpha)^2\big(2(3+\alpha)^2(3+\alpha^2)^2 \\ &- 2(5+\alpha^2)NMx + (1+\alpha)(2+\alpha)NMx^2\big) + c_1^2(1+\alpha)^4Mx \\ &\big(-8(3+\alpha)^2(3+\alpha^2) + 4(5+\alpha^2)^2Mx - 4(1+\alpha)(2+\alpha)(5+\alpha^2) \\ &Mx^2 + (1+\alpha)^2(2+\alpha)^2Mx^3\big) + 4c_1^3(1+\alpha)^3(2+\alpha)NMyZ \\ &- 4c_1(1+\alpha)^5(2+\alpha)M^2x\big(-2(5+\alpha^2) + (1+\alpha)(2+\alpha)x\big)yZ \\ &- 4(1+\alpha)^6M^2\big((3+\alpha)^2x^2 - (2+\alpha)^2y^2Z^2\big)\Big|.\end{aligned}$$

Simplifying the above expression by using triangle inequality after substituting $M := 4 - c_1^2$ and $Z := 1 - x^2$, and using the fact that $|y| \leq 1$, we get

$$
\begin{aligned}
|a_4^2 - a_3^2| \leq & \frac{1}{16(1+\alpha)^6(2+\alpha)^2(3+\alpha)^2}\Big(c_1^6 N^2 + 2c_1^4(1+\alpha)^2\big(2(3+\alpha)^2(3+\alpha^2)^2 \\
& + 2(4-c_1^2)(5+\alpha^2)N|x| + (4-c_1^2)(1+\alpha)(2+\alpha)N|x|^2\big) \\
& + c_1^2(4-c_1^2)(1+\alpha)^4|x|\big(8(3+\alpha)^2(3+\alpha^2) + 4(4-c_1^2)(5+\alpha^2)^2|x| \\
& + 4(4-c_1^2)(1+\alpha)(2+\alpha)(5+\alpha^2)|x|^2 + (4-c_1^2)(1+\alpha)^2(2+\alpha)^2 \\
& |x|^3\big) + 4c_1^3(4-c_1^2)(1+\alpha)^3(2+\alpha)N(1-|x|^2) \\
& + 4c_1(4-c_1^2)^2(1+\alpha)^5(2+\alpha)|x|\big(2(5+\alpha^2) + (1+\alpha)(2+\alpha)|x|\big) \\
& (1-|x|^2) + 4(4-c_1^2)^2(1+\alpha)^6\big((3+\alpha)^2|x|^2 + (2+\alpha)^2(1-|x|^2)^2\big)\Big) \\
(37) \qquad & =: F(c_1, |x|) = F(c,\mu) (\text{say}).
\end{aligned}
$$

We shall further maximize the function $F(c,\mu)$ in (37) for $(c,\mu) \in [0,2] \times [0,1]$. Differentiating $F(c,\mu)$ partially with respect to μ, we get

$$
\begin{aligned}
\frac{\partial F}{\partial \mu} = & \frac{1}{16(1+\alpha)^4(2+\alpha)^2(3+\alpha)^2}\Big(- 8c^3(1+\alpha)^3(2+\alpha)MN\mu + 4c^4(1+\alpha)^2 \\
& MN\big(5+\alpha^2 + (1+\alpha)(2+\alpha)\mu\big) + 4(1+\alpha)^6 M^2\big(2(3+\alpha)^2\mu \\
& + 4(2+\alpha)^2\mu(1-\mu^2)\big) + c\big(8(1+\alpha)^5(2+\alpha)(5+\alpha^2)M^2 \\
& + 8(1+\alpha)^6(2+\alpha)^2M^2\mu - 24(1+\alpha)^5(2+\alpha)(5+\alpha^2)M^2\mu^2 \\
& - 16(1+\alpha)^6(2+\alpha)^2M^2\mu^3\big) + c^2\big(8(1+\alpha)^2(3+\alpha)^2(3+\alpha^2) \\
& + 8(1+\alpha)^2(5+\alpha^2)^2M\mu + 12(1+\alpha)^3(2+\alpha)(5+\alpha^2)M\mu^2 \\
& + 4(1+\alpha)^4(2+\alpha)^2M\mu^3\big)\Big).
\end{aligned}
$$

For $0 < \mu < 1$, and for any fixed $c \in [0,2]$, we observe that $\partial F/\partial \mu > 0$. Thus $F(c,\mu)$ is an increasing function of μ, and for $c \in [0,2]$, $F(c,\mu)$ has a maximum value at $\mu = 1$. Thus, we have

$$
\max_{0 \leq \mu \leq 1} F(c,\mu) = F(c,1) = G(c). \tag{38}
$$

Then equations (37) and (38) yield

$$
\begin{aligned}
G(c) = & \frac{1}{4(1+\alpha)^2(2+\alpha)^2(3+\alpha)^2}\Big(2(1+\alpha)^4(3+\alpha)^2M^2 - 6c(1+\alpha)^3(2+\alpha) \\
& (4+\alpha+\alpha^2)M^2 + c^2\big(2(3+\alpha)^2(3+\alpha^2) + 3(4+\alpha+\alpha^2) \\
& (7+3\alpha+2\alpha^2)M\big) - 2c^3(1+\alpha)(2+\alpha)MN + c^4(7+3\alpha+2\alpha^2)MN\Big).
\end{aligned}
$$

Now, we need to maximize $G(c)$ for $c \in [0,2]$. Substituting $M = 4 - c^2$ and differentiating $G(c)$ with respect to c, we get

$$\begin{aligned}G'(c) =&\frac{1}{16(1+k)^6(2+\alpha)^2(3+\alpha)^2)}\Big(- 16c(4-c^2)(1+\alpha)^6(3+\alpha)^2\\&+c^2(4-c^2)(1+\alpha)^4(-2c(1+\alpha)^2(2+\alpha)^2 - 8c(1+\alpha)(2+\alpha)(5+\alpha^2)\\&- 8c(5+\alpha^2)^2) - 2c^3(1+\alpha)^4\big((4-c^2)(1+\alpha)^2(2+\alpha)^2\\&+8(3+\alpha)^2(3+\alpha^2) + 4(4-c^2)(1+\alpha)(2+\alpha)(5+\alpha^2)\\&+4(4-c^2)(5+\alpha^2)^2\big) + 2c(4-c^2)(1+\alpha)^4((4-c^2)(1+\alpha)^2(2+\alpha)^2\\&+8(3+\alpha)^2(3+\alpha^2) + 4(4-c^2)(1+\alpha)(2+\alpha)(5+\alpha^2)\\&+4(4-c^2)(5+\alpha^2)^2) + 6c^5N^2 + 2c^4(1+\alpha)^2(-2c(1+\alpha)(2+\alpha)N\\&- 4c(5+\alpha^2)N) + 8c^3(1+\alpha)^2\big(2(3+\alpha)^2(3+\alpha^2)^2\\&+(4-c^2)(1+\alpha)(2+\alpha)N + 2(4-c^2)(5+\alpha^2)N\big)\Big).\end{aligned}$$

Now for $0 \le c < c_0$, $G'(c) > 0$ and therefore

$$\begin{aligned}\max_{0\le c\le 2} G(c) &= G(2)\\&= \frac{64(1+\alpha)^2(3+\alpha)^2(3+\alpha^2)^2 + 64(12-7\alpha+19\alpha^2-\alpha^3+\alpha^4)^2}{16(1+\alpha)^6(2+\alpha)^2(3+\alpha)^2}. \qquad (39)\end{aligned}$$

whereas, for $c_0 < c \le 2$, maximum of $G(c)$ exists at $c_0 \in [0,2]$, where c_0 is a root of $G'(c) = 0$ and we omit the details. □

Remark 3.8. *Since $\mathcal{M}(0) \equiv \mathcal{S}^*$, it follows that $|T_2(3)| \le 25$ for starlike functions and this was proved in* [4].

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