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ON IMPROVEMENTS OF SOME INTEGRAL INEQUALITIES

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Abstract. In this paper, improved power-mean integral inequality, which provides a better approach than power-mean integral inequality, is proved. Using Hölder-İşcan integral inequality and improved power-mean integral inequality, some inequalities of Hadamard's type for functions whose derivatives in absolute value at certain power are quasi-convex are given. In addition, the results obtained are compared with the previous ones. Then, it is shown that the results obtained together with identity are better than those previously obtained.

1. Introduction

The famous Young inequality for two scalars is the t-weighted arithmetic-geometric mean inequality. This inequality says that if $x, y > 0$ and $t \in [0,1]$, then

$$x^t y^{1-t} \leq tx + (1-t)y \tag{1}$$

with equality if and only if $x = y$.

Definition 1.1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$f\left(tx + (1-t)y\right) \leq tf\left(x\right) + (1-t)f\left(y\right)$$

valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \varnothing$.

Theorem 1.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. See [2, 3, 4, 6], *for the results of the*

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*Corresponding

generalization, improvement and extention of the famous integral inequality (2).

Definition 1.3. *A function $f : [a, b] \to \mathbb{R}$ is said quasi-convex on interval $[a, b]$ if*

$$f\left(tx + (1-t)y\right) \le \sup\left\{f(x), f(y)\right\}$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

Theorem 1.4 (Hölder Inequality for Integrals [7])**.** *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a, b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a, b]$, $q \ge 1$ then*

$$\int_a^b |f(x)g(x)|\, dx \le \left(\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\, dx\right)^{\frac{1}{q}}, \tag{3}$$

with equality holding if and only if $A\,|f(x)|^p = B\,|g(x)|^q$ almost everywhere, where A and B are constants.

If we get $|f|\,|g| = \left(|f|^{1/p}\right)\left(|f|^{1/q}\,|g|\right)$ in the Hölder inequality, then we obtain the following power-mean integral inequality as a simple result of the Hölder inequality:

Theorem 1.5 (Power-mean Integral Inequality)**.** *Let $q \ge 1$. If f and g are real functions defined on $[a, b]$ and if $|f|$, $|f|\,|g|^q$ are integrable functions on $[a, b]$ then*

$$\int_a^b |f(x)g(x)|\, dx \le \left(\int_a^b |f(x)|\, dx\right)^{1-\frac{1}{q}} \left(\int_a^b |f(x)|\,|g(x)|^q\, dx\right)^{\frac{1}{q}}. \tag{4}$$

In [1], Alomari et al used the following lemma to obtain main results for quasi-convex functions.

Lemma 1.6. *Let $I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \\ &= \frac{b-a}{4}\left[\begin{array}{c} \int_0^1 (-t) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt \\ +\int_0^1 t f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) dt \end{array}\right]. \end{aligned} \tag{5}$$

In [1], Alomari et al., obtained the following result for quasi-convex functions using Lemma 1.6 and Hölder integral inequality.

Theorem 1.7. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is an quasi-convex on*

$[a,b]$, for $q>1$, then the following inequality holds:

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left[\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right], \end{aligned} \tag{6}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In addition, in [1], Alomari et al., obtained the following result for quasi-convex functions using Lemma 1.6 and power-mean integral inequality.

Theorem 1.8 ([1])**.** *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is an quasi-convex on $[a,b]$, for $q\geq 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & \frac{b-a}{8}\left[\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{7}$$

Recently, in [5], İşcan gave a refinement of the Hölder integral inequality as following:

Theorem 1.9 (Hölder-İşcan Integral Inequality [5])**.** *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then*

$$\begin{aligned} &i.)\ \int_a^b |f(x)g(x)|\,dx \\ \leq\ & \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\} \end{aligned} \tag{8}$$

$$\text{ii.)}\quad \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\left|f(x)\right|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\left|g(x)\right|^q dx\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_a^b (x-a)\left|f(x)\right|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\left|g(x)\right|^q dx\right)^{\frac{1}{q}}\right\}$$
$$\le \left(\int_a^b \left|f(x)\right|^p dx\right)^{\frac{1}{p}}\left(\int_a^b \left|g(x)\right|^q dx\right)^{\frac{1}{q}}. \tag{9}$$

One of the biggest goals of the researchers working on inequalities is finding the best approach. In this paper, firstly improved power-mean integral inequality, which provides a better approach than power-mean integral inequality, has been proved. Then, we obtained a refinement of the inequality (6) using Hölder-İşcan integral inequality. Later, we get some new Hermite- Hadamard type inequalities using Lemma 1.6, power-mean integral inequality and improved power-mean integral inequality.

2. Main results

An refinement of power-mean integral inequality as a result of the Hölder-İşcan integral inequality can be given as follows:

Theorem 2.1 (Improved power-mean integral inequality)**.** *Let $q \geq 1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|$, $|f|\,|g|^q$ are integrable functions on $[a,b]$ then*

$$\text{i.)}\int_a^b \left|f(x)g(x)\right| dx$$
$$\le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\left|f(x)\right| dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)\left|f(x)\right|\left|g(x)\right|^q dx\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_a^b (x-a)\left|f(x)\right| dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)\left|f(x)\right|\left|g(x)\right|^q dx\right)^{\frac{1}{q}}\right\}. \tag{10}$$

$$ii.)\frac{1}{b-a}\left\{\begin{array}{c}\left(\int_a^b (b-x)\left|f(x)\right|dx\right)^{1-\frac{1}{q}}\\ \left(\int_a^b (b-x)\left|f(x)\right|\left|g(x)\right|^q dx\right)^{\frac{1}{q}}\end{array}\right.$$

$$+\left(\int_a^b (x-a)\left|f(x)\right|dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)\left|f(x)\right|\left|g(x)\right|^q dx\right)^{\frac{1}{q}}\Bigg\}$$

$$\leq \left(\int_a^b \left|f(x)\right|dx\right)^{1-\frac{1}{q}}\left(\int_a^b \left|f(x)\right|\left|g(x)\right|^q dx\right)^{\frac{1}{q}}. \tag{11}$$

Proof. i.) Firstly, let $q>1$. By using of Hölder inequality in (3), it is easily seen that

$$\begin{aligned}&\int_a^b \left|f(x)g(x)\right|dx\\ &= \frac{1}{b-a}\left\{\int_a^b \left|(b-x)^{1/p}f^{1/p}(x)(b-x)^{1/q}f^{1/q}(x)g(x)\right|dx\right.\\ &\left.+\int_a^b \left|(x-a)^{1/p}f^{1/p}(x)(x-a)^{1/q}f^{1/q}(x)g(x)\right|dx\right\}\end{aligned}$$

$$\begin{aligned}\leq \frac{1}{b-a}&\left\{\left(\int_a^b (b-x)\left|f(x)\right|dx\right)^{1-1/q}\left(\int_a^b (b-x)\left|f(x)\right|\left|g(x)\right|^q dx\right)^{1/q}\right.\\ &\left.+\left(\int_a^b (x-a)\left|f(x)\right|dx\right)^{1-1/q}\left(\int_a^b (x-a)\left|f(x)\right|\left|g(x)\right|^q dx\right)^{1/q}\right\}.\end{aligned}$$

For $q=1$, it is seen that the inequality (10) holds.

ii.) First let us consider the case

$$\left(\int_a^b \left|f(x)\right|dx\right)^{1-\frac{1}{q}}\left(\int_a^b \left|f(x)\right|\left|g(x)\right|^q dx\right)^{\frac{1}{q}}=0.$$

Then, $f(x)=0$ for almost everywhere $x\in[a,b]$ or $g(x)=0$ for almost everywhere $x\in[a,b]$. Thus, we have

$$\int_a^b \left|f(x)g(x)\right|dx=0.$$

Therefore, the inequality (11) is trivial in this case. Finally, we consider the case

$$S=\left(\int_a^b \left|f(x)\right|dx\right)^{1-\frac{1}{q}}\left(\int_a^b \left|f(x)\right|\left|g(x)\right|^q dx\right)^{\frac{1}{q}}\neq 0.$$

Then

$$\begin{aligned}
&\frac{1}{(b-a)S}\left\{\left(\int_a^b (b-x)|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)|f(x)||g(x)|^q\,dx\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_a^b (x-a)|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)|f(x)||g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}\\
\leq\ &\frac{1}{b-a}\left\{\left(\frac{\int_a^b (b-x)|f(x)|\,dx}{\int_a^b |f(x)|\,dx}\right)^{1-\frac{1}{q}}\left(\frac{\int_a^b (b-x)|g(x)|^q\,dx}{\int_a^b |f(x)||g(x)|^q\,dx}\right)^{1/q}\right.\\
&\left.+\left(\frac{\int_a^b (x-a)|f(x)|\,dx}{\int_a^b |f(x)|\,dx}\right)^{1-\frac{1}{q}}\left(\frac{\int_a^b (x-a)|f(x)||g(x)|^q\,dx}{\int_a^b |f(x)||g(x)|^q\,dx}\right)^{1/q}\right\}.
\end{aligned}$$

Let $q>1$. Applying (1) on the right hand sides integrals of the last inequality

$$\begin{aligned}
&\frac{1}{(b-a)S}\left\{\left(\int_a^b (b-x)|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)|f(x)||g(x)|^q\,dx\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_a^b (x-a)|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)|f(x)||g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}\\
\leq\ &\frac{1}{b-a}\left\{\frac{(q-1)\int_a^b (b-x)|f(x)|\,dx}{q\int_a^b |f(x)|\,dx}+\frac{\int_a^b (b-x)|g(x)|^q\,dx}{q\int_a^b |f(x)||g(x)|^q\,dx}\right.\\
&\left.+\frac{(q-1)\int_a^b (x-a)|f(x)|\,dx}{q\int_a^b |f(x)|\,dx}+\frac{\int_a^b (x-a)|g(x)|^q\,dx}{q\int_a^b |f(x)||g(x)|^q\,dx}\right\}\\
=\ &1-\frac{1}{q}+\frac{1}{q}=1.
\end{aligned}$$

ii.) It can be seen easily that the inequality holds for $q=1$. Thus, the proof is completed. □

Remark 2.2. *The inequality (11) show that the inequality (10) is better than the inequality (4).*

Remark 2.3. *If we get* $|f|\,|g|=\left(|f|^{1/p}\right)\left(|f|^{1/q}|g|\right)$ *in the Hölder-İşcan inequality, then we can easily obtain the improved power-mean integral inequality as a simple result of the Hölder-İşcan inequality.*

Theorem 2.4. *Let* $f:I^{\circ}\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* I° *such that* $f'\in L[a,b]$*, where* $a,b\in I^{\circ}$ *with* $a<b$*. If* $|f'|^q$ *is an quasi-convex*

on the interval $[a,b]$, *for* $q>1$ *then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{12}$$
$$\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}+1\right]$$
$$\times\left[\begin{array}{c}\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\\ +\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\end{array}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1.6, Hölder-İşcan integral inequality and the quasi convexity of the function $|f'|^q$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{4}\left[\int_0^1|-t|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1|t|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right]$$
$$\leq \frac{b-a}{4}\left[\left(\int_0^1(1-t)\left|-t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 t\left|-t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$
$$+\frac{b-a}{4}\left[\left(\int_0^1(1-t)\left|t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 t\left|t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$\times\left[\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\right]$$

$$
\begin{aligned}
&+\frac{b-a}{4}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{q},\left|f'(a)\right|^{q}\right\}\right)^{\frac{1}{q}} \\
&\times\left[\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\right] \\
= &\ \frac{b-a}{4}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}+1\right] \\
&\times\left[\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}}+\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{q},\left|f'(a)\right|^{q}\right\}\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

where

$$
\int_0^1 (1-t)\left|t\right|^p dt = \tfrac{1}{(p+1)(p+2)}, \quad \int_0^1 (1-t)\, dt = \tfrac{1}{2},
$$
$$
\int_0^1 t\left|t\right|^p dt = \tfrac{1}{p+2}, \quad \int_0^1 t dt = \tfrac{1}{2}.
$$

This completes the proof of the Theorem. □

Remark 2.5. *The inequality (12) is better than the inequality (6). For this, we need to show that*

$$
\beta(p) := \left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}+1\right] \le \alpha(p) := \left(\frac{1}{p+1}\right)^{\frac{1}{p}}.
$$

If we write as $\beta(p) = \alpha(p) M(p)$*, then* $M(p) = \left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\left(\frac{1}{p+2}\right)^{\frac{1}{p}}+\left(\frac{p+1}{p+2}\right)^{\frac{1}{p}}\right]$*. Therefore, by using concavity of the function* $h:[0,\infty)\to\mathbb{R}, h(x)=x^s, 0< s\le 1$*, we have*

$$
\begin{aligned}
M(p) &= 2^{\frac{1}{p}}\left[\frac{1}{2}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}+\frac{1}{2}\left(\frac{p+1}{p+2}\right)^{\frac{1}{p}}\right] \\
&\le 2^{\frac{1}{p}}\left(\frac{\frac{1}{p+2}+\frac{p+1}{p+2}}{2}\right)^{\frac{1}{p}} \\
&= 1.
\end{aligned}
$$

Hence, $\beta(p) \le \alpha(p)$*.*

Theorem 2.6. *Let* $f: I^\circ \subset \mathbb{R}\to\mathbb{R}$ *be a differentiable mapping on* I°*,* $a, b \in I^\circ$ *with* $a<b$*.* $f' \in L[a,b]$*, where* $a,b\in I$ *with* $a<b$*. If* $|f'|^q$ *is an quasi-convex on the interval* $[a,b]$*,* $q\ge 1$ *then the following inequality holds:*

$$
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \tag{13}
$$
$$
\times\left[\begin{array}{c}\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}} \\ +\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{q},\left|f'(a)\right|^{q}\right\}\right)^{\frac{1}{q}}\end{array}\right].
$$

Proof. From Lemma 1.6, power-mean integral inequality and the definition of quasi-convexity of the function $|f'|^q$, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\;& \frac{b-a}{4}\left[\int_0^1 |-t|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1 |t|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right] \\
\leq\;& \frac{b-a}{4}\left[\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |-t|^q\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&+\frac{b-a}{4}\left[\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t|^q\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
\leq\;& \frac{b-a}{4}\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \\
&+\frac{b-a}{4}\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \\
=\;& \frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

where $\int_0^1 |t|^q = \frac{1}{q+1}$. This completes the proof of the Theorem. □

Corollary 2.7. *If we take $q = 1$ in the inequality (13), then we get the following inequality:*

$$
\begin{aligned}
(14)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\;& \frac{b-a}{8}\left[\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'(b)\right|\right\}+\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'(a)\right|\right\}\right].
\end{aligned}
$$

The inequality (14) coincides with the inequality (7) in Theorem 1.8.

Theorem 2.8. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is an* *quasi-convex on the interval* $[a,b]$, $q\geq 1$ *then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[\left(\frac{1}{q+1}\right)^{\frac{1}{q}}+1\right] \\
&\times\left[\begin{array}{c}\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}} \\ +\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\end{array}\right].
\end{aligned} \tag{15}$$

Proof. From Lemma 1.6, improved power-mean integral inequality and the definition of quasi-convexity of the function $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 |-t|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1 |t|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right] \\
\leq\ & \frac{b-a}{4}\left[\left(\int_0^1(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)\left|-t\right|^q\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|-t\right|^q\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&+\frac{b-a}{4}\left[\left(\int_0^1(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)\left|t\right|^q\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|t\right|^q\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[\left(\frac{1}{q+1}\right)^{\frac{1}{q}}+1\right]\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}} \\
&+\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[\left(\frac{1}{q+1}\right)^{\frac{1}{q}}+1\right]\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[\left(\frac{1}{q+1}\right)^{\frac{1}{q}}+1\right] \\
&\times\left[\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}+\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$\int_0^1 (1-t)\,|t|^q\,dt = \frac{1}{(q+1)(q+2)}, \qquad \int_0^1 t\,|t|^q\,dt = \frac{1}{q+2}.$$

This completes the proof of the Theorem. □

Corollary 2.9. *If we take $q = 1$ in the inequality (15), then we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \tag{16}$$
$$\leq \frac{b-a}{8}\left[\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|, |f'(b)|\right\} + \sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|, |f'(a)|\right\}\right].$$

The inequality (16) coincides with the inequality (7) in Theorem 1.8 for $q = 1$.

Remark 2.10. *The inequality (15) is better than the inequality (13) in Theorem 2.6. Proof can be made similar to Remark 2.5.*

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