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TUBULAR SURFACES WITH MODIFIED ORTHOGONAL FRAME IN EUCLIDEAN 3-SPACE

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Abstract. In this study, tubular surfaces that play an important role in technological designs in various branches are examined for the case of the base curve is not satisfying the fundamental theorem of the differential geometry. In order to give an alternative perspective to the researches on tubular surfaces, the modified orthogonal frame is used in this study. Firstly, the relationships between the Serret-Frenet frame and the modified orthogonal frame are summarized. Then the definitions of the tubular surfaces, some theorems, and results are given. Moreover, the fundamental forms, the mean curvature, and the Gaussian curvature of the tubular surface are calculated according to the modified orthogonal frame. Finally, the properties of parameter curves of the tubular surface with modified orthogonal frame are expressed and the tubular surface is drawn according to the Frenet frame and the modified orthogonal frame.

1. Introduction

In 1850, Gaspard Monge described for the first time the canal surfaces as the envelopes of a one-parameter family of the moving spheres $S(s)$ with a variable radius. Geometric and analytical properties of these surfaces and their different perspectives in surface theory were studied [5, 23]. If the center of a sphere $S(s)$ is taken as $\alpha(s)$ then the curve $\alpha(s)$ is called the center curve of the canal surface. For a canal surface, if the center curve $\alpha(s)$ is a straight line, the channel surface is called a revolution surface. Moreover, if the radius of the sphere-forming canal surface is constant, these canal surfaces are called the tubular surface. Tubular surfaces, which play an important role in computer drawings and material design, were studied by taking different frames in Euclidean, Minkowski, and Galilean spaces, [6]-[22]. The Frenet frame has not always been sufficient in solving some problems in studies related to curves and surfaces. Therefore, different frames were needed to solve these problems. While the Frenet frame is a frame on the regular curve, the Darboux frame

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is a nature-moving frame setting up on a regular surface. However, a new alternative frame to the Frenet frame called the Bishop frame was defined by L. R. Bishop in 1975. The main problem of the Frenet frame is that it cannot be identified at the points where the curvature is zero. For example, let α be a regular space curve. The Frenet frame is undefined when $\alpha''(s) = 0$. Also, the principal normal vector $N(s)$ of the Frenet frame may have a non-removable discontinuity at these points. This is why the Bishop frame is used. On the other hand, Sasai described the modified orthogonal frame as an alternative to the Frenet frame for the solution of the same problem [21]. This frame, which was defined to compensate for the deficiency of the Frenet frame, has been actively studying for the last few years, [2]-[19].

Our aim in this study to investigate the geometric properties of a tubular surface with a modified orthogonal frame. Also, the characterizations of the parameter curves of the tubular surface are examined. Thus, using the modified orthogonal frame in the study makes it different from previous studies and becomes a source for new researches.

2. Preliminaries

The mathematical measurements of the turning and twisting of a curve in E^3 have been given in [18]. First, let us consider unit speed curves. An important aspect of the differential geometry of a curve is to use the Frenet frame field $\{t, n, b\}$. The Frenet differentiation formula constituted with the help of these vectors is given as follows:

$$t' = \kappa n, n' = -\kappa t + \tau b, b' = -\tau n.$$

Here, κ and τ are the curvature and the torsion of the curve α, respectively.

The fundamental theorem of the differential geometry of curves in E^3 states that a curve is completely determined by its curvature κ and torsion τ functions. Specifically, suppose $\kappa(s)$ and $\tau(s)$ are given functions defined on some interval I, such that κ is positive and continuously differentiable and τ is continuous. Then there exists a curve $\alpha(s)$ defined on I for which s is arc-length such that κ and τ are, respectively, the curvature and torsion [8]. But, the Frenet frame is insufficient at the points where the curvature of the curve α is zero. Because at these points, the principal normal and binormal vectors of any the curve α become discontinuous. Sasai, who was searching for a solution to this problem, has defined the modified orthogonal frame as an alternative to Frenet frame [21]. If the curvature $\kappa(s)$ of the curve α doesn't vanish, then we define an orthogonal frame as follow:

$$T = \frac{d\alpha}{ds}, N = \frac{dT}{ds}, B = T \wedge N.$$

This frame becomes meaningless in the case of κ equals to zero. So, we can construct a new alternative framework by assuming that κ is different from zero. Under this assumption, the relation between a modified orthogonal frame $\{T, N, B\}$ and a Frenet frame $\{t, n, b\}$ is

$$T = t, N = \kappa n, B = \kappa b.$$

It is seen that the modified orthogonal frame holds the following equations with the aid of the Frenet frame

$$\langle T, N\rangle = \langle T, B\rangle = \langle N, B\rangle = 0,\ \langle T, T\rangle = 1,\ \langle N, N\rangle = \langle B, B\rangle = \kappa^2.$$

Due to these equations, the derivative equations of the modified orthogonal frame $\{T, N, B\}$ are given as

$$\begin{array}{l} T'(s) = N(s), \\ N'(s) = -\kappa^2 T(s) + \frac{\kappa'}{\kappa} N(s) + \tau B(s), \\ B'(s) = -\tau N(s) + \frac{\kappa'}{\kappa} B(s). \end{array}$$

Here " $'$ " refers to the differential according to the parameter s and $\tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2}$ is the torsion of the curve α. There are several different ways to formulate a surface concept mathematically. One of these surfaces is given with

$$\varphi(s, v) = \beta(s) + v\delta(s).$$

This surface generated with the help of a director curve and straight lines, which called the rulings, are called ruled surfaces. Moreover, these straight lines may be curves as well. The first fundamental form allows us to calculate the angle, area, and length on the surface. However, the concept that expresses the deviation of the surface from the tangent plane is the second fundamental form. φ_s and φ_v are tangent vectors of a surface $\varphi(s, v)$ and the coefficients of these forms are

$$\begin{array}{l} E = \langle \varphi_s, \varphi_s\rangle, \mathrm{F} = \langle \varphi_s, \varphi_v\rangle, \mathrm{G} = \langle \varphi_v, \varphi_v\rangle, \\ e = \langle \varphi_{ss}, U\rangle, f = \langle \varphi_{sv}, U\rangle, \mathrm{g} = \langle \varphi_{vv}, U\rangle, \end{array}$$

respectively. Here, $U = \frac{\varphi_s \wedge \varphi_v}{\|\varphi_s \wedge \varphi_v\|}$ is the normal vector field of the surface.

Owing to the fact that the Gaussian curvature depends solely on the metric which on the coefficients of the first fundamental form, it is invariant under isometric deformation. The Gaussian curvature plays a special role in the theory of surfaces, and many formulas are available for its computation, [20]. The mean curvature measures also the change resulting from the contraction and expansion of the surface. The Gaussian curvature and the mean curvature of the surface $\varphi(s, v)$ are

$$\begin{array}{l} K = \frac{eg - f^2}{EG - F^2}, \\ H = \frac{1}{2}\frac{Eg - 2Ff + Ge}{EG - F^2}, \end{array}$$

respectively.

3. Properties of Tubular Surfaces with Modified Orthogonal Frame

A canal surface is defined as the envelope of the moving sphere of variable radius. If the radius is constant, this canal surface is called the tubular surface. The parametric equation of the tubular surface is given as follows

$$\varphi(s,v)=\alpha(s)+r\left(\cos vN(s)+\sin vB(s)\right)$$

where $v\in[0,2\pi)$, r is the radius of the tubular surface and the curve $\alpha(s)$ is the center curve of the tubular surface. Also, the vectors N and B are perpendicular to the curve at the point $\alpha(s)$ of the curve α.

The derivatives according to parameters s and v of the tubular surface $\varphi(s,v)$ are, respectively,

(1)
$$\begin{aligned}\varphi_s&=\left(1-r\kappa^2\cos v\right)T+r\left(\tfrac{\kappa'}{\kappa}\cos v-\tau\sin v\right)N+r\left(\tau\cos v+\tfrac{\kappa'}{\kappa}\sin v\right)B,\\ \varphi_v&=r\left(-\sin vN+\cos vB\right).\end{aligned}$$

From equation (1), the coefficients of the first fundamental form are found as

$$E=\left(1-r\kappa^2\cos v\right)^2+r^2\kappa^2\left(\left(\frac{\kappa'}{\kappa}\right)^2+\tau^2\right),\mathrm{F}=r^2\kappa^2\tau,\mathrm{G}=r^2\kappa^2. \tag{2}$$

Moreover, considering equations of (1), the normal vector field of the tubular surface $\varphi(s,v)$ is obtained as

$$U=\frac{\left(r\frac{\kappa'}{\kappa}\right)T-\cos v\left(1-r\kappa^2\cos v\right)N-\sin v\left(1-r\kappa^2\cos v\right)B}{A}, \tag{3}$$

where $A=\kappa^2\sqrt{\left(r\left(\frac{-1}{\kappa}\right)'\right)^2+\left(1-r\kappa^2\cos v\right)^2}\neq0$ and, here, $r\kappa^2\cos v\neq1$ or κ is not a constant.

If the unit normal vector at any point of a surface $\varphi(s,v)$ vanishes, i.e. $\varphi_s\wedge\varphi_v=0$ at any points, then these points are called the singular points of the surface. So the following result is obvious.

Corollary 3.1. *If $r\kappa^2\cos v=1$ and κ is non-zero constant then there is no singular point on the tubular surface.*

The second-order partial derivatives of tubular surface $\varphi(s,v)$ are found by

(4)
$$
\begin{aligned}
\varphi_{ss} &= \left(-3r\kappa\kappa' \cos v + r\kappa^2\tau \sin v\right) T \\
&\quad + \left(\left(1 - r\kappa^2 \cos v\right) + r\left(\left(\tfrac{\kappa''}{\kappa} - \tau^2\right)\cos v - \left(\tau' + 2\tfrac{\kappa'}{\kappa}\tau\right)\sin v\right)\right) N \\
&\quad + r\left(\left(\tau' + 2\tfrac{\kappa'}{\kappa}\tau\right)\cos v + \left(\tfrac{\kappa''}{\kappa} - \tau^2\right)\sin v\right) B, \\
\varphi_{sv} &= \left(r\kappa^2 \sin v\right) T + r\left(-\tau\cos v - \tfrac{\kappa'}{\kappa}\sin v\right) N + r\left(\tfrac{\kappa'}{\kappa}\cos v - \tau \sin v\right) B, \\
\varphi_{vv} &= -r\left(\cos v N + \sin v B\right).
\end{aligned}
$$

From the equations (3) and (4), the coefficients of the second fundamental form are

(5)
$$
\begin{aligned}
e &= \tfrac{1}{A}\left(\left(r^2\kappa'\right)\left(-3\kappa'\cos v + \kappa\tau\sin v\right) - \kappa^2\cos v\left(1 - r\kappa^2\cos v\right)^2 - r\kappa^2\left(1 - r\kappa^2\cos v\right)\left(\tfrac{\kappa''}{\kappa} - \tau^2\right)\right), \\
f &= \tfrac{1}{A}\left(r^2\kappa\kappa'\sin v + r\kappa^2\tau\left(1 - r\kappa^2\cos v\right)\right), \\
g &= \tfrac{1}{A}\left(r\kappa^2\left(1 - r\kappa^2\cos v\right)\right).
\end{aligned}
$$

The Gaussian and mean curvatures of the tubular surface $\varphi(s,v)$ with the help of equations (2) and (5) are obtained as

$$
K = \frac{\left\{\left(1-r\kappa^2\cos v\right)\begin{pmatrix} \left(r^2\kappa'\right)\left(-3\kappa'\cos v + \kappa\tau\sin v\right) \\ -\kappa^2\cos v\left(1 - r\kappa^2\cos v\right)^2 \\ -r\kappa^2\left(1 - r\kappa^2\cos v\right)\left(\frac{\kappa''}{\kappa} - \tau^2\right) \end{pmatrix} - r\left(r\kappa'\sin v + \kappa\tau\left(1 - r\kappa^2\cos v\right)\right)^2\right\}}{A^2 r\left(\left(1-r\kappa^2\cos v\right)^2 + \left(r\kappa'\right)^2\right)},
$$

$$
H = \frac{\left(\left(1-\kappa^2 r\cos v\right)^2\left(1-2r\kappa^2\cos v\right) - r^3\kappa'\left(3\kappa'\cos v + \kappa\tau\sin v\right) + r^2\left(\left(\kappa'\right)^2 - \kappa''\kappa\right)\left(1 - r\kappa^2\cos v\right)\right)}{2Ar\left(\left(1-r\kappa^2\cos v\right)^2 + \left(r\kappa'\right)^2\right)},
$$

respectively.

Let's give some theorems about geometric interpretation of parametric curves of the tubular surface $\varphi(s,v)$.

Theorem 3.2.

i. The $s-$parameter curves of the tubular surface $\varphi(s,v)$ are geodesic curves if and only if

$$
r = \frac{\sin v}{\kappa^2\sin v\cos v + \left(\tau' + 2\frac{\kappa'}{\kappa}\tau\right)},\ \kappa^2\sin v\cos v + \left(\tau' + 2\tfrac{\kappa'}{\kappa}\tau\right) \neq 0.
$$

ii. The $v-$parameter curves of the tubular surface $\varphi(s,v)$ are geodesic curves if and only if κ is constant.

Proof. In order for the parameter curves of the surface to be geodesic curves, the acceleration vectors of these curves must be perpendicular to the surface and therefore they are parallel to the normal vector of the surface.

i. From the equations (3) and (4), we get

$$U \wedge \varphi_{ss} =$$
$$\frac{1}{A}\left\{\begin{array}{l} \left(1 - r\kappa^2 \cos v\right)\left(-r\left(\tau' + 2\frac{\kappa'}{\kappa}\tau\right) + \sin v\left(1 - r\kappa^2 \cos v\right)\right) T \\ + \left(\begin{array}{l} r\kappa \sin v\left(1 - r\kappa^2 \cos v\right)\left(3\kappa' \cos v - \kappa\tau \sin v\right) \\ -r^2 \frac{\kappa'}{\kappa}\left(\left(\tau' + 2\frac{\kappa'}{\kappa}\tau\right)\cos v + \left(\frac{\kappa''}{\kappa} - \tau^2\right)\sin v\right) \end{array}\right) N \\ + \left(\begin{array}{l} r\kappa \cos v\left(1 - r\kappa^2 \cos v\right)\left(-3\kappa' \cos v + \kappa\tau \sin v\right) + \\ r\frac{\kappa'}{\kappa}\left(1 - r\kappa^2 \cos v\right) + r^2\frac{\kappa'}{\kappa}\left(\left(\frac{\kappa''}{\kappa} - \tau^2\right)\cos v - \left(\tau' + 2\frac{\kappa'}{\kappa}\tau\right)\sin v\right) \end{array}\right) B \end{array}\right\}.$$

Since T, N and B are linearly independent, this means that $U \wedge \varphi_{ss} = 0$ if and only if

$$\begin{array}{l} \left(1 - r\kappa^2 \cos v\right)\left(-r\left(\tau' + 2\frac{\kappa'}{\kappa}\tau\right) + \sin v\left(1 - r\kappa^2 \cos v\right)\right) = 0, \\ -r\kappa \sin v\left(1 - r\kappa^2 \cos v\right)\left(-3\kappa' \cos v + \kappa\tau \sin v\right) \\ -r^2\frac{\kappa'}{\kappa}\left(\left(\tau' + 2\frac{\kappa'}{\kappa}\tau\right)\cos v + \left(\frac{\kappa''}{\kappa} - \tau^2\right)\sin v\right) = 0, \\ r\kappa \cos v\left(1 - r\kappa^2 \cos v\right)\left(-3\kappa' \cos v + \kappa\tau \sin v\right) + r\frac{\kappa'}{\kappa}\left(1 - r\kappa^2 \cos v\right) \\ +r^2\frac{\kappa'}{\kappa}\left(\left(\frac{\kappa''}{\kappa} - \tau^2\right)\cos v - \left(\tau' + 2\frac{\kappa'}{\kappa}\tau\right)\sin v\right) = 0. \end{array} \tag{6}$$

When these last two equations are taken into consideration together and the necessary operations are done, we have

$$r\frac{\kappa'}{\kappa}\left(-r\left(\tau' + 2\frac{\kappa'}{\kappa}\tau\right) + \sin v\left(1 - r\kappa^2 \cos v\right)\right) = 0.$$

So, from this equation and in the first equation of the equation (6), we get

$$r = \frac{\sin v}{\kappa^2 \sin v \cos v + \left(\tau' + 2\frac{\kappa'}{\kappa}\tau\right)}, \quad \kappa^2 \sin v \cos v + \left(\tau' + 2\frac{\kappa'}{\kappa}\tau\right) \neq 0.$$

ii. From the equations (3) and (4), we have $U \wedge \varphi_{vv} = \frac{1}{A}\left\{\left(-\frac{\kappa'}{\kappa}r^2\right)\left(\sin vN + \cos vB\right)\right\}$. Since N and B are linearly independent, this means that $U \wedge \varphi_{vv} = 0$ if and only if κ is constant. As a result, the $v-$parameter curves are geodesic curves. □

Theorem 3.3.
i. The $s-$parameter curves of the tubular surface $\varphi(s, v)$ are asymptotic curves if and only if

$$r = \frac{-2\kappa^4 \cos v^2 + \kappa^2\left(\frac{\kappa''}{\kappa} - \tau^2\right) \pm \kappa\sqrt{\kappa^2\left(\frac{\kappa''}{\kappa} - \tau^2\right)^2 + 4\kappa'\kappa\tau \sin v \cos v - 12\kappa'^2 \cos v^2}}{2\left(\left(\kappa^4\left(\frac{\kappa''}{\kappa} - \tau^2\right) - 3\kappa'^2\right)\cos v + \kappa\kappa'\tau \sin v - \kappa^6 \cos v^3\right)}.$$

ii. The $v-$parameter curves of the tubular surface $\varphi(s, v)$ are asymptotic curves if and only if $r\kappa^2 \cos v = 1$ and κ is not constant.

Proof. If the parameter curves on the surface are asymptotic curves, the normal curvature of the parameter curves must be zero everywhere. Therefore, $\langle \varphi_{ss}, U\rangle = 0$ and $\langle \varphi_{vv}, U\rangle = 0$ must be provided for the s and $v-$parameter curves.

i. From equation (5), we know

$$e = \langle \varphi_{ss}, U \rangle =$$
$$\frac{1}{A}\left(\left(r^2\kappa'\right)\left(-3\kappa'\cos v + \kappa\tau\sin v\right) - \kappa^2\cos v\left(1 - r\kappa^2\cos v\right)^2 - r\kappa^2\left(1 - r\kappa^2\cos v\right)\left(\frac{\kappa''}{\kappa} - \tau^2\right)\right).$$

$s-$parameter curves of the tubular surface $\varphi(s,v)$ are asymptotic curves if and only if $e = 0$. In this case, when necessary operations are taken in the equation above, we get

$$r = \frac{-2\kappa^4\cos v^2 + \kappa^2\left(\frac{\kappa''}{\kappa} - \tau^2\right) \pm \kappa\sqrt{\kappa^2\left(\frac{\kappa''}{\kappa} - \tau^2\right)^2 + 4\kappa'\kappa\tau\sin v\cos v - 12\kappa'^2\cos v^2}}{2\left(\left(\kappa^4\left(\frac{\kappa''}{\kappa} - \tau^2\right) - 3\kappa'^2\right)\cos v + \kappa\kappa'\tau\sin v - \kappa^6\cos v^3\right)}.$$

ii. From equation (5), we know that

$$\mathrm{g} = \langle \varphi_{vv}, U \rangle = \frac{1}{A}\left(r\kappa^2\left(1 - r\kappa^2\cos v\right)\right).$$

If $r\kappa^2\cos v = 1$, $v-$parameter curves of the tubular surface $\varphi(s,v)$ are asymptotic curves. In that case, κ has not to be constant. But if κ is also constant, the normal vector of the tubular surface does not vanish.

□

Theorem 3.4. *The s and $v-$parameter curves of the tubular surface $\varphi(s,v)$ are lines of curvature if and only if $\tau = 0$ and κ is constant.*

Proof. If the parameter curves of a surface are lines of curvature, then $F = f = 0$. In that case, from the equations (2) and (5), we get

$$r^2\kappa^2\tau = 0$$

and

$$\kappa\kappa' r^2\sin v + \kappa^2\tau r\left(1 - \kappa^2 r\cos v\right) = 0.$$

$F = f = 0$ if $\tau = 0$ and, κ is constant or $v = k\pi$, $(k \in Z)$. So, the s and $v-$parameter curves of the tubular surface $\varphi(s,v)$ are lines of curvature.

□

Example 3.5. *We choose the eight curve, which is also known as Gerono lemniscate curve as the center curve* [9]. *In this example, the graphics of the tubular surfaces with the base curve of eight curves by using both the Frenet frame and the modified orthogonal frames have been given for comparison. First, let us consider the curve $\alpha(s)$ given by the parametric equation*

$$\alpha(s) = (\sin(s), \sin(s)\cos(s), s).$$

The elements of the Frenet trihedron of the curve $\alpha(s)$ are obtained as

$$t(s)=\left(\frac{\sqrt{2}\cos(s)}{\sqrt{4+\cos(2s)+\cos(4s)}},\frac{\sqrt{2}\cos(2s)}{\sqrt{4+\cos(2s)+\cos(4s)}},\frac{\sqrt{2}}{\sqrt{4+\cos(2s)+\cos(4s)}}\right),$$

$$n(s)=\begin{pmatrix}\frac{\sin(s)(-1+4\cos(2s)+\cos(4s))}{\sqrt{4+\cos(2s)+\cos(4s)}\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}},\\ -\frac{\sin(2s)(6+\cos(2s))}{\sqrt{4+\cos(2s)+\cos(4s)}\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}},\\ \frac{\sin(2s)+2\sin(4s)}{\sqrt{4+\cos(2s)+\cos(4s)}\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}}\end{pmatrix},$$

$$b(s)=\begin{pmatrix}\frac{2\sqrt{2}\sin(2s)}{\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}},\\ -\frac{\sqrt{2}\sin(s)}{\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}},\\ -\frac{3\sin(s)+\sin(3s)}{\sqrt{2}\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}}\end{pmatrix}.$$

The curvature of the unit speed curve $\alpha(s)$ are found as

$$\kappa(s)=\frac{2\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}}{(4+\cos(2s)+\cos(4s))^{3/2}}.$$

Besides the curvature $\kappa(s)=\frac{2\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}}{(4+\cos(2s)+\cos(4s))^{3/2}}$ is not differentiable, the principal normal and binormal vectors are discontinuous at $s=0$ since $n_+\neq n_-$ and $b_+\neq b_-$ for $n_+=\lim\limits_{s\to0^+}n(s)$, $n_-=\lim\limits_{s\to0^-}n(s)$ and $b_+=\lim\limits_{s\to0^+}b(s)$, $b_-=\lim\limits_{s\to0^-}b(s)$. Looking for a solution to this problem, let us consider the modified orthogonal frame of Sasai, [21]. The elements of the modified orthogonal frame of the unit speed curve $\alpha(s)$ are obtained as

$$T(s)=\left(\frac{\sqrt{2}\cos(s)}{\sqrt{4+\cos(2s)+\cos(4s)}},\frac{\sqrt{2}\cos(2s)}{\sqrt{4+\cos(2s)+\cos(4s)}},\frac{\sqrt{2}}{\sqrt{4+\cos(2s)+\cos(4s)}}\right),$$
$$N(s)=\left(\frac{2\sin(s)(-1+4\cos(2s)+\cos(4s))}{(4+\cos(2s)+\cos(4s))^2},\frac{-\sin(2s)(6+\cos(2s))}{(4+\cos(2s)+\cos(4s))^2},\frac{2(\sin(2s)+2\sin(4s))}{(4+\cos(2s)+\cos(4s))^2}\right),$$
$$B(s)=\left(\frac{4\sqrt{2}\sin(2s)}{(4+\cos(2s)+\cos(4s))^{3/2}},\frac{-2\sqrt{2}\sin(s)}{(4+\cos(2s)+\cos(4s))^{3/2}},\frac{-\sqrt{2}(3\sin(s)+\sin(3s))}{(4+\cos(2s)+\cos(4s))^{3/2}}\right).$$

Now let's draw the graphs of tubular surfaces whose equations are

$$\phi_F(s,v)=\alpha(s)+r(\cos v n(s)+\sin v b(s))$$

and

$$\varphi_M(s,v)=\alpha(s)+r(\cos v N(s)+\sin v B(s))$$

according to the Frenet frame and the modified orthogonal frame, respectively.

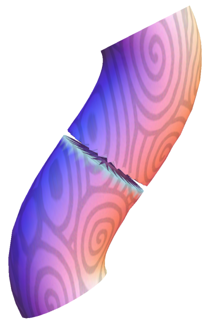

FIGURE 1. Tubular surface obtained by Frenet elements of base curve

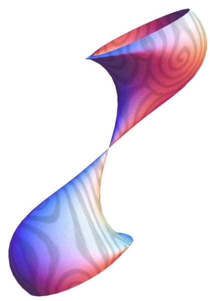

FIGURE 2. Tubular surface obtained by modified orthogonal frame of base curve

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