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THE PROJECTIVE MODULE $P^{(2)}$ OVER THE AFFINE COORDINATE RING OF THE 2-SPHERE S^2

Sanghee Kim

Abstract. It is known that the rank 2 stably free syzygy module $P^{(2)}$ is not free. This algebraic fact was proved analytically, but this remarkable fact still lacks of a simple algebraic proof. The main purpose of this paper is to give a partially algebraic proof by making use of a theorem whose proof is quite topological, and the further properties of the module will be discussed.

1. Introduction

There is a famous result which states that the polynomial sections of the tangent bundle of the $(n-1)$-sphere is free as a module over the coordinate ring if and only if $n = 1, 2, 4$, or 8. All known proofs are topological or analytic, the $n = 3$ case (see [9, Proposition 17.7]) being a special case of the known Hairy Ball Theorem. There is a nice summary of this in [3, Example 19.17]. Using [2, Theorem 4.3.8], whose proof is quite topological and does most of heavy lifting, we prove the $n = 3$ case.

Unless otherwise stated, every *ring* R is a commutative ring with identity, and every *module* is a unitary R-module.

Each section is divided into two parts: the first half concerning modules over rings, and the second half concerning the projective module $P^{(2)}$ over the Affine coordinate ring of the 2-sphere S^2.

In section 2, we define the rank of the finitely generated projective module over a Noetherian ring. We deal with the Affine coordinate ring

$$R = \mathbb{R}[X, Y, Z]/\langle X^2 + Y^2 + Z^2 - 1\rangle$$

of the real 2-sphere $S^2 = \{(a, b, c) \in \mathbb{R}^3 \,|\, a^2 + b^2 + c^2 = 1\}$. We get a result saying that the rank 2 stably free syzygy module $P^{(2)}$ over R does not contain any two elements $\mathbf{f}$ and $\mathbf{g}$ of R^3 having the property that $\|\mathbf{f} \times \mathbf{g}\|$ is a unit in R (Thereorem 2.3). Using the result, we give a partially algebraic proof that the module $P^{(2)}$ over R is not free (Corollary 2.5).

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In section 3, we find two sets of the generators of $P^{(2)}$ (Lemma 3.15 and Theorem 3.16). We get a known result saying that the syzygy module is indecomposable. The proof can be done by direct computation.

In section 4, we deal with maximal submodules of $P^{(2)}$, and then the Affine coordinate ring

$$R(\mathbb{C}) = \mathbb{C}[X, Y, Z]/\langle X^2 + Y^2 + Z^2 - 1\rangle$$

of the complex 2-sphere $S^2(\mathbb{C}) = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \alpha^2+\beta^2+\gamma^2 = 1\}$. We find all maximal ideals of the ring $R(\mathbb{C})$ (Theorem 4.8) and get a result (Theorem 4.10).

2. Syzygy Modules

Let M be a finitely generated module over a ring R. Then M has a minimal generating set Ω, that is, M is generated by Ω but by no proper subset of Ω. Moreover, every minimal generating set for M has the same number of elements.This number is denoted by $\mu(M)$.

Theorem 2.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then every finitely generated projective module over R is free. More precisely, if P is a finitely generated projective module over R, then $P \cong R^{\mu(P)}$.*

Proof. See [8, Proposition 2.3.2], [6, Corolary 3.5] and [9, Theorem 10.4]. □

Let P be a finitely generated projective module over a Noetherian ring R, and let $\mathfrak{p} \in Spec(R)$. Then $P_\mathfrak{p}$ is a finitely generated projective module over a Noetherian ring $R_\mathfrak{p}$ with unique maximal ideal $\mathfrak{p}R_\mathfrak{p}$. By Theorem 2.1, $P_\mathfrak{p}$ is free over $R_\mathfrak{p}$, and

$$P_\mathfrak{p} \cong R_\mathfrak{p}^{\mu(P_\mathfrak{p})}.$$

We define $rk(P) : Spec(R) \to \mathbb{N}$ by $rk(P)(\mathfrak{p}) = \mu(P_\mathfrak{p})$. We write also $rk_\mathfrak{p}(P)$ instead of $rk(P)(\mathfrak{p})$. $rk(P)$ is called the *rank* (*map*).

Throughout the remainder of this section, R will denote the Affine coordinate ring

$$\mathbb{R}[X, Y, Z]/\langle X^2 + Y^2 + Z^2 - 1\rangle$$

of the real 2-sphere $S^2 = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1\}$.

Alternately, R may be thought of as the ring of polynomial functions defined on S^2. To see this, let $f \in R$. Then there exists a polynomial $F \in \mathbb{R}[X, Y, Z]$ such that $f = F + \langle X^2+Y^2+Z^2-1\rangle$. For $\mathbf{p} \in S^2$, we define $f(\mathbf{p}) := F(\mathbf{p}) \in \mathbb{R}$. Assume that there exists $G \in \mathbb{R}[X, Y, Z]$ such that $f = G+\langle X^2+Y^2+Z^2-1\rangle$. Then $F - G \in \langle X^2 + Y^2 + Z^2 - 1\rangle$, so $F(\mathbf{p}) - G(\mathbf{p}) = 0$, i.e., $F(\mathbf{p}) = G(\mathbf{p})$, for all $\mathbf{p} \in S^2$. Thus the value of $f(\mathbf{p})$ is well-defined.

It is known in [7, p.35] that the Affine coordinate ring R of the real 2-sphere S^2 is a unique factorization domain (or briefly UFD). In particular, it is known in [9, Proposition 17.7] that the ring R is an integral domain.

We will use lower case letters to denote images of elements from $\mathbb{R}[X, Y, Z]$ in R. For example, write

$$\begin{aligned} x = &\quad X + \langle X^2 + Y^2 + Z^2 - 1\rangle, \\ y = &\quad Y + \langle X^2 + Y^2 + Z^2 - 1\rangle, \\ z = &\quad Z + \langle X^2 + Y^2 + Z^2 - 1\rangle. \end{aligned} \tag{2.1}$$

Then $R = \mathbb{R}[x, y, z]$.

Define a map $(x\,y\,z) : R^3 \to R$ by

$$(x\,y\,z)(f, g, h) = xf + yg + zh.$$

Since $x^2 + y^2 + z^2 = 1$, the map $(x\,y\,z)$ is surjective. In fact, for any $f \in R$,

$$(xf, yf, zf) \in R^3$$

and

$$(x\,y\,z)(xf, xg, xh) = x(xf) + y(yf) + z(zf) = (x^2 + y^2 + z^2)f = f.$$

It can be easily seen that $(x\,y\,z)$ is an R-homomorphism. So, we can get an exact sequence

$$0 \longrightarrow Ker(x\,y\,z) \longrightarrow R^3 \xrightarrow{(x\,y\,z)} R \longrightarrow 0. \tag{2.2}$$

$Ker(x\,y\,z)$ is the solution space of the surjective R-homomorphism $(x\,y\,z)$ and it is usually denoted by $P^{(2)}$. This R-module is called a *syzygy module* (see [13, p. 17]).

Lemma 2.2. *Let* $\mathbf{f} = (f_1, f_2, f_3)$ *and* $\mathbf{g} = (g_1, g_2, g_3)$ *be in* R^3*, and let*

$$A = \begin{pmatrix} x & y & z \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{pmatrix}.$$

If $\mathbf{f}$ *and* $\mathbf{g}$ *are in* $P^{(2)}$*, then the following statements are true.*

1. $det(A) = \pm\|\mathbf{f} \times \mathbf{g}\|$.
2. $det(A)$ *is a unit in* R *if and only if* $\|\mathbf{f} \times \mathbf{g}\|$ *is a unit in* R.

Proof. (1)

$$AA^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{f}\cdot\mathbf{f} & \mathbf{f}\cdot\mathbf{g} \\ 0 & \mathbf{g}\cdot\mathbf{f} & \mathbf{g}\cdot\mathbf{g} \end{pmatrix}$$

so that

$$\begin{aligned} (det(A))^2 &= det(AA^t) \\ &= (\mathbf{f}\cdot\mathbf{f})(\mathbf{g}\cdot\mathbf{g}) - (\mathbf{f}\cdot\mathbf{g})^2 \\ &= \|\mathbf{f} \times \mathbf{g}\|^2. \end{aligned}$$

This means that $det(A) = \pm\|\mathbf{f} \times \mathbf{g}\|$.

(2) It is proved by (1). □

Theorem 2.3. *$P^{(2)}$ does not contain any two elements $\mathbf{f}$ and $\mathbf{g}$ of R^3 having the property that $\|\mathbf{f}\times\mathbf{g}\|$ is a unit in R.*

Proof. Suppose on the contrary that $P^{(2)}$ contains such two elements $\mathbf{f} = (f_1, f_2, f_3)$ and $\mathbf{g} = (g_1, g_2, g_3)$, where $f_1, f_2, f_3, g_1, g_2, g_3 \in R$. Consider the following matrix

$$A = \begin{pmatrix} x & y & z \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{pmatrix}.$$

By Lemma 2.2, $det(A)$ is a unit in R. There exists an element f in R such that $f\,det(A) = 1_R$. Consider the following matrix

$$\tilde{A} = \begin{pmatrix} x & y & z \\ ff_1 & ff_2 & ff_3 \\ g_1 & g_2 & g_3 \end{pmatrix}.$$

Then $\tilde{A} \in SL_3(R)$. This contradicts to [2, Theorem 4.3.8]. □

Example 2.4. Notice that $(-y, x, 0), (-z, 0, x) \in P^{(2)}$. Consider the following matrix

$$A = \begin{pmatrix} x & y & z \\ -y & x & 0 \\ -z & 0 & x \end{pmatrix}.$$

Then $det(A) = x$. Write $\mathbf{f} = (-y, x, 0)$ and $\mathbf{g} = (-z, 0, x)$. Then $\mathbf{f}\times\mathbf{g} = (x^2, xy, xz)$, so $\|\mathbf{f}\times\mathbf{g}\| = x$, which is not a unit in R.

$P^{(2)}$ contains properly a projective R-submodule of rank 2. In fact, the two elements $\mathbf{f}$, $\mathbf{g}$ of $P^{(2)}$ in Example 2.4 are linearly independent over R, so $R\mathbf{f}\oplus R\mathbf{g} \subseteq P^{(2)}$. Moreover, $(0, -z, y)$ does not belong to $R\mathbf{f}\oplus R\mathbf{g}$, but it does belong to $P^{(2)}$. Thus $R\mathbf{f}\oplus R\mathbf{g} \subsetneq P^{(2)}$. $\{(x, y, z), \mathbf{f}, \mathbf{g}\}$ can not generate R^3.

Since R is R-projective, the sequence (2.2) splits. That is, there is an R-homomorphism $s : R \to R^3$ such that $(x\,y\,z)\circ s = id_R$. Such an s is so called a *section* of $(x\,y\,z)$. In fact, if we define a map $s : R \to R^3$ by

$$s(f) = (fx, fy, fz),$$

where $f \in R$, then s satisfies $(x\,y\,z)\circ s = id_R$. Moreover, we can show that $P^{(2)} \oplus s(R) = R^3$ and $s(R) \cong R$. Hence $P^{(2)}$ is projective, rank 2, stably free over R. However, $P^{(2)}$ is not isomorphic to R^2. We state this again and prove this.

The topological proof can be seen in [7, p.34], and [5, Proposition 3.1.10]. The analytic proof using the Hairy Ball Theorem can be seen in [9, Proposition 17.7]. We provide a new proof of this. The proof is much easier than the topological proof and the analytic proof.

Corollary 2.5. *$P^{(2)}$ is not free over R.*

Algebraic Proof. We have already known that the exact sequence (2.2) splits, so that there exists an R-homomorphism $s : R \to R^3$ such that $(x\,y\,z)\circ s = id_R$. In fact, $s : R \to R^3$ is defined by $s(f) = (fx,\, fy,\, fz)$, where $f \in R$. In particular, $s(1) = (x,\, y,\, z)$.

Suppose that $P^{(2)}$ is R-free. Then $P^{(2)}$ has an R-free basis $\{\mathbf{f},\, \mathbf{g}\}$ over R, where $\mathbf{f},\, \mathbf{g} \in R^3$, so that $P^{(2)} = R\mathbf{f} \oplus R\mathbf{g}$. Thus,

$$\begin{aligned} R^3 &= s(R) \oplus P^{(2)} \\ &= Rs(1) \oplus R\mathbf{f} \oplus R\mathbf{g} \\ &= R(x,\, y,\, z) \oplus R\mathbf{f} \oplus R\mathbf{g}. \end{aligned}$$

There are elements a_{ij} $(1 \le i,\, j \le 3)$ such that

$$\begin{aligned} (1,\, 0,\, 0) &= a_{11}(x,\, y,\, z) + a_{12}\mathbf{f} + a_{13}\mathbf{g}, \\ (0,\, 1,\, 0) &= a_{21}(x,\, y,\, z) + a_{22}\mathbf{f} + a_{23}\mathbf{g}, \\ (0,\, 0,\, 1) &= a_{31}(x,\, y,\, z) + a_{32}\mathbf{f} + a_{33}\mathbf{g}. \end{aligned}$$

Now write $\mathbf{f} = (f_1,\, f_2,\, f_3)$ and $\mathbf{g} = (g_1,\, g_2,\, g_3)$, where $f_1,\, f_2,\, f_3, g_1,\, g_2,\, g_3 \in R$. Then we get

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x & y & z \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Taking their determinants on both sides of this matrix equation, we can see that the determinant of the second matrix is a unit in R. (This shows that the unimodular matrix $(x\,y\,z)$ is completed to a matrix whose determinant is a unit in R.) By Lemma 2.2, $\|\mathbf{f} \times \mathbf{g}\|$ is a unit in R. This contradicts to Theorem 2.3. □

Question How can we prove [2, Theorem 4.3.8] or Theorem 2.3 algebraiclally? If we prove either one of these, then the $n = 3$ case has a simple algebraic proof.

3. The Indecomposability of $P^{(2)}$

In this section we deal with the indecomposability of $P^{(2)}$, and then find the two sets of generators of $P^{(2)}$.

Let R be a ring. Assume that for two submodules M' and M'' of an R-module M, $M'_{\mathfrak{m}} \cong M''_{\mathfrak{m}}$ for all $\mathfrak{m} \in Max(R)$. Then we can not say that $M' \cong M''$. For example, see [8, Example 1.2.1].

An element a in R is called a *zero-divisor* if there is a nonzero element $b \in R$ such that $ab = 0$. Let's $Z(R)$ denote the set of all zero-divisors of R. Then notice that $0 \in Z(R)$. It is known that

$$Z(R) = \cup_{\substack{\mathfrak{p} \in Spec(R) \\ \mathfrak{p} \subseteq Z(R)}} \mathfrak{p}.$$

For example, let $R = \mathbb{Z}/\langle 6 \rangle$. Then

$$Z(R) = \{0, 2, 3, 4\} = \{0, 2, 4\} \cup \{0, 3\} = \langle 2 \rangle \cup \langle 3 \rangle.$$

Proposition 3.1. *Let R be a ring. Then the nilradical $\sqrt{0_R}$ of the ring R is contained in $Z(R)$.*

Proof. There are two ways to prove this.
(Method I) Use the definitions of $\sqrt{0_R}$ and $Z(R)$.
(Method II)

$$\sqrt{0_R} = \cap_{\mathfrak{p} \in Spec(R)} \mathfrak{p} \subseteq \cup_{\substack{\mathfrak{p} \in Spec(R) \\ \mathfrak{p} \subseteq Z(R)}} \mathfrak{p} = Z(R).$$

□

Corollary 3.2. *Let R be a ring. Then every nonzero zero-divisor of R is not nilpotent.*

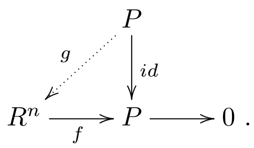
Let R be a ring, and S a multiplicatively closed subset of R. Define a map $\alpha : R \to R_S$ by $\alpha(r) = r/1$. Then α is an R-homomorphism. However, α is not injective, in general. For example, let's $R = \mathbb{Z}/\langle 6 \rangle$, as before. Define a map $\alpha : R \to R_{\langle 3 \rangle}$ by $\alpha(r) = r/1$. Then α is an R-homomorphism. However, it is not injective, because $3 \neq 0$ in R, but $3/1 = 0/1$ in $R_{\langle 3 \rangle}$ noting that $2 \in \mathbb{Z}_6 \backslash \langle 3 \rangle$ and $2 \cdot 3 = 0$ in R.

Lemma 3.3. *Let R be a ring, and let P be a finitely generated projective R-module. Let $S = R \backslash Z(R)$. Then the following two statements are true:*

1. *S is a saturated multiplicatively closed subset of R.*
2. *P_S can be given an R-module structure.*
3. *If we define $\alpha : P \to P_S$ by $\alpha(x) = x/1$, where $x \in P$, then α is an R-monomorphism.*

Proof. It is easy to prove that (1) and (2) are true.

(3) Say, $P = \langle x_1, \cdots, x_n \rangle$. Define a map $f : R^n \to P$ by $f(a_1, \cdots, a_n) = a_1x_1 + \cdots + a_nx_n$. Then f is an R-epimorphism. Consider the following diagram :

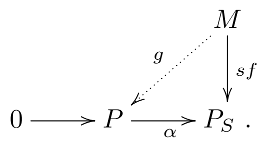


Since P is projective, there exists an R-homomorphism $g : P \to R^n$ such that $f \circ g = id$, so that g is an R-monomorphism. Define a map $\alpha : P \to P_S$ by $\alpha(x) = x/1$. Then α is an R-homomorphism. Assume that $x/1 = 0$ in P_S, where $x \in P$. Then there exists an element $s \in S$ such that $sx = 0$ in P. $sg(x) = g(sx) = g(0) = 0$. Let's write $g(x) = (b_1, \cdots, b_n)$. Then for all $i \in \{1, \cdots, n\}$, $sb_i = 0$. Since $s \in S$, we must have $a_i = 0$. $g(x) = (0, \cdots, 0)$ Since g is injective, we can get $x = 0$. This shows that α is injective. □

Corollary 3.4. *Let R be a ring, and let $S = R\backslash Z(R)$. Then the statements are true:*

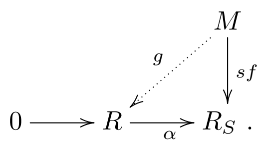
1. *The ring R_S can be given an R-module structure.*
2. *The mapping $\alpha : R \to R_S$ defined by $\alpha(r) = r/1$ is an R-monomorphism.*

Theorem 3.5. *Let M be a finitely generated module over a ring R, let P be a finitely generated projective R-module. Let $S = R\backslash Z(R)$. Then for every R-monomorphism $f : M \to P_S$, there exists an element $s \in S$ and an R-monomorphism $g : M \to P$ such that the following diagram is commutative :*



Proof. Let $x_1, \cdots, x_n$ be generators of M. Then
$$\langle f(x_1), \cdots, f(x_n)\rangle \subseteq P_S,$$
so that there exist elements $p_1, \cdots, p_n \in P$, and $s_1, \cdots, s_n \in S$ such that
$$f(x_1) = p_1/s_1, \cdots, f(x_n) = p_n/s_n.$$
Let $s = s_1 \cdots s_n$. Then $s \in S$, and there exist $q_1, \cdots, q_n \in P$ such that
$$f(x_1) = q_1/s, \cdots, f(x_n) = q_n/s.$$
Define a map $g : M \to P$ by $g(x_1) = q_1, \cdots, g(x_n) = q_n$. Then g is an R-homomorphism. Moreover, $\alpha \circ g(x_i) = g(x_i)/1 = q_i/1 = s(q_i/s) = sf(x_i)$ for all $i \in \{1, \cdots, n\}$, so that $\alpha \circ g = sf$. Assume now that $g(x) = 0$, where $x \in M$. Then $s(f(x)) = (sf)(x) = (\alpha \circ g)(x) = \alpha(0) = 0$. Write $f(x) = p/t$, where $p \in P$ and $t \in S$. Then $sp/t = s(f(x)) = 0$ in P_S. There exists an element $u \in S$ such that $u(sp) = 0$. $us \in S$ and $(us)p = 0$. Thus $p/1 = 0$ in P_S. By Lemma 3.3 (3), $p = 0$. Thus $f(x) = 0$. Since f is injective, $x = 0$. This shows that g is injective. □

Corollary 3.6. *Let M be a finitely generated module over a ring R, and let $S = R\backslash Z(R)$. Then for every R-monomorphism $f : M \to R_S$, there exists an element $s \in S$ and an R-monomorphism $g : M \to R$ such that the following diagram is commutative :*



We now turn our attention to a finitely generated projective module P over a Noetherian ring R. If P has constant rank n, then we show that P can be embedded in R^n. In particular, if P has constant rank 1, then it is known that

P is R-isomorphic to a projective ideal of R. We proceed to prove this directly using Corollary 3.6.

Lemma 3.7. *Let $\mathfrak{p}$ be a prime ideal of a ring R, let M be an R-module, and let n be a positive integer such that $M_\mathfrak{p} \cong (R_\mathfrak{p})^n (= R_\mathfrak{p} \oplus \cdots \oplus R_\mathfrak{p})$. Then there exists a submodule N of M generated by n elements such that $N_\mathfrak{p} = M_\mathfrak{p}$.*

Proof. $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module of rank n. Let $x_1/s_2, \cdots, x_n/s_n \in M_\mathfrak{p}$ be an $R_\mathfrak{p}$-free basis for $M_\mathfrak{p}$. Let $N = Rx_1 + \cdots + Rx_n$. Then $N_\mathfrak{p} = M_\mathfrak{p}$. □

Corollary 3.8. *Let R be a ring. Let M be an R-module and let n be a positive integer such that $M_\mathfrak{p} \cong (R_\mathfrak{p})^n$ for every $\mathfrak{p}$ of $Spec(R)$. Then M is generated by n elements over R.*

Proof. This can be shown if we use Lemma 3.7 and [10, Lemma 9.15]. □

Lemma 3.9. *Let R be a ring, and let P, Q be R-modules. If P is a projective R-module and $f : Q \to P$ is an R-epimorphism, then there exists an R-homomorphism $g : P \to Q$ such that $f \circ g = id_P$, so that g is an R-monomorphism.*

Lemma 3.10. *Let R be a ring, and let M be an R-module, and n be a positive integer such that $M_\mathfrak{p} \cong (R_\mathfrak{p})^n$ for every $\mathfrak{p}$ of $Spec(R)$. If M is projective over R, then it can be embedded in R^n.*

Proof. By Corollary 3.8, M is generated by n elements over R, say by $x_1, \cdots, x_n$. Define a map $f : R^n \to M$ by $f(r_1, \cdots, r_n) = r_1x_1 + \cdots + r_nx_n$. Then f is an R-epimorphism. Since M is projective, it follows from Lemma 3.9 that M can be embedded in R^n. □

Theorem 3.11. *Let R be a Noetherian ring. Let P be a finitely generated projective R-module of constant rank n. Then P can be embedded in R^n.*

Proof. Use Lemma 3.10 to prove this. □

Corollary 3.12 ([8], Lemma 3.2.1)**.** *Let R be a Noetherian ring, and let P be a finitely generated projective R-module of constant rank 1. Then the following two statements are true.*

1. *$P \cong I$ for some projective ideal (also called invertible ideal) I of R.*
2. *If I is a principle ideal of R in (1), then $P \cong R$.*

Proof. (1) Use Theorem 3.11 to prove (1).
(2) Since $rk\, P = 1$, it follows from (1) that

$$(0 :_R I)_\mathfrak{p} = 0 :_{R_\mathfrak{p}} I_\mathfrak{p} = 0 :_{R_\mathfrak{p}} P_\mathfrak{p} = 0 :_{R_\mathfrak{p}} R_\mathfrak{p} = 0$$

for all $\mathfrak{p} \in Spec(R)$. By the Local-Global property, $0 :_R I = 0$. Thus, since I is principal, $P \cong I \cong R/(0 :_R I) \cong R$. □

Corollary 3.12 (1) can be proved alternatively by making use of Corollary 3.6 as follows.

Alternative proof of Corollary 3.12 (1). Let $S = R\backslash Z(R)$. Define a map $\alpha : P \to P_S$ by $\alpha(x) = x/1$. Then by Lemma 3.3(3) α is an R-monomorphism. Moreover, note that $P_S \cong P \otimes_R R_S$. Then for every $\mathfrak{p} \in Spec(R)$,

$$(P_S)_\mathfrak{p} \cong (P \otimes_R R_S)_\mathfrak{p} \cong P_\mathfrak{p} \otimes_{R_\mathfrak{p}} (R_S)_\mathfrak{p} \cong R_\mathfrak{p} \otimes_{R_\mathfrak{p}} (R_S)_\mathfrak{p} \cong (R_S)_\mathfrak{p},$$

because P is of rank one. Since P_S is projective over R_S, it follows from Lemma 3.10 that there exists an R_S-monomorphism $\beta : P_S \to R_S$.

Consider the composite map

$$P \xrightarrow{\alpha} P_S \xrightarrow{\beta} R_S .$$

Let $f = \beta \circ \alpha$. Then f is an R-monomorphism. By Corollary 3.6, there exists an element $s \in S$ and an R-monomorphism $g : P \to R$ such that the following diagram is commutative :

$$\begin{array}{ccccc} & & & & P \\ & & & \overset{g}{\swarrow} & \downarrow {\scriptstyle sf} \\ 0 & \longrightarrow & R & \xrightarrow[\alpha]{} & R_S \end{array}$$

where α is the natural R-monomorphism. Let $I = g(P)$. Then I is an ideal of R and $P \cong I$. □

Let R be a ring, I an ideal of R, and $S = R\backslash Z(R)$. Then S is a multiplicatively closed subset of R. Let $(R :_{R_S} I)$ denote the set $\{x \in R_S \mid Ix \subseteq R\}$. If $I(R :_{R_S} I) = R$, then I is called an *invertible ideal* of R. Assume, in particular, that R is an integral domain. Then $Z(R) = \{0\}$. Let $Frac(R)$ denote the field of fractions of R, as usual. Then $Frac(R) = R_{R\backslash\{0\}}$. Let I be an ideal of R. Then $I(R :_{Frac(R)} I) = R$ if and only if I is an invertible ideal.

Let $R = K[x_1, x_2, \cdots]$ be a polynomial ring with infinitely many indeterminates $x_1, x_2, \cdots$ over a field K. Then R is a unique factorization domain because if f is in R, then there exists a positive integer n such that $f \in K[x_1, x_2, \cdots, x_n]$, which is a unique factorization domain. However R is not Noetherian, because the ideal $\langle x_1, x_2, \cdots \rangle$ is not finitely generated.

Corollary 3.13 ([7], Theorem 1.3, p.72)**.** *Let R be a Noetherian, unique factorization domain and let P be a finitely generated projective R-module. If P has constant rank one, then $P \cong R$.*

Proof. By Corollary 3.12 (1), $P \cong I$ for some ideal of R. I is a projective ideal of R. Adopt the proof of [4, Theorem 6.8] to get $I(R :_{R_S} I) = R$. From this, we can show that I is principal. And then use Corollary 3.12 (2) to show that $P \cong R$. □

It is known (see [7, p.35]) that the Affine coordinate ring R of the real 2-sphere S^2 is a UFD. Using these facts, we can get Corollary 3.14 below, which is known, for example, in [7, p.34].

Corollary 3.14. *$P^{(2)}$ is indecomposable.*

Proof. Use Corollary 3.13 and Corollary 2.5 to show this. □

Let R be the Affine coordinate ring of the real 2-sphere. Then R is a Noetherian ring and R^3 is a Noetherian R-module. Thus $P^{(2)}$ is finitely generated over R. We are concerning the generators of $P^{(2)}$ to find two sets of its generators.

We have already known that the exact sequence (2.2) splits, and have shown that $R^3 = P^{(2)} \oplus R(x, y, z)$. There exist $\mathbf{u}, \mathbf{v}, \mathbf{w} \in P^{(2)}$ and $a, b, c \in R$ such that

$$\begin{aligned}(1, 0, 0) &= \mathbf{u} + a(x, y, z),\\ (0, 1, 0) &= \mathbf{v} + b(x, y, z),\\ (0, 0, 1) &= \mathbf{w} + c(x, y, z).\end{aligned}$$

Sending the elements on both sides of the equations, we can get $a = x$, $b = y$, $c = z$. Thus

$$\begin{aligned}(1, 0, 0) &= \mathbf{u} + x(x, y, z),\\ (0, 1, 0) &= \mathbf{v} + y(x, y, z),\\ (0, 0, 1) &= \mathbf{w} + z(x, y, z).\end{aligned}$$

Lemma 3.15. *Let R be an Affine coordinate ring of the real 2-sphere S^2. $P^{(2)}$ is generated by the following three elements*

$$\begin{aligned}\mathbf{u} &= (1, 0, 0) - x(x, y, z),\\ \mathbf{v} &= (0, 1, 0) - y(x, y, z),\\ \mathbf{w} &= (0, 0, 1) - z(x, y, z).\end{aligned}$$

Proof. It is easy to show that $\langle \mathbf{u}, \mathbf{v}, \mathbf{w}\rangle \subseteq P^{(2)}$. Conversely, let $\mathbf{f}$ be any element of $P^{(2)}$. Write $\mathbf{f} = (f_1, f_2, f_3)$, where $f_1, f_2, f_3 \in R$. Then

$$\begin{aligned}\mathbf{f} &= (f_1, f_2, f_3)\\ &= f_1(1, 0, 0) + f_2(0, 1, 0) + f_3(0, 0, 1)\\ &= f_1(\mathbf{u} + x(x, y, z)) + f_2(\mathbf{v} + y(x, y, z)) + f_3(\mathbf{w} + z(x, y, z))\\ &= f_1\mathbf{u} + f_2\mathbf{v} + f_3\mathbf{w} + (f_1x + f_2y + f_3z)(x, y, z)\\ &= f_1\mathbf{u} + f_2\mathbf{v} + f_3\mathbf{w},\end{aligned}$$

which belongs to $\langle \mathbf{u}, \mathbf{v}, \mathbf{w}\rangle$. Thus $P^{(2)} \subseteq \langle \mathbf{u}, \mathbf{v}, \mathbf{w}\rangle$. This shows that

$$P^{(2)} = \langle \mathbf{u}, \mathbf{v}, \mathbf{w}\rangle.$$

□

Since $P^{(2)}$ is indecomposable and it is generated by the three elements $\mathbf{u}$, $\mathbf{v}, \mathbf{w}$, the sum $R\mathbf{u} + R\mathbf{v} + R\mathbf{w}$ is not direct, but the sum of any two of $R\mathbf{u}$, $R\mathbf{v}$, $R\mathbf{w}$ is direct. For example, the sum $R\mathbf{u} + R\mathbf{v}$ is direct.

Theorem 3.16. *Let R be an Affine coordinate ring of the real 2-sphere S^2. $P^{(2)}$ is also generated by the following three elements*

$$\mathbf{f} = (-y,\, x,\, 0),$$
$$\mathbf{g} = (-z,\, 0,\, x),$$
$$\mathbf{h} = (0,\, -z,\, y).$$

Proof. With the same notations as in Lemma 3.15, we have the following

$$\mathbf{u} = -y\mathbf{f} - z\mathbf{g},$$
$$\mathbf{v} = x\mathbf{f} - z\mathbf{h},$$
$$\mathbf{w} = y\mathbf{h} + x\mathbf{g}.$$

Thus by Lemma 3.15, $P^{(2)} = \langle \mathbf{f},\, \mathbf{g},\, \mathbf{h} \rangle$. □

4. Maximal submodules of $P^{(2)}$

In this section we deal with maximal submodules of $P^{(2)}$.

Lemma 4.1. *A nonzero projective module has a maximal submodule.*

Proof. See [1, Proposition 17.14]. □

Corollary 4.2. *$P^{(2)}$ has a maximal submodule.*

Of course, Corollary 4.2 can be proved alternatively as follows: By Lemma 3.15, or by Theorem 3.16, $P^{(2)}$ is finitely generated. We now can use the Zorn lemma to show that $P^{(2)}$ has a maximal submodule.

Theorem 4.3 ([12], Theorem, p.169)**.** *Let R be a ring. If P is a projective R-module with unique maximal submodule, then P is indecomposable.*

Proof. It is known in the paper [12, Propostion 2]: Let R be a ring and M a right R-module with unique maximal submodule. Then either one of the following is true.

1. M is indecomposable.
2. There exist submodules M_1, M_2 of M such that $M = M_1 \oplus M_2$, M_1 has unique maximal submodule, and M_2 does not have maximal submodule.

Let P be a projective R-module with unique maximal submodule. Suppose that P is not indecomposable. Then there exist submodules P_1, P_2 of P such that $P = P_1 \oplus P_2$, P_1 has unique maximal submodule, P_2 does not have maximal submodule. Suppose that P_2 is nonzero. Then by Lemma 4.1 P_2 has a maximal submodule. This contradiction shows that P is indecomposable. □

We can not use this result to prove that $P^{(2)}$ is indecomposable, because we do not know whether $P^{(2)}$ has unique maximal submodule.

If two positive integers m, n are relatively prime, then the residue class ring $\mathbb{Z}/mn\mathbb{Z}$ of the ring $\mathbb{Z}$ of integers is decomposable because

$$\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}.$$

However the formal power series ring $F[[x]]$, where F is a field, has unique maximal ideal $\langle x \rangle$, so it is indecomposable by Theorem 4.3. A ring is called a *local ring* if it is a Noetherian ring with unique maximal ideal. For example, the formal power series ring $F[[x]]$, where F is a field, is a local ring. The ring $\mathbb{Z}$ of integers is a Noetherian ring, but it is not local, because it has infinitely many maximal ideals $\langle 2 \rangle$, $\langle 3 \rangle$, $\langle 5 \rangle$, $\cdots$.

Lemma 4.4. *Let a, b, $c \in \mathbb{R}$. With the same notations as in (2.1) we have that following*

$$\mathbb{R}[x, y, z]/\langle x-a, y-b, z-c \rangle$$
$$\cong \mathbb{R}[X, Y, Z]/(\langle X-a, Y-b, Z-c \rangle + \langle X^2+Y^2+Z^2-1 \rangle).$$

Proof. Use the third isomorphism theorem for rings to prove this. □

Lemma 4.5. *Let $(a, b, c) \in S^2$. Then $\langle X^2+Y^2+Z^2-1 \rangle \subseteq \langle X-a, Y-b, Z-c \rangle$ in the ring $\mathbb{R}[X, Y, Z]$ of polynomials with coefficients in $\mathbb{R}$ in indeterminates X, Y, Z.*

Lemma 4.6. [10, Exercise 3.15] *Let F be a field and let $a_1, \cdots, a_n \in F$. Then the ideal*

$$\langle X_1 - a_1, \cdots, X_n - a_n \rangle$$

of the ring $F[X_1, \cdots, X_n]$ (of polynomials with coefficients in F in indeterminates $X_1, \cdots, X_n$) is maximal.

Theorem 4.7. *If R is the Affine coordinate ring of the real 2-sphere S^2, then for every $(a, b, c) \in S^2$, $\langle x-a, y-b, z-c \rangle \in Max(R)$.*

Proof. We can use Lemma 4.4 - Lemma 4.6 to prove this result. □

• Any maximal ideal in the polynomial ring $K[X_1, \cdots, X_n]$ over a field K is generated by n elements (see [6, Exercise 3.1] and [8, Exercise 6.1.2]).

• (Weak Nullstellensatz) If K is an algebraically closed field, then an ideal M is maximal in $K[X_1, \cdots, X_n]$ if and only if there exit $a_1, \cdots, a_n \in K$ such that $M = \langle X_1 - a_1, \cdots, X_n - a_n \rangle$ (see [6, Corollary 3.3.6] and [10, Theorem 14.6]).

• The complex number field $\mathbb{C}$ is algebraically closed, so an ideal M is maximal in $\mathbb{C}[X, Y, Z]$ if and only if there exit α, β, $\gamma \in \mathbb{C}$ such that $M = \langle X-\alpha, Y-\beta, Z-\gamma \rangle$.

Let's denote x, y, z like in (2.1). Then

$$\mathbb{C}[x, y, z] = \mathbb{C}[X, Y, Z]/\langle X^2+Y^2+Z^2-1 \rangle.$$

Let $Max(R)$ denote the set of all maximal ideals of a ring R.

Theorem 4.8. *Let* $S^2(\mathbb{C}) = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \alpha^2 + \beta^2 + \gamma^2 = 1\}$. *Then*

$$Max(\mathbb{C}[x, y, z]) = \{\langle x - \alpha, y - \beta, z - \gamma\rangle \mid (\alpha, \beta, \gamma) \in S^2(\mathbb{C})\}.$$

Proof. We can adopt the proof of Lemma 4.4 - Lemma 4.6 to show

$$\{\langle x - \alpha, y - \beta, z - \gamma\rangle \mid (\alpha, \beta, \gamma) \in S^2(\mathbb{C})\} \subseteq Max(\mathbb{C}[x, y, z]).$$

Conversely, let $\mathfrak{m} \in Max(\mathbb{C}[x, y, z])$. Then there exists an ideal $\mathfrak{M}$ in $\mathbb{C}[X, Y, Z]$ with $\langle X^2+Y^2+Z^2-1\rangle \subseteq \mathfrak{M}$ such that $\mathfrak{M}/\langle X^2+Y^2+Z^2-1\rangle = \mathfrak{m}$. Moreover, by the third isomorphism theorem for rings, $\mathfrak{M}$ is a maximal ideal of $\mathbb{C}]X, Y, Z]$. By the Weak Nullstellensatz, there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\mathfrak{M} = \langle X-\alpha, Y-\beta, Z-\gamma\rangle$. Since $X^2+Y^2+Z^2-1 \in \mathfrak{M} = \langle X-\alpha, Y-\beta, Z-\gamma\rangle$, we can see that $\alpha^2 + \beta^2 + \gamma^2 - 1 = 0$, so that $(\alpha, \beta, \gamma) \in \mathbb{S}^2(\mathbb{C})$ and $\mathfrak{m} = \langle x - \alpha, y - \beta, z - \gamma\rangle$. □

Let R be a ring, and let M be an R-module. $Soc(M)$ is defined to be the sum of all simple R-submodules of M.

Lemma 4.9. *Let* $S^2(\mathbb{C}) = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \alpha^2 + \beta^2 + \gamma^2 = 1\}$, *and let*

$$R(\mathbb{C}) = \mathbb{C}[X, Y, Z]/\langle X^2 + Y^2 + Z^2 - 1\rangle,$$

which is called the Affine coordinate ring of the complex 2-sphere $S^2(\mathbb{C})$. *If* M *is a simple* $R(\mathbb{C})$*-module, then as* $R(\mathbb{C})$*-modules,*

$$Soc(M) \cong R(\mathbb{C})/\langle x - \alpha, y - \beta, z - \gamma\rangle$$

for some $(\alpha, \beta, \gamma) \in S^2(\mathbb{C})$.

Theorem 4.10. *Let* $R(\mathbb{C})$ *be the Affine coordinate ring of the complex 2-sphere* $S^2(\mathbb{C})$. *Let* L *be a maximal* $R(\mathbb{C})$*-submodule of* $P^{(2)}$. *Then the following are true.*

1. $P^{(2)}/L$ *is* $R(\mathbb{C})$*-isomorphic to* $R/\langle x-\alpha, y-\beta, z-\gamma\rangle$ *for some* $(\alpha, \beta, \gamma) \in S^2(\mathbb{C})$.
2. *The injective envelope* $E(P^{(2)}/L)$ *of the* $R(\mathbb{C})$*-module* $P^{(2)}/L$ *is an indecomposable injective* $R(\mathbb{C})$*-module.*

Proof. See [11, Theorem 2.32 Corollary]. □

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Sanghee Kim
Department of Mathematics, Chosun University,
Kwangju 61452, South Korea.
E-mail : shkim8599@chosun.ac.kr