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SOME NOTES ON NEARLY COSYMPLECTIC MANIFOLDS

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Abstract. In this paper, we study some symmetric and recurrent conditions of nearly cosymplectic manifolds. We prove that Ricci-semisymmetric and Ricci-recurrent nearly cosymplectic manifolds are Einstein and conformal flat nearly cosymplectic manifold is locally isometric to Riemannian product $\mathbb{R} \times N$, where N is a nearly Kähler manifold.

1. Introduction

In modern mathematics, the geometry of manifolds have a growing interest and become an important tool for some subject of physical science and many other areas of mathematics. In the last century, various classes of the manifolds have been studied and developed by many researchers. One of the important special class of manifolds expressed based on the Kähler Manifolds is the notion of nearly Kähler manifolds which are also defined as one of the classes almost Hermitian manifolds such that a complex metric structure $(\bar{M}, J, \bar{g})$ satisfies the condition $(\nabla_E J)\, E = 0$, for every vector field on $\bar{M}$ (here ∇ denotes the Levi-Civita connection). In 2002, Nagy introduced notable classifications of nearly Kähler manifold such that one of the result of this study that $6-$dimensional nearly Kähler manifold play an important role in Rham decomposition. It is known that the odd dimensional counter parts of nearly Kähler manifolds are nearly cosymplectic manifolds. Nearly cosymplectic structure $(\psi, \xi, \eta, \bar{g})$ on almost contact manifold satisfying $(\nabla_E \psi)\, E = 0$ then it is called a nearly cosymplectic manifold. Blair [2], Blair and Showers [3] improved this notion and studied important properties, then Nicola [11], Endo [6] and others made important contributions to the literature on this subject ([1], [5], [15], [16], [17]).

In 1950, Walker [18] introduced the notion of recurrent manifolds. In geometry and topology, recurrent structures take an important place. In [4], the authors De and Guha introduced the idea of generalized recurrent manifold. Then Ruse showed that if the associated 1-form B becomes zero, then the manifold

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reduces to a recurrent manifold [9]. Furthermore, the notion of recurrent space studied by several authors such as Ricci-recurrent manifolds by Patterson [12], 2-recurrent manifolds by Lichnerowicz [8] , projective 2-recurrent manifolds by D. Ghosh [7] and others.

A Riemannian manifold $(\bar{M}, \bar{g})$ is called generalized recurrent [4] if its curvature tensor R satisfies the condition

$$(\nabla_W R)(E,F)G = \alpha(W)R(E,F)G + \beta(W)\left[\bar{g}(F,G)E - \bar{g}(E,G)F\right] \tag{1}$$

for all $E, F, G \in \Gamma(\bar{M})$. Where, α and β are non-vanishing 1-forms defined by $\alpha(W) = \bar{g}(W, \rho_1)$, $\beta(W) = \bar{g}(W, \rho_2)$ also ∇ denotes covariant differentiation with respect to the metric $\bar{g}$. Here ρ_1 and ρ_2 are vector fields associated with $1-$forms α and β respectively. A Riemannian manifold $(\bar{M}, \bar{g})$ is called a generalized Ricci-recurrent [4] if its Ricci tensor S of type $(0, 2)$ satisfies the condition

$$(\nabla_E S)(F,G) = \alpha(E)S(F,G) + (n-1)\beta(E)\bar{g}(F,G) \tag{2}$$

where α and β defined in (1).

The Weyl conformal curvature tensor is defined by

$$\begin{aligned} &C(E,F)G \\ &= R(E,F)G - \frac{1}{2n-1}[S(F,G)E - S(E,G)F + \bar{g}(F,G)QE - \bar{g}(E,G)QF] \\ &\quad + \frac{r}{2n(2n-1)}[\bar{g}(F,G)E - \bar{g}(E,G)F] \end{aligned} \tag{3}$$

for all $E, F, G \in \Gamma(\bar{M})$, where R is the curvature tensor, S is the Ricci tensor, and $r = tr(S)$ is scalar curvature [14]. If $C = 0$, then the manifold is called conformally flat manifold.

Motivated by the above studies, we show that Ricci-semisymmetric and Ricci-recurrent nearly cosymplectic manifolds are Einstein, besides conformally flat nearly cosymplectic manifold is locally isometric to the Riemannian product $\mathbb{R} \times N$, where N is a nearly Kähler manifold. Also, we obtain scalar curvature r of generalized recurrent nearly cosymplectic manifolds.

2. Preliminaries

Let $(\bar{M}, \psi, \xi, \eta, \bar{g})$ be an $(2n+1)-$ dimensional almost contact Riemannian manifold, an endomorphism ψ of tangent bundle of $\Gamma(\bar{M})$, a vector field ξ, called characteristic vector field, η dual form of ξ, and $\bar{g}$ is the Riemannian metric. Under the above condition almost contact structure $(\psi, \xi, \eta, \bar{g})$ satisfies following:

$$\psi\xi = 0, \quad \eta(\psi E) = 0, \quad \eta(\xi) = 1, \tag{4}$$

$$\psi^2 E = -E + \eta(E)\xi, \qquad \eta(E) = \bar{g}(E, \xi), \ , \tag{5}$$

$$\bar{g}(\psi E, \psi F) = \bar{g}(E, F) - \eta(E)\eta(F), \tag{6}$$

for any E, F tangent on $\bar{M}$. Almost contact manifold is called nearly cosymplectic manifold if the equality

$$(\nabla_E \psi)F + (\nabla_F \psi)E = 0, \ E, F \in \chi(\bar{M}) \tag{7}$$

holds [3]. In fact, from that condition equivalent to $(\nabla_E \psi)E = 0$. It is well known that the vector field ξ is Killing satisfies $\nabla_\xi \xi = 0$ and $\nabla_\xi \eta = 0$. The tensor field h of type $(1,1)$ defined by

$$\nabla_E \xi = hE \tag{8}$$

is skew symmetric and anti-commutes with ψ. It satisfies $h\xi = 0$, $\eta \circ h = 0$ and $\nabla_\xi \psi = \psi h = \frac{1}{3}\mathcal{L}_\xi \psi$. In a nearly cosymplectic manifolds some formulas given by [11], [6]:

$$\bar{g}((\nabla_E \psi)F, hG) = \eta(F)\bar{g}(h^2 E, \psi G) - \eta(E)\bar{g}(h^2 F, \psi G), \tag{9}$$

$$(\nabla_E h)F = \bar{g}(h^2 E, F)\xi - \eta(F)h^2 E, \tag{10}$$

$$tr(h^2) = constant, \tag{11}$$

$$R(F, G)\xi = \eta(F)h^2 G - \eta(G)h^2 F, \tag{12}$$

$$S(\xi, G) = -\eta(G)tr(h^2), \tag{13}$$

$$S(\psi F, G) = S(F, \psi G), \quad \psi Q = Q\psi, \tag{14}$$

$$S(\psi F, \psi G) = S(F, G) + \eta(F)\eta(G)tr(h^2), \tag{15}$$

$$\eta(R(E, F)V) = \eta(F)\bar{g}(h^2 E, V) - \eta(E)\bar{g}(h^2 F, V). \tag{16}$$

Definition 2.1. *An $n-$dimensional Riemann manifold $(\bar{M}, \bar{g})$ is said to be $\eta-$Einstein manifold if the Ricci tensor satisfies;*

$$S(E, F) = a\bar{g}(E, F) + b\eta(E)\eta(F) \tag{17}$$

for every $E, F \in \chi(\bar{M})$, where $a, b : \bar{M} \to R$ is a function. If $b = 0$ then manifold is Einstein manifold.

Proposition 2.2. *Let $(\bar{M}, \psi, \xi, \eta, \bar{g})$ be an $(2n+1)-$ dimensional nearly cosymplectic manifold. Then $h = 0$ if and only if $\bar{M}$ is locally isometric to the Riemann product $\mathbb{R} \times N$, where N is a nearly Kähler manifold* [11].

3. MAIN THEOREMS

Theorem 3.1. *Let $(\bar{M}, \psi, \xi, \eta, \bar{g})$ be an $(2n+1)-$ dimensional nearly cosymplectic manifold. If $\bar{M}$ is ricci-semisymmetric manifold, then $\bar{M}$ is an Einstein manifold.*

Proof. Let $\bar{M}$ be an $(2n+1)-$ dimensional nearly cosymplectic manifold and satisfy

$$R(E,F).S = 0. \tag{18}$$

Our assumption is equivalent to

$$(R.S)(E,F,U,V) = -S(R(E,F)U,V) - S(U,R(E,F)V) = 0 \tag{19}$$

for all E, F, U and V on $\bar{M}$. Putting $U = \xi$ in ((7)) we get

$$S(R(E,F)\xi, V) + S(\xi, R(E,F)V) = 0. \tag{20}$$

By the help of (12) and (13) we have

$$\eta(E)S(h^2F,V) - \eta(F)S(h^2E,V) - \eta(R(E,F)V)tr(h^2) = 0. \tag{21}$$

If equation of (16) is applied instead of $\eta(R(E,F)V)$, then we get

$$\eta(E)S(h^2F,V) - \eta(F)S(h^2E,V) - \left[\eta(F)g(h^2E,V) - \eta(E)g(h^2F,V)\right] tr(h^2) = 0.$$

If $E = \xi$ is taken and necessary abbreviations are made

$$S(h^2F,V) + \bar{g}(h^2F,V)tr(h^2) = 0.$$

Consequently we get

$$S(h^2F,V) = -\bar{g}(h^2F,V)tr(h^2). \tag{22}$$

□

Corollary 3.2. *Let $(\bar{M}, \psi, \xi, \eta, \bar{g})$ be an $(2n+1)-$ dimensional nearly cosymplectic manifold. If $\bar{M}$ is semisymmetric manifold, then $\bar{M}$ is an Einstein manifold.*

Proof. If $R(E,F).R = 0$, then $R(E,F).S = 0$. Then the result is reached due to the (3.1) theorem. □

Theorem 3.3. *Let $(\bar{M}, \psi, \xi, \eta, \bar{g})$ be a $(2n+1)-$ dimensional nearly cosymplectic manifold. If $\bar{M}$ is ricci-recurrent manifold, then $\bar{M}$ is an Einstein manifold.*

Proof. Let $\bar{M}$ be an $(2n+1)-$dimensional nearly cosymplectic manifold and define a function f on $\bar{M}$ which satisfies the condition $f^2 = \bar{g}(Q,Q)$ with $\bar{g}(QE,F) = S(E,F)$. Then

$$F(f^2) = F[\bar{g}(Q,Q)] = 2\bar{g}(\nabla_F Q, Q) \tag{23}$$

and $\bar{M}$ is ricci-recurrent from equation (2) we have

$$F(f^2) = 2f^2\alpha(Ff).$$

Also since $F(f^2) = 2f(Ff)$, we get

$$f(Ff) = f^2\alpha(F).$$

From here

$$Ff = f(\alpha F) \neq 0 \tag{24}$$

is easily seen. Then from equation (24) we get

$$E(F(f)) - F(E(f)) = \{E\alpha(F) - F\alpha(E)\}f.$$

Then we have

$$\left\{\nabla_E\nabla_F - \nabla_F\nabla_E - \nabla_{[E,F]}\right\} f = \{E\alpha(F) - F\alpha(E) - \alpha\,[E,F]\}f.$$

Because the left side of the equation is equal to zero and $f \neq 0$

$$d_\alpha(E,F) = 0. \tag{25}$$

This indicates that the α $(1 - form)$ is closed. From equation (2), we get

$$(\nabla_E\nabla_F S)(U,V) = \{E\alpha(F) - F\alpha(E)\}S(U,V).$$

Then using (25), we obtain

$$(R(E,F)S) = 2d_\alpha(E,F)S(U,V) = 0.$$

As a result, it is proved that a ricci-recurrent manifold is an Einstein manifold. □

Theorem 3.4. *Let $(\bar{M},\psi,\xi,\eta,\bar{g})$ be a $(2n+1)-$dimensional nearly cosymplectic manifold. If $\bar{M}$ is conformally flat, then $\bar{M}$ is locally isometric to the Riemann product $\mathbb{R}\times N$, where N is a nearly Kähler manifold.*

Proof. If $\bar{M}$ is conformally flat, the equation (3),

$$\begin{aligned} R(E,F)G &= \frac{1}{2n-1}[S(F,G)E - S(E,G)F \\ &\quad +\bar{g}(F,G)QE - \bar{g}(E,G)QF] \\ &\quad -\frac{r}{2n(2n-1)}[\bar{g}(F,G)E - \bar{g}(E,G)F]. \end{aligned} \tag{26}$$

Considering equations (4), (13) and taking $G=\xi$, we get

$$\begin{aligned} &\eta(E)h^2F - \eta(F)h^2E \\ =\ &\frac{1}{2n-1}\left[-tr(h^2)\eta(F)E + tr(h^2)\eta(E)F + \eta(F)QE - \eta(E)QF\right] \\ &-\frac{r}{2n(2n-1)}\left[\eta(F)E - \eta(E)F\right]. \end{aligned} \tag{27}$$

Again replacing $F=\xi$ in (27) and using (4), we obtain

$$(28)\qquad -h^2E=\frac{1}{2n-1}\left[tr(h^2)\eta(E)\xi+QE\right]-\frac{r}{2n(2n-1)}\left[E-\eta(E)\xi\right]$$

Taking inner product both sides equation (28) with F and using (6), we have

$$\begin{aligned}\frac{1}{2n-1}S(E,F) &= \frac{r}{2n(2n-1)}\bar{g}(\psi E,\psi F)-\bar{g}(h^2E,F)\\ (29)\qquad &\quad -\frac{1}{2n-1}tr(h^2)\eta(E)\eta(F).\end{aligned}$$

Let $\{e_1,e_2,...,e_{2n-1}\}$ be a local orthonormal basis of vector fields in M. Then by putting $E=F=e_i$ in (29) and summing up with respect to $i, 1\le i\le 2n-1$, we have

$$tr(h^2)=0.$$

It is clear from here $h=0$. So the manifold is the manifold is locally isometric to the Riemann product $\mathbb{R}\times N$, where N is a nearly Kähler manifold. The proof is complete. □

Corollary 3.5. *Under assumption Theorem* (3.4)*, if* $h=0$*, then the manifold is cosymplectic manifold.*

Theorem 3.6. *Let* $(\bar{M},\psi,\xi,\eta,\bar{g})$ *be an* $(2n+1)-$ *dimensional generalized recurrent nearly cosymplectic manifold then its scalar curvature* r *satisfies condition*

$$(30)\qquad \eta(\alpha)r+\eta(\beta)(n-1)(n-2)+2\eta(\alpha)tr(h^2)=0.$$

Proof. Let $\bar{M}$ be an $(2n+1)-$dimensional generalized recurrent nearly cosymplectic manifold. Using second Bianchi's identity in equation (1), we get

$$\begin{aligned}&\alpha(E)\bar{g}(R(F,G)W,U)+\beta(E)\left[\bar{g}(G,W)\bar{g}(F,U)-\bar{g}(F,W)\bar{g}(G,U)\right]\\ (31)\qquad &+\alpha(F)\bar{g}(R(G,E)W,U)+\beta(F)[\bar{g}(E,W)\bar{g}(G,U)-\bar{g}(G,W)\bar{g}(E,U)]\\ &+\alpha(G)\bar{g}(R(E,F)W,U)+\beta(G)\left[\bar{g}(F,W)\bar{g}(E,U)-\bar{g}(E,W)\bar{g}(F,U)\right]=0.\end{aligned}$$

Let $\{e_i\}$ is an local orthonormal basis of the tangent space at each point of the manifold. By virtue of (31), putting $F=U=e_i$ and taking summation over $i, 1\le i\le n$, we obtain

$$\begin{aligned}&\alpha(E)S(G,W)+\beta(E)\bar{g}(G,W)(n-1)+R(G,E,W,A)+\beta(G)\bar{g}(E,W)\\ (32)\qquad &-\beta(E)\bar{g}(G,W)-\alpha(G)S(E,W)+(1-n)\beta(G)\bar{g}(E,W)\\ =&\ 0.\end{aligned}$$

Again putting $G=W=e_i$ in (32), and taking summation over $i, 1\le i\le n$, we have

$$(33)\qquad r\alpha(E)+(n-1)(n-2)\beta(E)-2S(E,A)=0.$$

Finally, taking $E=\xi$ in (33), the result of (30) is obtained. □

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