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# FEKETE-SZEGÖ INEQUALITY FOR A SUBCLASS OF NON-BAZILEVIĆ FUNCTIONS INVOLVING CHEBYSHEV POLYNOMIAL

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**Abstract.** In this present work, we obtain certain coefficients of the subclass $\mathcal{H}_{\lambda,\gamma}(s,b,n)$ of non-Bazilević functions and estimate the relevant connection to the famous classical Fekete-Szegö inequality of functions belonging to this class.

## 1. Introduction and definitions

Let $\mathcal{A}$ denote the class of all analytic functions $f(z)$ in the open unit disk

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$$

have the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

and normalized with the condition

$$f(0) = 0 = f'(0) - 1.$$

Let $\mathcal{S}$ refer to the class of all univalent functions belong to the normalized analytic functions $\mathcal{A}$.

Let $f(z)$ and $g(z)$ be analytic functions in $\mathcal{U}$, we call that $f(z)$ is a subordinate to $g(z)$ in unit disk $\mathcal{U}$, written $f(z) \prec g(z)$, if there is a Schwarz function $w(z)$, which is analytic in unit disk $\mathcal{U}$ and satisfies $w(0) = 0$ and $|w(z)| < 1$, $(z \in \mathcal{U})$ such that $f(z) = g(w(z))$.

Furthermore, if the function $g(z)$ is univalent in $\mathcal{U}$, then we get the equivalent (see [9])

$$f(z) \prec g(z) \quad (z \in \mathcal{U}) \Longleftrightarrow f(0) = g(0) \quad and \quad f(\mathcal{U}) \subset g(\mathcal{U}).$$

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The Fekete-Szegö inequalities $\left|a_3 - \mu a_2^2\right|$ for the normalized Taylor series

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

is well-known in the geometric functions theory. Its origin was in the disproof by Fekete and Szegö of the 1933 guess of Paley and Littlewood that the coefficients of odd univalent functions are limited by unity (see [11], has since got incredible attention, particularly in many classes of univalent functions). For that reason Fekete-Szegö function introduced by many authors were found in more classes of univalent functions (see [3, 10, 14, 16, 17, 18]).

The larger part research papers dealing with orthogonal polynomials of Chebyshev, contain mostly main results of first type $\mathrm{T}_k(t)$ and second type $\mathrm{U}_k(t)$ of Chebyshev polynomials and their various uses in more applications. Also, one can see the papers in ([1, 2, 4, 5, 8, 13, 6]). The first and second types of the Chebyshev polynomials are famous in the case of a certified variable $-1 < n < 1$ which are defined by the follows:

$$\mathrm{T}_k(n) = \cos k\theta.$$

$$\mathrm{U}_k(n) = \frac{\sin(k+1)\theta}{\sin\theta},$$

where $k$ refer to the degree of polynomial and $n = \cos\theta$.

Babalola [7] introduced a new class of $\gamma$-pseudo starlike function of order $\delta\,(0 \le \delta < 1)$ which satisfies the following analytic condition

$$\Re\left\{\frac{z(f'(z))^{\gamma}}{f(z)}\right\} > \delta \qquad (\gamma \ge 1,\ z \in \mathcal{U}), \tag{2}$$

Recently, Frasin [12] introduced and investigated the coefficient for subclasses of Sakaguchi functions which satisfied the following geometrical condition

$$\Re\left\{\frac{(s-b)zf'(z)}{f(sz)-f(bz)}\right\} > \beta \qquad (z \in \mathcal{U}), \tag{3}$$

where $s$ and $b$ are complex numbers with $s \ne b$ and $0 \le \beta < 1$.

Therefore, the authors benefited from equations (2) and (3) to introduce a new type of non-Bazilević as follows:

**Definition 1.1.** *A function $f \in \mathcal{A}$ given by* (1) *belongs to the subclass $\mathcal{H}_{\lambda,\gamma}(s,b,n)$, if it satisfy the next subordination:*

$$(f'(z))^{\gamma}\left(\frac{(s-b)z}{f(sz)-f(bz)}\right)^{\lambda} \prec G(z,n) = \frac{1}{1-2nz+z^2}, \tag{4}$$

*where $s$ and $b$ are complex numbers with $s \ne b$, $|b| \le 1$, $b \ne 1$, $\gamma \ge 1$, $0 \le \lambda \le 1, n \in (1/2, 1)$ and $z \in \mathcal{U}$.*

**Remark 1.2.** $(i)$ *If we take* $\gamma = 1$ *and* $\lambda = 0$ *in the Definition* 1.1, *then we get the class* $\mathcal{H}(n)$ *which consists of functions* $f \in \mathcal{A}$ *satisfying*

$$f'(z) \prec G(z,n) = \frac{1}{1 - 2nz + z^2}.$$

$(ii)$ *If we take* $\gamma = 1$ *and* $\lambda = 1$ *in the Definition* 1.1, *then we get the class* $\mathcal{H}(s,b,n)$ *which consists of functions* $f \in \mathcal{A}$ *satisfying*

$$\frac{(s-b)zf'(z)}{f(sz) - f(bz)} \prec G(z,n) = \frac{1}{1 - 2nz + z^2}.$$

We note that if $n = \cos\alpha$ with $\alpha \in (-\pi/3, \pi/3)$, then

$$G(z,n) := \frac{1}{1 - 2\cos\alpha z + z^2} = 1 + \sum_{k=1}^{\infty} \frac{\sin((k+1)\alpha)}{\sin\alpha} z^k \quad (z \in \mathcal{U}).$$

Thus

$$G(z,n) = 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + \cdots \quad (z \in \mathcal{U}).$$

From [19], we write

$$G(z,n) = 1 + \mathrm{U}_1(n)z + \mathrm{U}_2(n)z^2 + \cdots \quad (z \in \mathcal{U},\ n \in (-1,1)),$$

where

$$\mathrm{U}_{k-1} = \frac{\sin(k \arccos n)}{\sqrt{1-n^2}}, \quad (k \in \mathbb{N} = \{1,2,3,..\})$$

refer the second type of the Chebyshev polynomials. It is well-known that

$$\mathrm{U}_k(n) = 2n\mathrm{U}_{k-1}(n) - \mathrm{U}_{k-2}(n),$$

and

$$\mathrm{U}_1(n) = 2n, \quad \mathrm{U}_2(n) = 4n^2 - 1, \quad \mathrm{U}_3(n) = 8n^3 - 4n, \ldots. \tag{5}$$

The first type of the ordinary generating function for Chebyshev polynomials $\mathrm{T}_k(n)$, $n \in [-1,1]$, have the form

$$\sum_{k=0}^{\infty} \mathrm{T}_k(n) z^k = \frac{1 - nz}{1 - 2nz + z^2} \quad (z \in \mathcal{U}).$$

Relations between the first $\mathrm{T}_k(n)$ and second type $\mathrm{U}_k(n)$ of the Chebyshev polynomials are connected as following :

$$\frac{\mathrm{dT}_k(n)}{\mathrm{d}n} = k\mathrm{U}_{k-1}(n),$$
$$\mathrm{T}_k(n) = \mathrm{U}_k(n) - n\mathrm{U}_{k-1}(n),$$
$$2\mathrm{T}_k(n) = \mathrm{U}_k(n) - \mathrm{U}_{k-2}(n).$$

Let $\Omega$ be the class of functions of the form

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

satisfying $|w(z)| < 1$ for $z \in \mathcal{U}$.

To prove our results, we need the following lemma.

**Lemma 1.3.** [15] *If $w \in \Omega$, then for any complex number $\mu$,*

$$\left|c_2 - \mu c_1^2\right| \leq \max\{1; |\mu|\}.$$

*The result is sharp for the functions given by*

$$w(z) = z \quad or \quad w(z) = z^2.$$

Our aim of the present paper is to define a new subclass $\mathcal{H}_{\lambda,\gamma}(s,b,n)$ of non-Bazilević functions by applying the Chebyshev polynomial and to provide estimates for initial coefficients. In addition to that, the problem of Fekete-Szegö in this class is additionally explained.

## 2. Main Results

Throughout this paper, we assume that $s$ and $b$ are complex numbers with $s \neq b$, $|b| \leq 1$, $b \neq 1$,

$$\gamma \geq 1, \quad 0 \leq \lambda \leq 1, \quad \lambda(s+b) \neq 2\gamma$$

and $n \in (1/2, 1)$.

**Theorem 2.1.** *Let the function $f$ given by* (1) *belongs to the class $\mathcal{H}_{\lambda,\gamma}(s,b,n)$. Then*

$$|a_2| \leq \frac{2n}{|2\gamma - \lambda(s+b)|}$$

*and*

$$|a_3| \leq \frac{4n^2+2n-1}{|3\gamma - \lambda(s^2+sb+b^2)|} + \frac{2n^2\,|\lambda(s+b)\,[4\gamma-(\lambda+1)(s+b)] - 4\gamma(\gamma-1)|}{|2\gamma-\lambda(s+b)|^2\;|3\gamma-\lambda(s^2+sb+b^2)|}.$$

*Proof.* Let $f \in \mathcal{H}_{\lambda,\gamma}(s,b,n)$. Then by the principle of subordination there exists a Schwarz function $w(z)$ with $w(0)=0$, $|w(z)|<1$, and satisfying

$$(f'(z))^{\gamma}\left(\frac{(s-b)z}{f(sz)-f(bz)}\right)^{\lambda} = 1 + \mathrm{U}_1(n)w(z) + \mathrm{U}_2(n)w^2(z) + \cdots \quad . \tag{6}$$

Changing the values of $f'(z), f(sz)$ and $f(bz)$ with their equivalent series expressions of the series in (6), we get

$$\left(1+\sum_{k=2}^{\infty} k a_k z^{k-1}\right)^{\gamma}\left(\frac{(s-b)z}{[sz+\sum_{k=2}^{\infty} a_k (sz)^k] - [bz+\sum_{k=2}^{\infty} a_k (bz)^k]}\right)^{\lambda}$$
$$= \quad 1 + \mathrm{U}_1(n)w(z) + \mathrm{U}_2(n)w^2(z) + \cdots \tag{7}$$

by applying the binomial expansion on the left part of (7) subject to the following condition

$$\left|\sum_{k=2}^{\infty} k a_k z^k\right| < \gamma, \quad \left|\sum_{k=2}^{\infty} a_k (sz)^k\right| < \lambda \quad and \quad \left|\sum_{k=2}^{\infty} a_k (bz)^k\right| < \lambda.$$

Simplification, we have

$$\begin{aligned} &\left[1-\lambda(s+b)a_2z+\lambda\left(\frac{\lambda+1}{2}(s+b)^2a_2^2-(s^2+sb+b^2)a_3\right)z^2+\cdots\right] \\ &\times\left[1+2\gamma a_2z+\gamma\left(3a_3+2(\gamma-1)a_2^2\right)z^2+\cdots\right] \\ =\ & 1+\mathrm{U}_1(n)w(z)+\mathrm{U}_2(n)w^2(z)+\cdots \end{aligned}$$

or equivalently

$$\begin{aligned} &1+2\gamma a_2z-\lambda(s+b)a_2z+\gamma\left[3a_3+2(\gamma-1)a_2^2\right]z^2-2\gamma\lambda(s+b)a_2^2z^2 \\ &+\lambda\left[\frac{\lambda+1}{2}(s+b)^2a_2^2-(s^2+sb+b^2)a_3\right]z^2+\cdots \end{aligned}$$

$$= 1+\mathrm{U}_1(n)w(z)+\mathrm{U}_2(n)w^2(z)+\cdots . \tag{8}$$

Let the function $w$ be of the form

$$w(z)=c_1z+c_2z^2+c_3z^3+\cdots . \tag{9}$$

Then we get

$$|w(z)|=\left|c_1z+c_2z^2+c_3z^3+\cdots\right|<1 \quad (z\in\mathcal{U}) \tag{10}$$

where

$$|c_j|\leq 1 \quad (j\in\mathbb{N}). \tag{11}$$

From the equations (8) and (9), we have

$$\begin{aligned} &1+\left[2\gamma-\lambda(s+b)\right]a_2z \\ &+\Big\{\left[3\gamma-\lambda(s^2+sb+b^2)\right]a_3 \\ &-\left[2\lambda\gamma(s+b)-\frac{\lambda\left(\lambda+1\right)}{2}(s+b)^2-2\gamma(\gamma-1)\right]a_2^2\Big\}z^2+\cdots \end{aligned}$$

$$= 1+\mathrm{U}_1(n)c_1z+\left(\mathrm{U}_1(n)c_2+\mathrm{U}_2(n)c_1^2\right)z^2+\cdots \tag{12}$$

From (12), we get

$$\left[2\gamma-\lambda(s+b)\right]a_2=\mathrm{U}_1(n)c_1$$

or equivalently

$$a_2=\frac{\mathrm{U}_1(n)c_1}{2\gamma-\lambda(s+b)}. \tag{13}$$

From $(5), (11)$ and $(13)$, we get

$$|a_2|\leq\frac{2n}{|2\gamma-\lambda(s+b)|}. \tag{14}$$

In order to obtain the bound of $|a_3|$, from the equation (12), we have

$$\begin{aligned} &\left[3\gamma-\lambda(s^2+sb+b^2)\right]a_3 \\ =\ &\mathrm{U}_1(n)c_2+\mathrm{U}_2(n)c_1^2 \\ &+\left[2\lambda\gamma(s+b)-\frac{\lambda\left(\lambda+1\right)}{2}(s+b)^2-2\gamma(\gamma-1)\right]a_2^2 \end{aligned} \tag{15}$$

By applying the equation (13) in the equation (15), we get

$$\begin{aligned} & \left[3\gamma - \lambda(s^2 + sb + b^2)\right] a_3 \\ = & \ \mathrm{U}_1(n)c_2 \\ & + \left[\frac{\lambda(s+b)\left[4\gamma - (\lambda+1)(s+b)\right] - 4\gamma(\gamma-1)}{2\left[2\gamma - \lambda(s+b)\right]^2}\mathrm{U}_1^2(n) + \mathrm{U}_2(n)\right] c_1^2. \end{aligned} \tag{16}$$

Furthermore, by using the equations (5) and (11) in the equation (16), we obtain

$$|a_3| \le \frac{4n^2+2n-1}{|3\gamma - \lambda(s^2+sb+b^2)|} + \frac{2n^2\left|\lambda(s+b)\left[4\gamma-(\lambda+1)(s+b)\right] - 4\gamma(\gamma-1)\right|}{|2\gamma-\lambda(s+b)|^2 \, |3\gamma - \lambda(s^2+sb+b^2)|}. \tag{17}$$

The proof of theorem be complete. □

If we take $\gamma = 1$ and $\lambda = 0$ in the Theorem 2.1, we get the next result.

**Corollary 2.2.** *Let the function $f$ given by (1) belongs to the class $\mathcal{H}(n)$. Then*

$$|a_2| \le n$$

*and*

$$|a_3| \le \frac{4n^2+2n-1}{3}.$$

If we take $\gamma = 1$ and $\lambda = 1$ in the Theorem 2.1, we get the next result.

**Corollary 2.3.** *Let the function $f$ given by (1) belongs to the class $\mathcal{H}(s,b,n)$. Then*

$$|a_2| \le \frac{2n}{|s+b-2|}$$

*and*

$$|a_3| \le \frac{4n^2+2n-1}{|s^2+sb+b^2-3|} + \frac{4n^2\,|s+b|}{|s+b-2|\,|s^2+sb+b^2-3|}.$$

**Theorem 2.4.** *Let the function $f$ given by (1) belongs to the class $\mathcal{H}_{\lambda,\gamma}(s,b,n)$ and $\mu \in \mathbb{C}$. Then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{2n}{|3\gamma - \lambda(s^2+sb+b^2)|} \max\{1,\ |\tau|\},$$

*where*

$$\begin{aligned} \tau \ = \ & \frac{4n^2-1}{2n} + \frac{\lambda(s+b)\left[4\gamma-(\lambda+1)(s+b)\right] - 4\gamma(\gamma-1)}{\left[2\gamma-\lambda(s+b)\right]^2} n \\ & - \frac{2\left[3\gamma - \lambda(s^2+sb+b^2)\right] n}{\left[2\gamma-\lambda(s+b)\right]^2}\mu. \end{aligned}$$

*The inequalities are sharp.*

*Proof.* From the equations (13) and (15), we have

$$\left|a_3 - \mu a_2^2\right| = \frac{\mathrm{U}_1(n)}{\left|3\gamma - \lambda(s^2+sb+b^2)\right|}\left|c_2 + \upsilon c_1^2\right|$$

where

$$\begin{aligned}\upsilon &= \frac{\lambda(s+b)\left[4\gamma-(\lambda+1)(s+b)\right]-4\gamma(\gamma-1)-2\mu\left[3\gamma-\lambda(s^2+sb+b^2)\right]}{2\left[2\gamma-\lambda(s+b)\right]^2}\mathrm{U}_1(n)\\ &\quad+\frac{\mathrm{U}_2(n)}{\mathrm{U}_1(n)}\end{aligned}$$

Then, by Lemma 1.3 and equality (5) we deduce that

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2n}{\left|3\gamma - \lambda(s^2+sb+b^2)\right|}\max\left\{1,\ |\tau|\right\},$$

where

$$\begin{aligned}\tau &= \frac{4n^2-1}{2n}+\frac{\lambda(s+b)\left[4\gamma-(\lambda+1)(s+b)\right]-4\gamma(\gamma-1)}{\left[2\gamma-\lambda(s+b)\right]^2}n\\ &\quad-\frac{2\left[3\gamma-\lambda(s^2+sb+b^2)\right]n}{\left[2\gamma-\lambda(s+b)\right]^2}\mu.\end{aligned}$$

The proof of theorem be complete. □

If we set $\gamma = 1$ and $\lambda = 0$ in Theorem 2.4, we get the next result.

**Corollary 2.5.** *Let the function $f$ given by (1) belongs to the class $\mathcal{H}(n)$ and $\mu \in \mathbb{C}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2n}{3}\max\left\{1,\ \left|\frac{4n^2-1}{2n}-\frac{3n}{2}\mu\right|\right\}.$$

*The inequalities are sharp.*

If we set $\gamma = 1$ and $\lambda = 1$ in the Theorem 2.4, we get the next result.

**Corollary 2.6.** *Let the function $f$ given by (1) belongs to the class $\mathcal{H}(s,b,n)$ and $\mu \in \mathbb{C}$. Then*

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{2n}{\left|s^2+sb+b^2-3\right|}\\ &\quad\times\max\left\{1,\ \left|\frac{4n^2-1}{2n}+\frac{2(s+b)n}{\left[2-(s+b)\right]}-\frac{2\left[3-(s^2+sb+b^2)\right]n}{\left[2-(s+b)\right]^2}\mu\right|\right\}.\end{aligned}$$

*The inequalities are sharp.*

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