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ROUGH STATISTICAL CONVERGENCE IN 2-NORMED SPACES

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Abstract. In this study, we introduced the notions of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained statistical convergence criteria associated with this set in 2-normed space. Then, we proved that this set is closed and convex in 2-normed space. Also, we examined the relations between the set of statistical cluster points and the set of rough statistical limit points of a sequence in 2-normed space.

1. Introduction and Background

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [12] and Schoenberg [28].

The concept of 2-normed spaces was initially introduced by Gähler [13, 14] in the 1960's. Since then, this concept has been studied by many authors. Gürdal and Pehlivan [18] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Sarabadan and Talebi [26] studied statistical convergence and ideal convergence of sequences of functions in 2-normed spaces. Futhermore, a lot of development have been made in this area (see [1, 2, 7, 19, 20, 21, 22, 30, 27, 29, 31, 32, 33, 34, 35]).

The idea of rough convergence was first introduced by Phu [23] in finite-dimensional normed spaces. In [23], he showed that the set $\mathrm{LIM}^r x$ is bounded, closed, and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of $\mathrm{LIM}^r x$ on the roughness degree r. In another paper [24] related to this subject, he defined the rough continuity of linear operators and showed that every linear operator $f : X \to Y$ is r -continuous

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at every point $x \in X$ under the assumption $dimY < \infty$ and $r > 0$ where X and Y are normed spaces. In [25], he extended the results given in [23] to infinite-dimensional normed spaces. Aytar [5] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [6] studied that the r-limit set of the sequence is equal to the intersection of these sets and that r-core of the sequence is equal to the union of these sets. Recently, Arslan and Dündar [3, 4] introduced rough convergence and investigated some properties in 2-normed spaces.

In this paper, we note that our results and proof techniques presented in this paper are analogues of those in Phu's [23] paper. Namely, the actual origin of most of these results and proof techniques is them papers. The following our theorems and results are the extension of theorems and results in [3, 4, 23].

Now, we recall the some fundamental definitions and notations (See [1, 2, 3, 5, 6, 8, 9, 10, 11, 15, 16, 17, 18, 19, 20, 23, 24, 25, 30, 26]).

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.
(ii) $\|x, y\| = \|y, x\|$.
(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha \in \mathbb{R}$.
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|; \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$. The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space.

A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to L in X if $\lim_{n\to\infty} \|x_n - L, y\| = 0$, for every $y \in X$. In such a case, we write $\lim_{n\to\infty} x_n = L$ and call L the limit of (x_n).

Example 1.1. *Let $x = (x_n) = (\frac{n}{n+1}, \frac{1}{n})$ and $L = (1, 0)$. It is clear that (x_n) convergent to $L = (1, 0)$ in 2-normed space $X = \mathbb{R}^2$.*

Throughout the paper, let r be a nonnegative real number and $\mathbb{R}^n$ denotes the real n-dimensional space with the norm $\|.\|$. Consider a sequence $x = (x_n) \subset \mathbb{R}^n$.

The sequence $x = (x_n)$ is said to be r-convergent to L, denoted by $x_n \xrightarrow{r} L$ provided that

$$\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} : \ n \geq n_\varepsilon \Rightarrow \|x_n - L\| < r + \varepsilon.$$

The set

$$\mathrm{LIM}^r x = \{L \in \mathbb{R}^n : x_n \stackrel{r}{\longrightarrow} L\}$$

is called the r-limit set of the sequence $x = (x_n)$. A sequence $x = (x_n)$ is said to be r-convergent if $\mathrm{LIM}^r x \neq \emptyset$. In this case, r is called the convergence degree of the sequence $x = (x_n)$. For $r = 0$, we get the ordinary convergence.

Let K be a subset of the set of positive integers $\mathbb{N}$, and let us denote the set $\{k \in K : k \leq n\}$ by K_n. Then the natural density of K is given by

$$\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n},$$

where $|K_n|$ denotes the number of elements in K_n. Clearly, a finite subset has natural density zero, and we have $\delta(K^c) = 1 - \delta(K)$ where $K^c := \mathbb{N} \setminus K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$.

A sequence $x = (x_n)$ is said to be r-statistically convergent to L, denoted by $x_n \stackrel{r-st}{\longrightarrow} L$, provided that the set

$$\{n \in \mathbb{N} : \|x_n - L\| \geq r + \varepsilon\}$$

has natural density zero for $\varepsilon > 0$; or equivalently, if the condition $st - \limsup \|x_n - L\| \leq r$ is satisfied. In addition, we can write $x_n \stackrel{r-st}{\longrightarrow} L$ if and only if, the inequality $\|x_n - L\| < r + \varepsilon$ holds for every $\varepsilon > 0$ and almost all n.

Let (x_n) be a sequence in $(X, \|.,.\|)$ 2-normed linear space and r be a non-negative real number. (x_n) is said to be rough convergent (r-convergent) to L denoted by $x_n \stackrel{\|.,.\|}{\longrightarrow}_r L$ if

$$\forall\varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} : n \geq n_\varepsilon \Rightarrow \|x_n - L, z\| < r + \varepsilon \tag{1}$$

or equivalently, for every $z \in X$, if

$$\limsup \|x_n - L, z\| \leq r. \tag{2}$$

If (1) holds L is an r-limit point of (x_n), which is usually no more unique (for $r > 0$). So, we have to consider the so-called r-limit set (or shortly r-limit) of (x_n) defined by

$$\mathrm{LIM}_2^r x := \{L \in X : x_n \stackrel{\|.,.\|}{\longrightarrow}_r L\}. \tag{3}$$

The sequence (x_n) is said to be rough convergent if $\mathrm{LIM}_2^r x \neq \emptyset$. In this case, r is called a convergence degree of (x_n). For $r = 0$ we have the classical convergence in 2-normed space again. But our proper interest is case $r > 0$. There are several reasons for this interest. For instance, since an orginally convergent sequence (y_n) (with $y_n \to L$) in 2-normed space often cannot be determined (i.e., measured or calculated) exactly, one has to do with an approximated sequence (x_n) satisfying $\|x_n - y_n, z\| \leq r$ for all n and for every $z \in X$, where $r > 0$ is an upper bound of approximation error. Then, (x_n) is no more convergent in the classical sense, but for every $z \in X$,

$$\|x_n - L, z\| \leq \|x_n - y_n, z\| + \|y_n - L, z\| \leq r + \|y_n - L, z\|$$

implies that is r-convergent in the sense of (1).

Example 1.2. *Let $X = \mathbb{R}^2$. The sequence $x = (x_n) = ((-1)^n, 0)$ is not convergent in $(X, \|.,.\|)$ but it is rough convergent for every $z \in X$. It is clear that*

$$\mathrm{LIM}_2^r x = \{y = (y_1, y_2) \in X : |y_1| \le r - 1,\ |y_2| \le r\}.$$

In other words

$$\mathrm{LIM}_2^r x = \begin{cases} \emptyset & ,\ if\ \ r < 1 \\ \overline{B}_r((-1,0)) \cap \overline{B}_r((1,0)) & ,\ if\ \ r \ge 1, \end{cases}$$

where $\overline{B}_r(L) := \{y \in X : \|y - L, z\| \le r\}$.

Lemma 1.3 ([3], Theorem 2.2)**.** *Let $(X, \|.,.\|)$ be a 2-normed space and consider a sequence $x = (x_n) \in X$. The sequence (x_n) is bounded if and only if there exists an $r \ge 0$ such that $\mathrm{LIM}_2^r x \ne \emptyset$. For all $r > 0$, a bounded sequence (x_n) is always contains a subsequence x_{n_k} with*

$$\mathrm{LIM}_2^{(x_{n_k}),r} x_{n_k} \ne \emptyset.$$

Lemma 1.4 ([3], Theorem 2.3)**.** *Let $(X, \|.,.\|)$ be a 2-normed space and consider a sequence $x = (x_n) \in X$. For all $r \ge 0$, the r-limit set $\mathrm{LIM}_2^r x$ of an arbitrary sequence (x_n) is closed.*

Lemma 1.5 ([3], Theorem 2.4)**.** *Let $(X, \|.,.\|)$ be a 2-normed space and consider a sequence $x = (x_n) \in X$. If $y_0 \in \mathrm{LIM}_2^{r_0} x$ and $y_1 \in \mathrm{LIM}_2^{r_1} x$, then*

$$y_\alpha := (1-\alpha) y_0 + \alpha y_1 \in \mathrm{LIM}_2^{(1-\alpha) r_0 + \alpha r_1} x, \quad for \ \ \alpha \in [0, 1].$$

2. MAIN RESULTS

Definition 2.1. *Let $(X, \|.,.\|)$ be a 2-normed space. A sequence $x = (x_n)$ in X said to be rough statistically convergent ($r_2 st$-convergent) to L, denoted by $x_n \xrightarrow{\|.,.\|}_{r_2 st} L$, provided that the set*

$$\{n \in \mathbb{N} : \|x_n - L, z\| \ge r + \varepsilon\}$$

has natural density zero, for every $\varepsilon > 0$ and each nonzero $z \in X$; or equivalently, if the condition

$$st - \limsup \|x_n - L, z\| \le r$$

is satisfied. In addition, we can write $x_n \xrightarrow{\|.,.\|}_{r_2 st} L$, if and only if, the inequality

$$\|x_n - L, z\| < r + \varepsilon$$

holds for every $\varepsilon > 0$, each nonzero $z \in X$ and almost all n.

In this convergence, r is called the statistical convergence degree. For $r = 0$, rough statistically convergent coincide ordinary statistical convergence.

Similar to the idea of classical rough convergence, the idea of rough statistical convergence of a sequence can be interpreted as follows.

Suppose that a sequence $y = (y_n)$ in X is statistically convergent and cannot be measured or calculated exactly, on one has to do with an approximated (or statistically approximated) sequence $x = (x_n)$ in X satisfying

$$\|x_n - y_n, z\| \leq r$$

for all n and each nonzero $z \in X$, (or for almost all n, that is,

$$\delta(n \in \mathbb{N} : \|x_n - y_n, z\| \geq r) = 0.)$$

Then, the sequence $x = (x_n)$ is not statistically convergent anymore, but since the inclusion for each nonzero $z \in X$

$$\{n \in \mathbb{N} : \|y_n - L', z\| \geq \varepsilon\} \supseteq \{n \in \mathbb{N} : \|x_n - L', z\| \geq r + \varepsilon\} \tag{4}$$

holds and we have

$$\delta(\{n \in \mathbb{N} : \|y_n - L', z\| \geq r + \varepsilon\}) = 0,$$

and so

$$\delta(\{n \in \mathbb{N} : \|x_n - L', z\| \geq r + \varepsilon\}) = 0,$$

that is, the sequence x in X is r-statistically convergent in 2-normed space $(X, \|.,.\|)$.

In general, the rough statistical limit of a sequence $x = (x_n)$ may not be unique for the roughness degree $r > 0$. So, we have to consider the so-called r-statistically limit set of the sequence x in X, which is defined by

$$st - \mathrm{LIM}_2^r x := \{L \in X : x_n \xrightarrow{\|.,.\|}_{r_2 st} L\}. \tag{5}$$

The sequence x is said to be r-statistically convergent provided that $st - \mathrm{LIM}_2^r x \neq \emptyset$.

We have that $\mathrm{LIM}_2^r x = \emptyset$ for an unbounded sequence $x = (x_n)$. But such a sequence might be rough statistically convergent. For instance, define

$$x_n := \begin{cases} ((-1)^n, 0) & , \ if \ \ n \neq k^2 \ \ (k \in \mathbb{N}) \\ (n, n) & , \ otherwise \end{cases} \tag{6}$$

in $X = \mathbb{R}^2$. Because the set $\{1, 4, 9, 16, ...\}$ has natural density zero, we have

$$st - \mathrm{LIM}_2^r x := \begin{cases} \emptyset & , \ if \ \ r < 1, \\ \overline{B}_r((-1,0)) \cap \overline{B}_r((1,0)) & , \ if \ \ r \geq 1, \end{cases}$$

and $\mathrm{LIM}_2^r x = \emptyset$, for all $r \geq 0$.

From the example above, we have $\mathrm{LIM}_2^r x = \emptyset$ but $st - \mathrm{LIM}_2^r x \neq \emptyset$. Because a finite set of natural numbers has natural density zero, $\mathrm{LIM}_2^r x \neq \emptyset$ implies $st - \mathrm{LIM}_2^r x \neq \emptyset$ and so, we have

$$\mathrm{LIM}_2^r x \subseteq st - \mathrm{LIM}_2^r x.$$

That is, we have the fact

$$\{r \geq 0 : \mathrm{LIM}_2^r x \neq \emptyset\} \subseteq \{r \geq 0 : st - \mathrm{LIM}_2^r x \neq \emptyset\}$$

and so

$$\inf\{r \geq 0 : \mathrm{LIM}_2^r x \neq \emptyset\} \geq \inf\{r \geq 0 : st - \mathrm{LIM}_2^r x \neq \emptyset\}.$$

It also directly yields

$$diam(\mathrm{LIM}_2^r x) \leq diam(st - \mathrm{LIM}_2^r x).$$

As mentioned above, we cannot say that the rough statistical limit of a sequence is unique for the degree of roughness $r > 0$. The following conclusion related to this fact.

Theorem 2.2. *Let $x = (x_n)$ be a sequence in $(X, \|.,.\|)$. Then, we have*

$$diam(st - \mathrm{LIM}_2^r x) \leq 2r.$$

Also, generally, $diam(st - \mathrm{LIM}_2^r x)$ has no smaller bound.

Proof. Suppose that $diam(st - \mathrm{LIM}_2^r x) > 2r$. Then, there exist $y, t \in st - \mathrm{LIM}_2^r x$ such that $\|y - t, z\| > 2r$, for each nonzero $z \in X$. Choose $\varepsilon \in \left(0, \frac{\|y-t,z\|}{2} - r\right)$. Since $y, t \in st - \mathrm{LIM}_2^r x$ we have $\delta(A_1) = 0$ and $\delta(A_2) = 0$, where

$$A_1 = \{n \in \mathbb{N} : \|x_n - y, z\| \geq r + \varepsilon\} \text{ and } A_2 = \{n \in \mathbb{N} : \|x_n - t, z\| \geq r + \varepsilon\}$$

for every $\varepsilon > 0$ and each nonzero $z \in X$. By the properties of natural density, we have $\delta(A_1^c \cap A_2^c) = 1$ and so for all $n \in A_1^c \cap A_2^c$, and each nonzero $z \in X$, we can write

$$\|y - t, z\| \leq \|x_n - y, z\| + \|x_n - t, z\| < 2(r + \varepsilon) = \|y - t, z\|$$

which is a contradiction.

Now let's do the second part of the proof. Let a sequence $x = (x_n)$ in $(X, \|.,.\|)$ such that $st - \lim x = L$. Then, for every $\varepsilon > 0$ and each nonzero $z \in X$, we can write

$$\delta(\{n \in \mathbb{N} : \|x_n - L, z\| \geq \varepsilon\}) = 0.$$

So, we have

$$\|x_n - y, z\| \leq \|x_n - L, z\| + \|L - y, z\| \leq \|x_n - L, z\| + r$$

for each $y \in \overline{B}_r(L) := \{y \in X : \|y - L, z\| \leq r\}$ and for each nonzero $z \in X$. Then, for every $\varepsilon > 0$ and each nonzero $z \in X$ we get

$$\|x_n - y, z\| < r + \varepsilon,$$

for each $n \in \{n \in \mathbb{N} : \|x_n - L, z\| < \varepsilon\}$. Since the sequence x is statistically convergent to L, for each nonzero $z \in X$, we have

$$\delta(\{n \in \mathbb{N} : \|x_n - L, z\| < \varepsilon\}) = 1.$$

Hence, we have $y \in st - \mathrm{LIM}_2^r x$. As a result, we can write

$$st - \mathrm{LIM}_2^r x = \overline{B}_r(L).$$

Since $diam(\overline{B}_r(L)) = 2r$, this shows that in general, the upper bound $2r$ of the diameter of the set $st - \text{LIM}_2^r x$ can no longer be reduced. $\square$

By [[3], Theorem 2.2], there exists a nonnegative real number r such that $\text{LIM}_2^r x \neq \emptyset$ for a bounded sequence. Because the fact $\text{LIM}_2^r x \neq \emptyset$ implies $st - \text{LIM}_2^r x \neq \emptyset$, we have the following result.

Result 2.3. *If a sequence $x = (x_n)$ is bounded in $(X, \|.,.\|)$, then there exists a nonnegative real number r such that $st - \text{LIM}_2^r x \neq \emptyset$.*

The opposite implication of the above result is not valid. If we let the sequence to be statistically bounded in 2-normed space, then we have the converse of Result 2.3. Hence, we give the following theorem.

Theorem 2.4. *A sequence $x = (x_n)$ is statistically bounded in $(X, \|.,.\|)$ if and only if there exists a nonnegative real number r such that $st - \text{LIM}_2^r x \neq \emptyset$.*

Proof. Let $x = (x_n)$ be a statistically bounded sequence. Then, there exists a positive real number M such that for each nonzero $z \in X$,

$$\delta(\{n \in \mathbb{N} : \|x_n, z\| \geq M\}) = 0.$$

Now, we let $r' := \sup\{\|x_n, z\| : n \in A^c\}$, where $A := \{n \in \mathbb{N} : \|x_n, z\| \geq M\}$, for each nonzero $z \in X$. Then, the set $st - \text{LIM}_2^{r'} x$ contains the origin of X. Therefore, we have $st - \text{LIM}_2^{r'} x \neq \emptyset$. If $st - \text{LIM}_2^r x \neq \emptyset$, for some $r \geq 0$, then there exists an L such that $L \in st - \text{LIM}_2^r x$, that is,

$$\delta(\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\}) = 0,$$

for each $\varepsilon > 0$ and each nonzero $z \in X$. Then, we say that almost all x_n's are contained in some ball with any radius grater than r. So the sequence x is statistically bounded. $\square$

By [[3], Proposition 2.1], we know that if x' is a subsequence of $x = (x_n)$, then $\text{LIM}_2^r x \subseteq \text{LIM}_2^r x'$. But this fact does not hold in the theory of statistical convergence. For example, define

$$x_n := \begin{cases} (n, n) & , \ if \ \ n = k^3, (k \in \mathbb{N}) \\ (0, (-1)^n) & , \ otherwise, \end{cases}$$

in $X = \mathbb{R}^2$. Then, the sequence $x' := ((1,1), (8,8), (27,27), \cdots)$ is a subsequence of $x = (x_n)$. We have $st - \text{LIM}_2^r x = \overline{B}_r((0,-1)) \cap \overline{B}_r((0,1))$ and $st - \text{LIM}_2^r x' = \emptyset$, for $r \geq 1$.

So we can present the statistical analogue of Arslan and Dündar's result [[3], Proposition 2.1] in the following theorem without proof.

Theorem 2.5. *If $x' = (x_{n_k})$ is a nonthin subsequence of $x = (x_n)$ in $(X, \|.,.\|)$, then*

$$st - \text{LIM}_2^r x \subseteq st - \text{LIM}_2^r x'.$$

Now, we give the topological and geometrical properties of the r-statistical limit set of a sequence.

Theorem 2.6. *The r-statistical limit set of a sequence $x = (x_n)$ is closed in $(X, \|.,.\|)$.*

Proof. If $st - \mathrm{LIM}_2^r x = \emptyset$, proof is clear. Let $st - \mathrm{LIM}_2^r x \neq \emptyset$. Then, we can choose a sequence

$$(y_n) \subseteq st - \mathrm{LIM}_2^r x$$

such that $y_n \to L'$ for $n \to \infty$. If we show that $L' \in st - \mathrm{LIM}_2^r x$, then the proof will be complete.

Let $\varepsilon > 0$ be given. Because $y_n \to L'$, there exists an $n_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that

$$\|y_n - L', z\| < \frac{\varepsilon}{2},$$

for all $n > n_{\frac{\varepsilon}{2}}$ and each nonzero $z \in X$. Now choose an $n_0 \in \mathbb{N}$ such that $n_0 > n_{\frac{\varepsilon}{2}}$. Then, we can write $\|y_{n_0} - L', z\| < \frac{\varepsilon}{2}$. On the other hand, since $(y_n) \subseteq st - \mathrm{LIM}_2^r x$, we have $y_{n_0} \in st - \mathrm{LIM}_2^r x$, that is,

$$\delta(\{n \in \mathbb{N} : \|x_n - y_{n_0}, z\| \geq r + \frac{\varepsilon}{2}\}) = 0. \tag{7}$$

Now let us show that the inclusion

$$\{n \in \mathbb{N} : \|x_n - L', z\| < r + \varepsilon\} \supseteq \{n \in \mathbb{N} : \|x_n - y_{n_0}, z\| < r + \frac{\varepsilon}{2}\} \tag{8}$$

holds for each nonzero $z \in X$. Let $k \in \{n \in \mathbb{N} : \|x_n - y_{n_0}, z\| < r + \frac{\varepsilon}{2}\}$. Hence, for each nonzero $z \in X$ we have

$$\|x_k - y_{n_0}, z\| < r + \frac{\varepsilon}{2}$$

and so

$$\|x_k - L', z\| \leq \|x_k - y_{n_0}, z\| + \|y_{n_0} - L', z\| < r + \varepsilon,$$

that is,

$$k \in \{n \in \mathbb{N} : \|x_n - L', z\| < r + \varepsilon\},$$

which proves (8). From (7), we can say that the set on the right-hand side of (8) has natural density 1. Then the natural density of the set on the left-hand side of (8) is equal to 1. So for each nonzero $z \in X$, we get

$$\delta(\{n \in \mathbb{N} : \|x_n - L', z\| \geq r + \varepsilon\}) = 0,$$

which completes the proof. □

Theorem 2.7. *The r-statistical limit set of a sequence $x = (x_n)$ is convex in $(X, \|.,.\|)$.*

Proof. Let $y_0, y_1 \in st - \mathrm{LIM}_2^r x$ for the sequence $x = (x_n)$ and let $\varepsilon > 0$ be given. For each nonzero $z \in X$, define

$$A_1 := \{n \in \mathbb{N} : \|x_n - y_0, z\| \geq r + \varepsilon\} \text{ and } A_2 := \{n \in \mathbb{N} : \|x_n - y_1, z\| \geq r + \varepsilon\}.$$

Since $y_0, y_1 \in st - \mathrm{LIM}_2^r x$, we have $\delta(A_1) = \delta(A_2) = 0$. Therefore, we have

$$\|x_n - [(1 - \lambda)y_0 + \lambda y_1], z\| = \|(1 - \lambda)(x_n - y_0) + \lambda(x_n - y_1), z\| < r + \varepsilon$$

for each $n \in A_1^c \cap A_2^c$, each $\lambda \in [0,1]$ and for each nonzero $z \in X$. Since, $\delta(A_1^c \cap A_2^c) = 1$, we have

$$\delta(\{n \in \mathbb{N} : \|x_n - [(1-\lambda)(y_0) + \lambda y_1], z\| \geq r + \varepsilon\}) = 0,$$

that is,

$$[(1-\lambda)(y_0) + \lambda y_1] \in st - \mathrm{LIM}_2^r x,$$

for each nonzero $z \in X$. This proves the convexity of the set $st - \mathrm{LIM}_2^r x$. □

Theorem 2.8. *Let $(X, \|.,.\|)$ be a 2-normed space and $r > 0$. Then, a sequence $x = (x_n)$ is r-statistically convergent to L in X iff there exists a sequence $y = (y_n)$ in X such that $st - \lim y = L$ and $\|x_n - y_n, z\| \leq r$, for each $n \in \mathbb{N}$ and each nonzero $z \in X$.*

Proof. Let $x_n \xrightarrow{\|.,.\|}_{r_2 st} L$. Then, for each nonzero $z \in X$ we have

$$st - \limsup \|x_n - L, z\| \leq r. \tag{9}$$

Now, for each nonzero $z \in X$ we define

$$y_n := \begin{cases} L & , \ if \|x_n - L, z\| \leq r \\ x_n + r\frac{L - x_n}{\|x_n - L, z\|} & , \ otherwise. \end{cases} \tag{10}$$

Then, for each nonzero $z \in X$ we can write

$$\|y_n - L, z\| := \begin{cases} 0 & , \ if \|x_n - L, z\| \leq r \\ \|x_n - L, z\| - r & , \ otherwise, \end{cases} \tag{11}$$

and by definition of y_n, we have

$$\|x_n - y_n, z\| \leq r, \text{ for all } n \in \mathbb{N}.$$

By (9) and the definition of y_n, for all $n \in \mathbb{N}$ we have $st - \limsup \|y_n - L, z\| = 0$, which implies that $st - \lim y_n = L$.

Conversely, since $st - \lim y_n = L$, we have

$$\delta(\{n \in \mathbb{N} : \|y_n - L, z\| \geq \varepsilon\}) = 0,$$

for each $\varepsilon > 0$ and each nonzero $z \in X$ and so, it is easy to see that the inclusion

$$\{n \in \mathbb{N} : \|y_n - L, z\| \geq \varepsilon\} \supseteq \{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\}$$

holds. Since

$$\delta(\{n \in \mathbb{N} : \|y_n - L, z\| \geq \varepsilon\}) = 0,$$

for each nonzero $z \in X$, we have

$$\delta(\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\}) = 0,$$

which completes the proof. □

If we replace the condition

$$\text{“}\|x_n - y_n, z\| \leq r, \text{ for all } n \in \mathbb{N} \text{ and for each nonzero } z \in X,\text{”}$$

in the hypothesis of the above theorem with the condition

$$\text{“}\delta(\{n \in \mathbb{N} : \|x_n - y_n, z\| > r\}) = 0\text{”},$$

then the theorem will also be valid.

Definition 2.9. *Let $(X, \|.,.\|)$ be a 2-normed space. $c \in X$ is called a statistical cluster point of a sequence $x = (x_n)$ in X provided that the natural density of the set*

$$\{n \in \mathbb{N} : \|x_n - c, z\| < \varepsilon\}$$

is different from zero for every $\varepsilon > 0$ and each nonzero $z \in X$. We denote the set of all statistical cluster points of the sequence x by Γ_x^2.

Now, we give an important property of the set of rough statistical limit points of a sequence.

Lemma 2.10. *For an arbitrary $c \in \Gamma_x^2$ of a sequence $x = (x_n)$ in $(X, \|.,.\|)$, we have $\|L - c, z\| \leq r$, for all $L \in st - \mathrm{LIM}_2^r x$ and each nonzero $z \in X$.*

Proof. Assume on the contrary that there exists a point $c \in \Gamma_x^2$ and $L \in st - \mathrm{LIM}_2^r x$ such that

$$\|L - c, z\| > r,$$

for each nonzero $z \in X$. Define $\varepsilon := \frac{\|L-c,z\|-r}{3}$. Then, for each nonzero $z \in X$ we can write

$$\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\} \supseteq \{n \in \mathbb{N} : \|x_n - c, z\| < \varepsilon\}. \tag{12}$$

Since $c \in \Gamma_x^2$, for each nonzero $z \in X$ we have

$$\delta(\{n \in \mathbb{N} : \|x_n - c, z\| < \varepsilon\}) \neq 0.$$

Hence by (12), for each nonzero $z \in X$ we have

$$\delta(\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\}) \neq 0,$$

which contradicts the fact $L \in st - \mathrm{LIM}_2^r x$. □

Now we give two statistical convergence criteria associated with the rough statistical limit set.

Theorem 2.11. *A sequence $x = (x_n)$ statistically converges to L in $(X, \|.,.\|)$ if and only if $st - \mathrm{LIM}_2^r x = \overline{B}_r(L)$.*

Proof. We have proved the necessity part of this theorem in the proof of Theorem 2.2.

Sufficiency. Since $st - \mathrm{LIM}_2^r x = \overline{B}_r(L) \neq \emptyset$, then by Theorem 2.4 we can say that the sequence x is statistically bounded. Assume on the contrary that the sequence x has another statistical cluster point L' different from L. Then, the point

$$\overline{L} := L + \frac{r}{\|L - L', z\|}(L - L')$$

satisfies

$$\|\overline{L} - L', z\| = \left(\frac{r}{\|L - L', z\|} + 1\right) \|L - L', z\| = r + \|L - L', z\| > r.$$

Since L' is a statistical cluster point of the sequence x, by Lemma 2.10 this inequality implies that $\overline{L} \notin st-\text{LIM}_2^r x$. This contradicts the fact $\|\overline{L} - L, z\| = r$ and $st - \text{LIM}_2^r x = \overline{B}_r(L)$. Therefore, L is the unique statistical cluster point of the sequence x and so, we can say that the sequence x is statistically convergent to L. □

Theorem 2.12. *Let $(X, \|.,.\|)$ be a strictly convex space and $x = (x_n)$ be a sequence in this space. If there $t_1, t_2 \in st-\text{LIM}_2^r x$ such that $\|t_1 - t_2, z\| = 2r$, for each nonzero $z \in X$, this sequence is statistically convergent to $\frac{1}{2}(t_1 + t_2)$.*

Proof. Assume that $t \in \Gamma_x^2$. Then, $t_1, t_2 \in st - \text{LIM}_2^r x$ implies that

$$\|t_1 - t, z\| \leq r \text{ and } \|t_2 - t, z\| \leq r \tag{13}$$

for each nonzero $z \in X$, by Lemma 2.10. On the other hand, for each nonzero $z \in X$, we have

$$2r = \|t_1 - t_2, z\| \leq \|t_1 - t, z\| + \|t_2 - t, z\|, \tag{14}$$

and so

$$\|t_1 - t, z\| = \|t_2 - t, z\| = r,$$

combining the inequalities (13) and (14). Since for each nonzero $z \in X$,

$$\frac{1}{2}(t_2 - t_1) = \frac{1}{2}[(t - t_1) + (t_2 - t)] \tag{15}$$

and $\|t_1 - t_2, z\| = 2r$, we have

$$\|\frac{1}{2}(t_2 - t_1), z\| = r.$$

By the strict convexity of the space and from the equality (15), for each nonzero $z \in X$ we get

$$\frac{1}{2}(t_2 - t_1) = t - t_1 = t_2 - t,$$

which implies that

$$t = \frac{1}{2}(t_1 + t_2).$$

Hence, t is the unique statistical cluster point of the sequence $x = (x_n)$. On the other hand, the assumption $t_1, t_2 \in st - \text{LIM}_2^r x$ implies that $st - \text{LIM}_2^r x \neq \emptyset$.

By Theorem 2.4, the sequence x is statistically bounded. Consequently, the sequence x is statistically convergent, i.e.,

$$st - \lim x = \frac{1}{2}(t_1 + t_2).$$

□

The following theorem is the statistical extension of [[4], Theorem 2.5].

Theorem 2.13. *Let $(X, \|.,.\|)$ be a 2-normed space.*
(i) *If $c \in \Gamma_x^2$ then,*

$$st - \mathrm{LIM}_2^r x \subseteq \overline{B}_r(c). \tag{16}$$

(ii)

$$st - \mathrm{LIM}_2^r x = \bigcap_{c \in \Gamma_x^2} \overline{B}_r(c) = \{L \in X : \Gamma_x^2 \subseteq \overline{B}_r(L)\}. \tag{17}$$

Proof. (i) Let $L \in st - \mathrm{LIM}_2^r x$ and $c \in \Gamma_x^2$. Then, by Lemma 2.10, we have

$$\|L - c, z\| \leq r,$$

otherwise we get

$$\delta(\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\}) \neq 0,$$

for $\varepsilon := \frac{\|L-c,z\|-r}{3}$ and each nonzero $z \in X$. This contradicts the fact $L \in st - \mathrm{LIM}_2^r x$.

(ii) By the inclusion (16), we can write

$$st - \mathrm{LIM}_2^r x \subseteq \bigcap_{c \in \Gamma_x^2} \overline{B}_r(c). \tag{18}$$

Now let $y \in \bigcap_{c \in \Gamma_x^2} \overline{B}_r(c)$. Then for each nonzero $z \in X$, we have

$$\|y - c, z\| \leq r,$$

for all $c \in \Gamma_x^2$, which is equivalent to

$$\Gamma_x^2 \subseteq \overline{B}_r(y),$$

that is,

$$\bigcap_{c \in \Gamma_x^2} \overline{B}_r(c) \subseteq \{L \in X : \Gamma_x^2 \subseteq \overline{B}_r(L)\}. \tag{19}$$

Now let $y \notin st - \mathrm{LIM}_2^r x$. Then, there exists an $\varepsilon > 0$ such that for each nonzero $z \in X$,

$$\delta(\{n \in \mathbb{N} : \|x_n - y, z\| \geq r + \varepsilon\}) \neq 0,$$

which implies the existence of a statistical cluster points c of the sequence x with

$$\|y - c, z\| \geq r + \varepsilon,$$

that is, $\Gamma_x^2 \nsubseteq \overline{B}_r(y)$ and

$$y \notin \{L \in X : \Gamma_x^2 \subseteq \overline{B}_r(L)\}.$$

Hence, $y \in st - \text{LIM}_2^r x$ follows from

$$y \in \{L \in X : \Gamma_x^2 \subseteq \overline{B}_r(L)\},$$

that is,

$$\{L \in X : \Gamma_x^2 \subseteq \overline{B}_r(L)\} \subseteq st - \text{LIM}_2^r x. \tag{20}$$

Therefore, the inclusions (18)-(20) ensure that (17) holds. □

We end this work by giving the relation between the set of statistical cluster points and the set of rough statistical limit points of a sequence.

Theorem 2.14. *Let $x = (x_n)$ be a statistically bounded sequence in $(X, \|.,.\|)$. If $r = diam(\Gamma_x^2)$, then we have*

$$\Gamma_x^2 \subseteq st - \text{LIM}_2^r x.$$

Proof. Let $c \notin st - \text{LIM}_2^r x$. Then there exists an $\varepsilon' > 0$ such that, for each nonzero $z \in X$

$$\delta(\{n \in \mathbb{N} : \|x_n - c, z\| \geq r + \varepsilon'\}) \neq 0. \tag{21}$$

Since the sequence is statistically bounded and from the inequality (21), there exists another statistical cluster point c' such that, for each nonzero $z \in X$,

$$\|c - c', z\| > r + \widetilde{\varepsilon},$$

where $\widetilde{\varepsilon} := \frac{\varepsilon'}{2}$. So we get $diam(\Gamma_x^2) > r + \widetilde{\varepsilon}$, which proves the theorem. □

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