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# BEREZIN NUMBER INEQUALITIES VIA YOUNG INEQUALITY

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**Abstract.** In this paper, we obtain some new inequalities for the Berezin number of operators on reproducing kernel Hilbert spaces by using the Hölder-McCarthy operator inequality. Also, we give refine generalized inequalities involving powers of the Berezin number for sums and products of operators on the reproducing kernel Hilbert spaces.

## 1. Introduction

Recall that the reproducing kernel Hilbert space $\mathcal{H}=\mathcal{H}(\Omega)$ (shortly, RKHS) is the Hilbert space of complex-valued functions on some set $\Omega$ such that the evaluation functional $f\to f(\lambda)$ is bounded on $\mathcal{H}$ for every $\lambda\in\Omega$. Then, by Riesz representation theorem for each $\lambda\in\Omega$ there exists a unique vector $k_\lambda$ in $\mathcal{H}$ such that $f(\lambda)=\langle f,k_\lambda\rangle$ for all $f\in\mathcal{H}$. The function $k_\lambda$ is called the reproducing kernel of the space $\mathcal{H}$. It is well known that (see Aronzajn [1])

$$k_\lambda(z)=\sum_{n=0}^{\infty}\overline{e_n(\lambda)}e_n(z)$$

for any orthonormal basis $\{e_n(z)\}_{n\geq 0}$ of the space $\mathcal{H}(\Omega)$. The normalized reproducing kernel is defined by $\widehat{k}_\lambda:=\frac{k_\lambda}{\|k_\lambda\|_{\mathcal{H}}}$. For a bounded linear operator $A$ acting in the RKHS $\mathcal{H}$, its Berezin symbol $\widetilde{A}$ (see Berezin [5, 6]) is defined by the formula

$$\widetilde{A}(\lambda):=\left\langle A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\ (\lambda\in\Omega).$$

The Berezin symbol is a function that is bounded by norm of the operator. Karaev [19] defined the Berezin set and the Berezin number of operator $A$, respectively by

$$\operatorname{Ber}(A):=\operatorname{Range}\left(\widetilde{A}\right)=\left\{\widetilde{A}(\lambda):\lambda\in\Omega\right\}$$

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and

$$\operatorname{ber}(A) := \sup_{\lambda \in \Omega} \left| \widetilde{A}(\lambda) \right|.$$

It is clear from definitions that $\widetilde{A}$ is a bounded function, $\operatorname{Ber}(A)$ lies in the numerical range $W(A)$, and so $\operatorname{ber}(A)$ does not exceed the numerical radius $w(A)$ of operator $A$. Recall that the numerical range and the numerical radius of operator $A$ are defined, respectively, by

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

and

$$w(A) := \sup_{\|x\|=1} |\langle Ax, x\rangle|$$

(for more information, see [15, 17, 22, 23, 24, 25, 28, 36]). Berezin set and Berezin number of operators are new numerical characteristics of operators on the RKHS which are introduced by Karaev in [19]. For the basic properties and facts on these new concepts, see [2, 3, 4, 20, 31, 32].

Suppose that $B(\mathcal{H})$ denotes the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$. It is well-known that

$$\operatorname{ber}(A) \leq w(A) \leq \|A\| \tag{1.1}$$

and

$$\frac{\|A\|}{2} \leq w(A)$$

for any $A \in B(\mathcal{H})$. But, Karaev [20] showed that

$$\frac{\|A\|}{2} \leq \operatorname{ber}(A)$$

is not hold for every $A \in B(\mathcal{H})$. Also, Berezin number inequalities were given by using the other inequalities in [9, 10, 11, 12, 13, 14, 16, 33, 34, 35].

Kittaneh and El-Haddad in [24] showed that if $A \in B(\mathcal{H})$, then

$$w^r(A) \leq \frac{1}{2} \left\| |A|^{2\alpha r} + |A^*|^{2(1-\alpha)r} \right\|, \tag{1.2}$$

and

$$w^{2r}(A) \leq \frac{1}{2} \left\| \alpha |A|^{2r} + (1-\alpha) |A^*|^{2r} \right\|. \tag{1.3}$$

where $0 \leq \alpha \leq 1$, and $r \geq 1$ (also see, [8, 27]).

If we apply the inequality of (1.1) to (1.2) and (1.3), we have

$$\operatorname{ber}^r(A) \leq \frac{1}{2} \left\| |A|^{2\alpha r} + |A^*|^{2(1-\alpha)r} \right\|_{\operatorname{ber}} \tag{1.4}$$

and

$$\operatorname{ber}^{2r}(A) \leq \frac{1}{2} \left\| \alpha |A|^{2r} + (1-\alpha) |A^*|^{2r} \right\|_{\operatorname{ber}} \tag{1.5}$$

where $0 \leq \alpha \leq 1$, and $r \geq 1$, respectively.

The purpose of this paper is to establish improvement of the Hölder-McCarthy operator inequality in the some special case on the reproducing kernel Hilbert spaces by using a simple consequence of the Jensen inequality for the convex function $f(t) = t^r$ where $r \geq 1$. Some improvements of norm and Berezin number inequality for the sums and powers operators acting on reproducing kernel Hilbert spaces are also presented.

Among many techniques in obtaining numerical radius and Berezin number inequalities is the study of certain scalar ones. For example, a simple consequence of the Jensen inequality for convex function $f(t) = t^r$ where $r \geq 1$ [29] which states that if $a, b \geq 0$ and $0 \leq \alpha \leq 1$, then

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \leq (\alpha a^r + (1-\alpha) b^r)^{\frac{1}{r}}, \tag{1.6}$$

for $r \geq 1$. The following result is known as a generalized mixed Schwarz inequality : If $A \in B(\mathcal{H})$, $x, y \in \mathcal{H}$ be two vectors and $0 \leq \alpha \leq 1$, then

$$|\langle Ax, y\rangle|^2 \leq \left\langle |A|^{2\alpha} x, x \right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y \right\rangle. \tag{1.7}$$

The next inequality is spectral theorem for positive operators and Jensen inequality and known as the Hölder McCarthy inequality [29] which states that if $A$ is a positive operators in $B(\mathcal{H})$ and $x \in \mathcal{H}$ is an unit vector, then

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle \text{ for } r \geq 1$$

$$\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r \text{ for } 0 < r \leq 1. \tag{1.8}$$

Kian [21] gave an improvement of the Hölder McCarthy's inequality which states that if $A$ is a positive operators on $\mathcal{H}$ and $x \in \mathcal{H}$ is an unit vector, then

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle - \langle |A - \langle Ax, x\rangle|^r x, x\rangle, \text{ for } r \geq 2. \tag{1.9}$$

In 2009, Shebrawi and Albadwi [30] proved a generalization of the mixed Schwartz inequality, which assert

$$|\langle Ax, y\rangle| \leq \|f(|A|) x\| \, \|g(|A^*|) y\|, \tag{1.10}$$

for all $x, y \in \mathcal{H}$, $A \in B(\mathcal{H})$ and $f$ and $g$ be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t) g(t) = t$ for all $t \in [0, \infty)$.

The following result [30] is a consequence of the convexity the function $f(t) = t^r$, $r \geq 1$ which states that if $a_i$, $i = 1, 2, ..., n$, are positive real numbers, then

$$\left(\sum_{i=1}^{n} a_i\right)^r \leq n^{r-1} \sum_{i=1}^{n} a_i^r, \text{ for } r \geq 1. \tag{1.11}$$

## 2. The main results

In this section, we obtain an improvement of Hölder-McCarthy's operator inequality in this case when $r \geq 1$ and get some improvements of Berezin number inequalities for operators on reproducing kernel Hilbert spaces.

Our first result is a refinement of the first inequality in (1.5) for $r \geq 2$.

**Theorem 2.1.** *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS. If $A \in B(\mathcal{H})$, $0 \leq \alpha \leq 1$ and $r \geq 2$, then*

$$\operatorname{ber}^{2r}(A) \leq \left\| \alpha |A|^{2r} + (1-\alpha)|A^*|^{2r} \right\|_{\operatorname{ber}} - \inf_{\lambda \in \Omega} \mu(\lambda),$$

*where*

$$\mu(\lambda) = \alpha \left( \widetilde{\left| |A|^2 - \widetilde{|A|^2}(\lambda) \right|^r}(\lambda) \right) + (1-\alpha) \left( \widetilde{\left| |A^*|^2 - \widetilde{|A^*|^2}(\lambda) \right|^r}(\lambda) \right).$$

*Proof.* Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Then we have

$$\begin{aligned}
\left| \widetilde{A}(\lambda) \right|^2 &= \left| \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 \\
&\leq \left\langle |A|^{2\alpha} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle |A^*|^{2(1-\alpha)} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&\quad \text{(by inequality (1.7))} \\
&\leq \left\langle |A|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^\alpha \left\langle |A^*|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{1-\alpha} \\
&\quad \text{(by inequality (1.8))} \\
&\leq \left( \alpha \left\langle |A|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^r + (1-\alpha) \left\langle |A^*|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^r \right)^{1/r} \\
&\quad \text{(by inequality (1.6))} \\
&\leq \Bigg( \alpha \left( \left\langle |A|^{2r} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle - \left\langle \left| |A|^2 - \left\langle |A|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^r \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right) \\
&\quad + (1-\alpha) \left( \left\langle |A^*|^{2r} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle - \left\langle \left| |A^*|^2 - \left\langle |A^*|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^r \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right) \Bigg)^{1/r} \\
&\quad \text{(by inequality (1.9))}.
\end{aligned}$$

So,

$$\begin{aligned}
\left| \widetilde{A}(\lambda) \right|^{2r} \leq\ & \alpha \left( \left\langle |A|^{2r} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle - \left\langle \left| |A|^2 - \left\langle |A|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^r \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right) \\
&+ (1-\alpha) \left( \left\langle |A^*|^{2r} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle - \left\langle \left| |A^*|^2 - \left\langle |A^*|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^r \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right).
\end{aligned}$$

and

$$\begin{aligned}\sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right|^{2r} &\leq \sup_{\lambda\in\Omega}\left\langle|\alpha|\,|A|^{2r}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+(1-\alpha)\sup_{\lambda\in\Omega}\left\langle|A^*|^{2r}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\\ &-\inf_{\lambda\in\Omega}\left(\left\langle|\alpha|\left||A|^2-\left\langle|A|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^r\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right.\\ &\left.+(1-\alpha)\left\langle\left||A^*|^2-\left\langle|A^*|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^r\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right).\end{aligned}$$

which is equivalent to

$$\mathrm{ber}^{2r}(A)\leq\left\|\alpha|A|^{2r}+(1-\alpha)|A^*|^{2r}\right\|_{\mathrm{ber}}-\inf_{\lambda\in\Omega}\mu(\lambda),$$

where $\mu(\lambda)=\alpha\left(\widetilde{\left||A|^2-\widetilde{|A|^2}(\lambda)\right|^r}(\lambda)\right)+(1-\alpha)\left(\widetilde{\left||A^*|^2-\widetilde{|A^*|^2}(\lambda)\right|^r}(\lambda)\right).$ □

Recall that the Young inequality says that if $a,b\geq 0$, and $\alpha\in[0,1]$, then

$$(1-\alpha)a+\alpha b\geq a^{1-\alpha}b^\alpha.$$

Many mathematicians improved Young inequality and reverse. Kober [26], proved that for $a,b>0$

$$(1-\alpha)a+\alpha b\leq a^{1-\alpha}b^\alpha+(1-\alpha)\left(\sqrt{a}-\sqrt{b}\right)^2,\ \alpha\geq 1. \tag{2.1}$$

Our second result is a refinement of the Hölder-McCarthy inequality by using (2.1).

**Lemma 2.2.** *Let $A\in B(\mathcal{H})$ be a positive operator. Then for all $\lambda\in\Omega$*

$$\left(\widetilde{A}(\lambda)\right)^\alpha\leq\frac{1}{\mu}\widetilde{A^\alpha}(\lambda),\ \alpha\geq 1, \tag{2.2}$$

*where* $\mu=1+2(\alpha-1)\left(1-\frac{\widetilde{A^{1/2}}(\lambda)}{\left(\widetilde{A}(\lambda)\right)^{1/2}}\right)$.

*Proof.* Applying functional calculus for the positive operators $A$ in (2.1), we get

$$(1-\alpha)aI+\alpha A\leq a^{1-\alpha}A^\alpha+(1-\alpha)\left(aI+A-2\sqrt{a}A^{\frac{1}{2}}\right).$$

The above inequality is equivalent to

$$\begin{aligned}(1-\alpha)a+\alpha\left\langle A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle &\leq a^{1-\alpha}\left\langle A^\alpha\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\\ &+(1-\alpha)\left(a+\left\langle A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle-2\sqrt{a}\left\langle A^{\frac{1}{2}}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)\end{aligned} \tag{2.3}$$

for all $\lambda \in \Omega$. By substituting $a = \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$ in (2.3), we get

$$\widetilde{A}(\lambda) \le \left(\widetilde{A}(\lambda)\right)^{1-\alpha} \widetilde{A^\alpha}(\lambda) + 2(1-\alpha)\widetilde{A}(\lambda)\left(1 - \frac{\widetilde{A^{1/2}}(\lambda)}{\left(\widetilde{A}(\lambda)\right)^{1/2}}\right).$$

By rearranging terms, we have

$$\left(\widetilde{A}(\lambda)\right)^{\alpha}\left(1 + 2(\alpha-1)\left(1 - \frac{\widetilde{A^{1/2}}(\lambda)}{\left(\widetilde{A}(\lambda)\right)^{1/2}}\right)\right) \le \widetilde{A^\alpha}(\lambda). \tag{2.4}$$

By the Hölder-McCarthy inequality, $1 \ge 1 - \frac{\widetilde{A^{1/2}}(\lambda)}{\left(\widetilde{A}(\lambda)\right)^{1/2}} \ge 0$. Hence, the following chain of the inequalities are true :

$$\left(\widetilde{A}(\lambda)\right)^{\alpha} \le \left(\widetilde{A}(\lambda)\right)^{\alpha}\left(1 + 2(\alpha-1)\left(1 - \frac{\widetilde{A^{1/2}}(\lambda)}{\left(\widetilde{A}(\lambda)\right)^{1/2}}\right)\right) \le \widetilde{A^\alpha}(\lambda),$$

where $A$ is positive and $\alpha \ge 1$. One can easily see that

$$1 - \frac{\widetilde{A^{1/2}}(\lambda)}{\left(\widetilde{A}(\lambda)\right)^{1/2}} \ge \inf\left\{1 - \frac{\widetilde{A^{1/2}}(\lambda)}{\left(\widetilde{A}(\lambda)\right)^{1/2}} : \lambda \in \Omega\right\}.$$

So,

$$1+2(\alpha-1)\left(1 - \frac{\widetilde{A^{1/2}}(\lambda)}{\left(\widetilde{A}(\lambda)\right)^{1/2}}\right) \ge 1+2(\alpha-1)\inf\left\{1 - \frac{\widetilde{A^{1/2}}(\lambda)}{\left(\widetilde{A}(\lambda)\right)^{1/2}} : \lambda \in \Omega\right\}.$$

Then from inequality (2.4), we get the desired result

$$\left(\widetilde{A}(\lambda)\right)^{\alpha} \le \frac{1}{\mu}\widetilde{A^\alpha}(\lambda),\ \alpha \ge 1,$$

where $\mu = 1 + 2(\alpha-1)\left(1 - \frac{\widetilde{A^{1/2}}(\lambda)}{\left(\widetilde{A}(\lambda)\right)^{1/2}}\right)$. □

The following theorem is an improvement of inequality (1.4).

**Theorem 2.3.** *Let $A \in B(\mathcal{H})$ be an invertible operator, $0 < \alpha < 1$ and $r > 1$. If for each $\widehat{k}_\lambda$ a normalized reproducing kernel*

$$\mu(\lambda) = \left(1 + 2(r-1)\left(1 - \frac{\left(\widetilde{|A|^{\alpha}}(\lambda)\right)}{\left(\widetilde{|A|^{2\alpha}}(\lambda)\right)^{1/2}}\right)\right)$$

*and*

$$\nu(\lambda)=\left(1+2(r-1)\left(1-\frac{\left(\widetilde{|A^*|^{(1-\alpha)}}(\lambda)\right)}{\left(\widetilde{|A^*|^{2(1-\alpha)}}(\lambda)\right)^{1/2}}\right)\right),$$

*then*

$$\operatorname{ber}^r(A)\le\frac{1}{2\zeta}\left\||A|^{2\alpha r}+|A^*|^{2(1-\alpha)r}\right\|_{\operatorname{ber}},$$

*where* $\mu=\inf_{\lambda\in\Omega}\mu(\lambda)$, $\nu=\inf_{\lambda\in\Omega}\nu(\lambda)$ *and* $\zeta=\min\{\mu,\nu\}$.

*Proof.* Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Then

$$\begin{aligned}\left|\widetilde{A}(\lambda)\right|=\left|\left\langle A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^2&\le\left\langle|A|^{2\alpha}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^{\frac12}\left\langle|A^*|^{2(1-\alpha)}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^{\frac12}\\&\le\left(\frac{\left\langle|A|^{2\alpha}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^r+\left\langle|A^*|^{2(1-\alpha)}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^r}{2}\right)^{\frac1r}\\&\le\left(\frac12\left(\frac1\mu\left\langle|A|^{2\alpha}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^r+\frac1\nu\left\langle|A^*|^{2(1-\alpha)}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^r\right)\right)^{\frac1r}.\end{aligned}$$

Hence,

$$\left|\widetilde{A}(\lambda)\right|^r\le\frac{1}{2\zeta}\left\langle\left(|A|^{2\alpha r}+|A^*|^{2(1-\alpha)r}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle.$$

By taking the supremum over $\lambda\in\Omega$ above inequality, we have

$$\operatorname{ber}^r(A)\le\frac{1}{2\zeta}\left\||A|^{2\alpha r}+|A^*|^{2(1-\alpha)r}\right\|_{\operatorname{ber}},$$

which is an improvement of inequality (1.4). □

### 2.1. Berezin inequalities for sums and products of operators

In this subsection, we present Berezin operator norm inequalities and a related Berezin inequality for the sum and product of operators on reproducing kernel Hilbert spaces.

Now, we recall that some general result for the product of operators from [18].

Dragomir in ([7], Theorem 2) showed that for $A,B\in B(\mathcal{H})$, $\alpha\in(0,1)$ and $r\ge1$

$$|\langle Ax,By\rangle|^{2r}\le\alpha\left\langle(A^*A)^{\frac{r}{\alpha}}x,x\right\rangle+(1-\alpha)\left\langle(B^*B)^{\frac{r}{1-\alpha}}y,y\right\rangle,\tag{2.5}$$

where $x,y\in\mathcal{H}$ with $\|x\|=\|y\|=1$.

Let $A,B\in B(\mathcal{H})$. The Schwarz inequality states that

$$|\langle Ax,By\rangle|^2\le\langle Ax,Ax\rangle\langle By,By\rangle,\text{ for all }x,y\in\mathcal{H}.$$

We get the following refinements of inequality (2.5) for $r\ge2$.

**Theorem 2.4.** *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS. If $A, B \in B(\mathcal{H})$, $0 \le \alpha \le 1$ and $r \ge 2$, then*

$$\operatorname{ber}^{2r}(B^*A) \le \left\| \alpha (A^*A)^{r/\alpha} + (1-\alpha)(B^*B)^{r/(1-\alpha)} \right\|_{\text{ber}} - \inf_{\lambda \in \Omega} \mu(\lambda), \tag{2.6}$$

*where*

$$\mu(\lambda) = \alpha \left( \left| (A^*A)^{1/\alpha} - \widetilde{(A^*A)^{1/\alpha}}(\lambda) \right|^r \right)^{\sim}(\lambda)$$
$$+ (1-\alpha) \left( \left| (B^*B)^{1/(1-\alpha)} - \widetilde{(B^*B)^{1/(1-\alpha)}}(\lambda) \right|^r \right)^{\sim}(\lambda).$$

*Proof.* For a normalized reproducing kernel $\widehat{k}_\lambda$, we have

$$\left| \left\langle (B^*A)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2$$
$$= \left| \left( \widetilde{B^*A}(\lambda) \right) \right|^2$$
$$\le \left\langle (A^*A)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle (B^*B)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$$

(by the Schwarz inequality)

$$= \left\langle \left( (A^*A)^{r/\alpha} \right)^\alpha \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle \left( (B^*B)^{r/(1-\alpha)} \right)^{1-\alpha} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$$
$$\le \left\langle (A^*A)^{\frac{1}{\alpha}} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^\alpha \left\langle (B^*B)^{r/(1-\alpha)} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{1-\alpha}$$

(by inequality (1.8))

$$\le \left( \alpha \left\langle (A^*A)^{\frac{1}{\alpha}} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^r + (1-\alpha) \left\langle (B^*B)^{1/(1-\alpha)} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^r \right)^{1/r}$$

(by inequality (1.6))

$$\le \left( \alpha \left\langle (A^*A)^{r/\alpha} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle - \alpha \left\langle \left| (A^*A)^{1/\alpha} - \left\langle (A^*A)^{1/\alpha} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^r \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right.$$
$$+ (1-\alpha) \left\langle (B^*B)^{\frac{r}{1-\alpha}} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$$
$$\left. - (1-\alpha) \left\langle \left| (B^*B)^{1/(1-\alpha)} - \left\langle (B^*B)^{1/(1-\alpha)} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^r \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right)^{1/r}$$

(by inequality (1.9)).

So,

$$\left| \left( \widetilde{B^*A}(\lambda) \right) \right|^{2r} \le \alpha \left\langle (A^*A)^{r/\alpha} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + (1-\alpha) \left\langle (B^*B)^{r/(1-\alpha)} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$$
$$- \alpha \left\langle \left| (A^*A)^{1/\alpha} - \left\langle (A^*A)^{1/\alpha} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^r \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$$
$$- (1-\alpha) \left\langle \left| (B^*B)^{1/(1-\alpha)} - \left\langle (B^*B)^{1/(1-\alpha)} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^r \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$$

and

$$\sup_{\lambda\in\Omega}\left|\left(\widetilde{B^*A}(\lambda)\right)\right|^{2r} \le \sup_{\lambda\in\Omega}\alpha\left\langle (A^*A)^{r/\alpha}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle + \sup_{\lambda\in\Omega}(1-\alpha)\left\langle (B^*B)^{r/(1-\alpha)}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle$$
$$- \inf_{\lambda\in\Omega}\left(\alpha\left(\left|(A^*A)^{1/\alpha} - \widetilde{(A^*A)^{1/\alpha}}(\lambda)\right|^r\right)^{\sim}(\lambda) + (1-\alpha)\left(\left|(B^*B)^{1/(1-\alpha)} - \widetilde{(B^*B)^{1/(1-\alpha)}}(\lambda)\right|^r\right)^{\sim}(\lambda)\right)$$

which is equivalent to

$$\operatorname{ber}^{2r}(B^*A) \le \left\|\alpha(A^*A)^{r/\alpha} + (1-\alpha)(B^*B)^{r/(1-\alpha)}\right\|_{\text{ber}} - \inf_{\lambda\in\Omega}\mu(\lambda),$$

where

$$\mu(\lambda) = \alpha\left(\left|(A^*A)^{1/\alpha} - \widetilde{(A^*A)^{1/\alpha}}(\lambda)\right|^r\right)^{\sim}(\lambda) + (1-\alpha)\left(\left|(B^*B)^{1/(1-\alpha)} - \widetilde{(B^*B)^{1/(1-\alpha)}}(\lambda)\right|^r\right)^{\sim}(\lambda).$$

□

**Corollary 2.5.** *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS. For $A, B \in B(\mathcal{H})$, $0 < \alpha < 1$ and $r \ge 2$, the following norm inequalities and Berezin inequalities hold:*

*(i)* $\|A\|_{\text{ber}}^{2r} \le \alpha\left\|(A^*A)^{r/\alpha}\right\|_{\text{ber}} + (1-\alpha) - \inf_{\lambda\in\Omega}|\zeta(\lambda)|$;

*(ii)* $\|A^2\|_{\text{ber}}^{2r} \le \alpha\left\|(A^*A)^{r/\alpha}\right\|_{\text{ber}} + (1-\alpha)\left\|(A^*A)^{r/(1-\alpha)}\right\|_{\text{ber}} - \inf_{\lambda\in\Omega}\zeta(\lambda) - \inf_{\lambda\in\Omega}\mu(\lambda)$;

*(iii)* $\operatorname{ber}^{2r}(A) \le \left\|\alpha(A^*A)^{r/\alpha} + (1-\alpha)\right\|_{\text{ber}} - \inf_{\lambda\in\Omega}\zeta(\lambda)$;

*(iv)* $\operatorname{ber}^{2r}(A^2) \le \left\|\alpha(A^*A)^{r/\alpha} + (1-\alpha)\right\|_{\text{ber}} - \inf_{\lambda\in\Omega}\nu(\lambda)$ *where*

$$\zeta(\lambda) = \alpha\left(\left(\left|(A^*A)^{1/\alpha} - \widetilde{(A^*A)^{1/\alpha}}(\lambda)\right|^r\right)^{\sim}(\lambda)\right),$$

$$\mu(\lambda) = (1-\alpha)\operatorname{ber}^r\left((AA^*)^{1/(1-\alpha)} - \widetilde{(AA^*)^{1/(1-\alpha)}}(\lambda)\right)$$

$$\nu(\lambda) = \alpha\left(\left|(A^*A)^{1/\alpha} - \widetilde{(A^*A)^{1/\alpha}}(\lambda)\right|^r\right)^{\sim}(\lambda) + (1-\alpha)\left(\left|(AA^*)^{1/(1-\alpha)} - \widetilde{(AA^*)^{1/(1-\alpha)}}(\lambda)\right|^r\right)^{\sim}(\lambda).$$

We need the following lemma which a generalization of the inequality (1.2).

**Theorem 2.6.** *([30]) Let $A_i$, $X_i$, $B_i \in B(\mathcal{H})$, $i = 1, 2, ..., n$, and let $f$ and $g$ be nonnegative functions on $[0,\infty)$ which are continuous and satisfy the relation $f(t)\,g(t) = t$ for all $t \in [0,\infty)$. Then*

$$w^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{2}\left\|\sum_{i=1}^{n}\left(\left(B_i^* f^2(|X_i|)\,B_i\right)^r + \left(A_i^* g^2(|X_i^*|)\,A_i\right)^r\right)\right\|, \; r \geq 1.$$

We refine the above inequality for $r \geq 1$ by applying a refinement of the Hölder-McCarthy inequality.

**Theorem 2.7.** *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS, $A_i$, $X_i$, $B_i \in B(\mathcal{H})$, $i = 1, 2, ..., n$, be invertible operators and let $f$ and $g$ be nonnegative functions on $[0,\infty)$ which are continuos and satisfy in $f(t)\,g(t) = t$ for all $t \in [0,\infty)$. Then, for all $r > 1$,*

$$\operatorname{ber}^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n}\left(\left(B_i^* f^2(|X_i|)\,B_i\right)^r + \left(A_i^* g^2(|X_i^*|)\,A_i\right)^r\right)\right\|_{\text{ber}},$$

*where* $\mu = \min\{\zeta,\gamma\}$, $\zeta = \inf_{\lambda\in\Omega}\left\{1 + 2(r-1)\left(1 - \frac{\widetilde{\left(B_i^* f^2(|X_i|)B_i\right)^{1/2}}(\lambda)}{\left(\widetilde{\left(B_i^* f^2(|X_i|)B_i\right)}(\lambda)\right)^{1/2}}\right)\right\}$ *and* $\gamma = \inf_{\lambda\in\Omega}\left\{1 + 2(r-1)\left(1 - \frac{\widetilde{\left(A_i^* g^2(|X_i^*|)A_i\right)^{\frac{1}{2}}}(\lambda)}{\left(\widetilde{\left(A_i^* g^2(|X_i^*|)A_i\right)}(\lambda)\right)^{\frac{1}{2}}}\right)\right\}$.

*Proof.* Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Then we have

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^{n} A_i^* X_i B_i\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^r \\
&= \left|\sum_{i=1}^{n}\left\langle\left(A_i^* X_i B_i\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^r \\
&\leq \sum_{i=1}^{n}\left|\left\langle\left(A_i^* X_i B_i\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^r = \left(\sum_{i=1}^{n}\left|\left\langle X_i B_i\widehat{k}_\lambda, A_i\widehat{k}_\lambda\right\rangle\right|\right)^r \\
&\leq \left(\sum_{i=1}^{n}\left\langle f^2(|X_i|)\,B_i\widehat{k}_\lambda, B_i\widehat{k}_\lambda\right\rangle^{\frac{1}{2}}\left\langle g^2(|X_i^*|)\,A_i\widehat{k}_\lambda, A_i\widehat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^r \\
&\text{(by inequality (1.10))}
\end{aligned}$$

$$= n^{r-1} \sum_{i=1}^{n} \left\langle B_i^* f^2 (|X_i|) B_i \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{\frac{r}{2}} \left\langle A_i^* g^2 (|X_i^*|) A_i \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{\frac{r}{2}}$$

(by inequality (1.11))

$$= \frac{n^{r-1}}{2} \sum_{i=1}^{n} \left\langle B_i^* f^2 (|X_i|) B_i \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{r} + \left\langle A_i^* g^2 (|X_i^*|) A_i \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{r}$$

(by inequality AM-GM)

$$= \frac{n^{r-1}}{2} \sum_{i=1}^{n} \left( \frac{1}{\zeta} \left\langle \left(B_i^* f^2 (|X_i|) B_i\right)^r \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left\langle \left(A_i^* g^2 (|X_i^*|) A_i\right)^r \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right)$$

(by (2.2))

$$= \frac{n^{r-1}}{2\mu} \sum_{i=1}^{n} \left\langle \left( \left(B_i^* f^2 (|X_i|) B_i\right)^r + \left(A_i^* g^2 (|X_i^*|) A_i\right)^r \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$$

$$= \frac{n^{r-1}}{2\mu} \left\langle \sum_{i=1}^{n} \left( \left(B_i^* f^2 (|X_i|) B_i\right)^r + \left(A_i^* g^2 (|X_i^*|) A_i\right)^r \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle. \tag{2.7}$$

So,

$$\left| \left( \widetilde{\sum_{i=1}^{n} A_i^* X_i B_i} \right) (\lambda) \right|^r \le \frac{n^{r-1}}{2\mu} \left\langle \sum_{i=1}^{n} \left( \left(B_i^* f^2 (|X_i|) B_i\right)^r + \left(A_i^* g^2 (|X_i^*|) A_i\right)^r \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$$

and by taking supremum over $\lambda \in \Omega$,

$$\sup_{\lambda \in \Omega} \left| \left( \widetilde{\sum_{i=1}^{n} A_i^* X_i B_i} \right) (\lambda) \right|^r$$

$$\le \sup_{\lambda \in \Omega} \frac{n^{r-1}}{2\mu} \left\langle \sum_{i=1}^{n} \left( \left(B_i^* f^2 (|X_i|) B_i\right)^r + \left(A_i^* g^2 (|X_i^*|) A_i\right)^r \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$$

which is equivalent to

$$\mathrm{ber}^r \left( \sum_{i=1}^{n} A_i^* X_i B_i \right) \le \frac{n^{r-1}}{2\mu} \left\| \sum_{i=1}^{n} \left( \left(B_i^* f^2 (|X_i|) B_i\right)^r + \left(A_i^* g^2 (|X_i^*|) A_i\right)^r \right) \right\|_{\mathrm{ber}},$$

as required. □

If we assume that $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, $0 < \alpha < 1$, in Theorem 2.7, then we get the following corollary.

**Corollary 2.8.** *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS and $A_i$, $X_i$, $B_i \in B(\mathcal{H})$, $i = 1$, 2, ..., $n$, be invertible operators, $r > 1$ and $0 < \alpha < 1$. Then*

$$\mathrm{ber}^r \left( \sum_{i=1}^{n} A_i^* X_i B_i \right) \le \frac{n^{r-1}}{2\mu} \left\| \sum_{i=1}^{n} \left( B_i^* |X_i|^{2\alpha} B_i \right)^r + \left( A_i^* \left|X_i^*\right|^{2(1-\alpha)} A_i \right)^r \right\|_{\mathrm{ber}},$$

*where* $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf_{\lambda\in\Omega}\left\{1 + 2(r-1)\left(1 - \frac{\widetilde{\left(B_i^* |X_i|^{2\alpha} B_i\right)^{1/2}}(\lambda)}{\left(\widetilde{\left(B_i^* |X_i|^{2\alpha} B_i\right)}(\lambda)\right)^{1/2}}\right)\right\},$$

$$\gamma = \inf_{\lambda\in\Omega}\left\{1 + 2(r-1)\left(1 - \frac{\widetilde{\left(A_i^* |X_i^*|^{2(1-\alpha)} A_i\right)^{1/2}}(\lambda)}{\widetilde{\left(A_i^* |X_i^*|^{2(1-\alpha)} A_i\right)}(\lambda)^{1/2}(\lambda)}\right)\right\}.$$

*In particular,*

$$\operatorname{ber}\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \le \frac{1}{2}\left\|\sum_{i=1}^{n}\left(B_i^* |X_i| B_i + A_i^*(|X_i^*|) A_i\right)\right\|_{\operatorname{ber}}.$$

Setting $A_i = B_i = I$, $i = 1, 2, ..., n$, in Theorem 2.7, the following inequalities for sums of operators are obtained.

**Corollary 2.9.** *Let* $\mathcal{H} = \mathcal{H}(\Omega)$ *be a RKHS and* $X_i \in B(\mathcal{H})$, $i = 1, 2, ...,$ $n$ *be invertible operators and let* $f$ *and* $g$ *be continuous nonnegative functions on* $[0, \infty)$ *such that* $f(t)g(t) = t$ *for all* $t \in [0, \infty)$. *Then, for* $r > 1$,

$$\operatorname{ber}^r\left(\sum_{i=1}^{n} X_i\right) \le \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n}\left(f^{2r}(|X_i|) + g^{2r}(|X_i^*|)\right)\right\|_{\operatorname{ber}},$$

*where* $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf_{\lambda\in\Omega}\left\{1 + 2(r-1)\left(1 - \frac{\widetilde{f(|X_i|)}(\lambda)}{\left(\widetilde{f^2(|X_i|)}(\lambda)\right)^{1/2}}\right)\right\},$$

$$\gamma = \inf_{\lambda\in\Omega}\left\{1 + 2(r-1)\left(1 - \frac{\widetilde{g(|X_i^*|)}(\lambda)}{\left(\widetilde{g^2(|X_i^*|)}(\lambda)\right)^{1/2}}\right)\right\}.$$

*In particular*

$$\operatorname{ber}^r\left(\sum_{i=1}^{n} X_i\right) \le \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n} |X_i|^{2\alpha r} + |X_i^*|^{2(1-\alpha)r}\right\|_{\operatorname{ber}}, \ \alpha \in (0,1)$$

*where* $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf_{\lambda\in\Omega}\left\{1 + 2(r-1)\left(1 - \frac{\widetilde{|X_i|^{\alpha}}(\lambda)}{\left(\widetilde{|X_i|^{2\alpha}}(\lambda)\right)^{1/2}}\right)\right\},$$

$$\gamma = \inf_{\lambda\in\Omega}\left\{1 + 2(r-1)\left(1 - \frac{\widetilde{|X_i^*|^{(1-\alpha)}}(\lambda)}{\left(\widetilde{|X_i^*|^{2(1-\alpha)}}(\lambda)\right)^{1/2}}\right)\right\}.$$

Next, we present some Berezin inequalities for products of operators. Put $X_i = I$, $i = 1, 2, ..., n$, in Theorem 2.7, to get the following.

**Corollary 2.10.** *Let* $\mathcal{H} = \mathcal{H}(\Omega)$ *be a RKHS and* $A_i$, $B_i \in B(\mathcal{H})$, $i = 1, 2, ..., n$, *be invertible operators and* $r \geq 1$. *Then*

$$\mathrm{ber}^r\left(\sum_{i=1}^{n} A_i^* B_i\right) \leq \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n} |B_i|^{2r} + |A_i|^{2r}\right\|_{\mathrm{ber}},$$

*where* $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf_{\lambda\in\Omega}\left\{1 + 2(r-1)\left(1 - \frac{\widetilde{|B_i|}(\lambda)}{\left(\widetilde{|B_i|}(\lambda)\right)^{1/2}}\right)\right\},$$

$$\gamma = \inf_{\lambda\in\Omega}\left\{1 + 2(r-1)\left(1 - \frac{\widetilde{|A_i|}(\lambda)}{\left(\widetilde{|A_i|}(\lambda)\right)^{1/2}}\right)\right\}.$$

*In particular*

$$\mathrm{ber}\left(\sum_{i=1}^{n} A_i^* B_i\right) \leq \frac{1}{2\mu}\left\|\sum_{i=1}^{n} (B_i^* B_i) + (A_i^* A_i)\right\|_{\mathrm{ber}}.$$

Letting $n = 1$ in Corollary 2.10, then

$$\mathrm{ber}^r(A^* B) \leq \frac{1}{2\mu}\left\|(B^* B)^r + (A^* A)^r\right\|_{\mathrm{ber}},$$

where $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf_{\lambda\in\Omega}\left\{1 + 2(r-1)\left(1 - \frac{\widetilde{(B^*B)^{\frac{1}{2}}}(\lambda)}{\left(\widetilde{(B^*B)}(\lambda)\right)^{1/2}}\right)\right\},$$

$$\gamma = \inf_{\lambda\in\Omega}\left\{1 + 2(r-1)\left(1 - \frac{\widetilde{(A^*A)^{\frac{1}{2}}}(\lambda)}{\left(\widetilde{(A^*A)}(\lambda)\right)^{1/2}}\right)\right\}.$$

## References

[1] N. Aronzajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc., **68** (1950), 337-404.

[2] M. Bakherad, *Some Berezin number inequalities for operator matrices*, Czechoslov. Math. J., **68(4)** (2018), 997-1009.

[3] M. Bakherad and M.T. Garayev, *Berezin number inequalities for operators*, Concr. Oper., **6(1)** (2019), 33-43.

[4] H. Başaran, M. Gürdal and A.N. Güncan, *Some operator inequalities associated with Kantorovich and Hölder-McCarthy inequalities and their applications*, Turkish J. Math., **43(1)** (2019), 523-532.

[5] F.A. Berezin, *Covariant and contravariant symbols for operators*, Math. USSR-Izvestiya, **6** (1972), 1117-1151.

[6] F.A. Berezin, *Quantization*, Math. USSR-Izvestiya, **8** (1974), 1109-1163.

[7] S. S. Dragomir, *Vector inequalities for powers of some operators in Hilbert spaces*, Filomat, **23(1)** (2009), 69-83.

[8] M. Fujii and R. Nakamoto, *Refinements of Hölder-McCarthy inequality and Young inequality*, Adv. Oper. Theory, **1(2)** (2016), 184-188.

[9] M. T. Garayev, *Berezin symbols, Hölder-McCarthy and Young inequalities and their applications*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., **43(2)** (2017), 287-295.

[10] M. Garayev, F. Bouzeffour, M. Gürdal and C. M. Yangöz, *Refinements of Kantorovich type, Schwarz and Berezin number inequalities*, Extracta Math., **35** (2020), 1-20.

[11] M. T. Garayev, M. Gürdal and A. Okudan, *Hardy-Hilbert's inequality and a power inequality for Berezin numbers for operators*, Math. Inequal. Appl., **19** (2016), 883-891.

[12] M. T. Garayev, M. Gürdal and S. Saltan, *Hardy type inequaltiy for reproducing kernel Hilbert space operators and related problems*, Positivity, **21** (2017), 1615-1623.

[13] M. T. Garayev, H. Guedri, M. Gürdal and G. M. Alsahli, *On some problems for operators on the reproducing kernel Hilbert space, Linear Multilinear Algebra*, **69** (11) (2021), 2059-2077.

[14] M. Garayev, S. Saltan, F. Bouzeffour and B. Aktan, *Some inequalities involving Berezin symbols of operator means and related questions*, RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat., **114(85)** (2020), 1-17.

[15] K. E. Gustafson and D. K. M. Rao, *Numerical Range*, Springer-Verlag, New York, 1997.

[16] M. Hajmohamadi, R. Lashkaripour and M. Bakherad, *Improvements of Berezin number inequalities*, Linear Multilinear Algebra, **68(6)** (2020), 1218-1229.

[17] Z. Heydarbeygi and M. Amyari, *Some refinements of the numerical radius inequalities via Young inequality*, Kragujevac J. Math., **45(2)** (2021), 191-202.

[18] J. A. R. Holbrook, *Multiplicative properties of the numerical radius in operator theory*, J. Reine Angew. Math., **237** (1969), 166-174.

[19] M. T. Karaev, *Berezin symbol and invertibility of operators on the functional Hilbert spaces*, J. Funct. Anal., **238** (2006), 181-192.

[20] M. T. Karaev, *Reproducing kernels and Berezin symbols techniques in various questions of operator theory*, Complex Anal. Oper. Theory, **7** (2013), 983-1018.

[21] M. Kian, *Operator Jensen inequality for superquadratic functions*, Linear Algebra Appl., **456** (2014), 82-87.

[22] F. Kittaneh, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math., **158(1)** (2003), 11-17.

[23] F. Kittaneh, *Numerical radius inequalities for Hilbert space operators*, Studia Math., **168(1)** (2005), 73-80.

[24] F. Kittaneh and M. El-Haddad, *Numerical radius inequalities for Hilbert space operators II*, Studia Math., **182(2)** (2007), 133-140.

[25] F. Kittaneh, M. S. Moslehian and T. Yamazaki, *Cartesian decomposition and numerical radius inequalities*, Linear Algebra Appl., **471** (2015), 46-53.

[26] H. Kober, *On the arithmetic and geometric means and on Hölder's inequality*, Proc. Amer. Math. Soc., **9** (1958), 452-459.

[27] C-S. Lin and Y. J. Cho, *On Hölder-McCarthy-type inequalities with powers*, J. Korean Math. Soc., **39(3)** (2002), 351-361.

[28] S. S. Sahoo, N. Das and D. Mishra, *Berezin number and numerical radius inequalities for operators on Hilbert spaces*, Advances in Oper. Theory, **5** (2020), 714-727

[29] M. Sattari, M. S. Moslehian and T. Yamazaki, *Some generalized numerical radius inequalities for Hilbert space operators*, Linear Algebra Appl., **470** (2015), 216-227.

[30] K. Shebrawi and H. Albadwi, *Numerical radius and operator norm inequalities*, J. Inequal. Appl., **2009** (2009), Article ID 492154, 11 pages.

[31] R. Tapdigoglu, *On the description of invariant subspaces in the space $C_n[0,1]$*, Houston J. Math., **39(1)** (2013), 169-176.

[32] R. Tapdigoglu, *New Berezin symbol inequalities for operators on the reproducing kernel Hilbert space*, Oper. Matrices, (accepted) 2020.

[33] U. Yamancı, and M. Gürdal, *On numerical radius and Berezin number inequalities for reproducing kernel Hilbert space*, New York J. Math., **23** (2017), 1531-1537.

[34] U. Yamancı, M. Gürdal and M. T. Garayev, *Berezin number inequality for convex function in reproducing kernel Hilbert space*, Filomat, **31** (2017), 5711-5717.

[35] U. Yamancı, R. Tunç and M. Gürdal, *Berezin numbers, Grüss type inequalities and their applications*, Bull. Malays. Math. Sci. Soc., **43** (2020), 2287-2296.

[36] A. Zamani, *Some lower bounds for the numerical radius of Hilbert space operators*, Adv. Oper. Theory, **2(2)** (2007), 98-107.

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