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INVARIANT CONVERGENCE IN FUZZY NORMED SPACES

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Abstract. In this study, we defined the notions of invariant convergence and invariant Cauchy sequences in fuzzy normed spaces. Also, we investigated some properties of invariant convergence and relations between invariant convergence and invariant Cauchy sequences in fuzzy normed spaces.

1. Introduction and Background

Banach [2] defined the generalized limit as an application of Hahn-Banach theorem on the set of all bounded real valued sequences. It is also known as Banach limit. Later, Lorentz [18] offered that if all Banach limits of a given bounded sequence are equal, it is called almost convergent. In further studies [6, 24], invariant mean and invariant convergence are given as a more general case of Banach limit and almost convergence. Also, several authors including Schaefer [27], Mursaleen and Edely [21], Mursaleen [22, 23], Savaş [25, 26] studied on invariant convergent sequences.

The idea of fuzzy set was initially introduced by Zadeh [30] to deal with imprecise phenomena as an alternative to classical set theory. After that, several classical concepts were reconstructed. Fuzzy topological spaces [3, 19], fuzzy metric [12, 15, 17], fuzzy norm [1, 4, 11, 16] are just some of the examples. Felbin's fuzzy norm [11], which is associated with Kaleva and Seikkala [15] type metric space by assigning a non-negative fuzzy real number to each element of a linear space, forms the basis of this study. Das and Das [5] studied fuzzy topology generated by fuzzy norm. Diamond and Kloeden [8] investigated the metric spaces of fuzzy sets-theory and applications. Fang and Huang [10] studied on the level convergence of a sequence of fuzzy numbers. Also some other authors [9,13,14] studied the notions of fuzzy numbers and fuzzy normed space.

Now, we recall the basic notions and some important definitions used in our paper (See [1, 4, 6, 7, 9, 11, 13, 16, 18, 20–24, 27–30]).

A fuzzy number is a fuzzy set provided that

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(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$;

(iii) u is upper semi-continuous;

(iv) $cl\{x \in \mathbb{R} : u(x) > 0\}$ is a compact set.

Let $L(\mathbb{R})$ be the set of all fuzzy number. $\mathbb{R}$ can be embedded in $L(\mathbb{R})$ since each $r \in \mathbb{R}$ can be considered a fuzzy real number $\tilde{r}$ defined by

$$\widetilde{r}(t) = \begin{cases} 1, & t = r, \\ 0, & t \neq r. \end{cases}$$

For $u \in L(\mathbb{R})$, the α-level set of u is defined by

$$[u]_\alpha = \left\{ \begin{array}{ll} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0,1] \\ cl\{x \in \mathbb{R} : u(x) > \alpha\}, & \text{if } \alpha = 0. \end{array} \right.$$

The α-level set of a fuzzy number denoted by $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ is a non-empty, bounded and closed interval for each $\alpha \in [0,1]$ where $u_\alpha^- = -\infty$ and $u_\alpha^+ = \infty$ are admissible.

If $u \in L(\mathbb{R})$ and $u(x) = 0$ for $x < 0$, then u is called a non-negative fuzzy number. Let $L^*(\mathbb{R})$ denote the set of all non-negative fuzzy number. It is easy to see $\tilde{0} \in L^*(\mathbb{R})$.

A partial ordering $\preceq$ in $L(\mathbb{R})$ is defined by for $u, v \in L(\mathbb{R})$,

$$u \preceq v \quad \text{iff} \quad u_\alpha^- \leq v_\alpha^- \quad \text{and} \quad u_\alpha^+ \leq v_\alpha^+ \quad \text{for all} \quad \alpha \in [0,1].$$

Arithmetic equations addition, multiplication and multiplication with a scaler on $L(\mathbb{R})$ are defined by

(i) $(u \oplus v)(t) = \sup_{s \in \mathbb{R}} \{u(s) \wedge v(t-s)\}, \quad t \in \mathbb{R}$

(ii) $(u \odot v)(t) = \sup_{s \in \mathbb{R} s \neq 0} \{u(s) \wedge v(t/s)\}, \quad t \in \mathbb{R}$

(iii) For $k \in \mathbb{R}^+$, ku is defined as $ku(t) = u(t/k)$ and $0u(t) = \tilde{0}$, $t \in \mathbb{R}$.

Let $u, v \in L(\mathbb{R})$ and $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$, $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$. Arithmetic equations in terms of α-level sets are defined by

(i) $[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+]$,

(ii) $[u \odot v]_\alpha = [u_\alpha^- . v_\alpha^-, u_\alpha^+ . v_\alpha^+]$, $u, v \in L^*(\mathbb{R})$,

(iii) $[ku]_\alpha = k[u]_\alpha = \begin{cases} [ku_\alpha^-, ku_\alpha^+], & k \geq 0, \\ [ku_\alpha^+, ku_\alpha^-], & k < 0. \end{cases}$

For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ is defined by

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} \max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\}.$$

One can see that

$$D(u, \tilde{0}) = \sup_{0 \leq \alpha \leq 1} \max\{|u_\alpha^-|, |u_\alpha^+|\} = \max\{|u_0^-|, |u_0^+|\}.$$

Obviously, $D(u, \tilde{0}) = u_\alpha^+$ when $u \in L^*(\mathbb{R})$.

A sequence (u_n) in $L(\mathbb{R})$ is called convergent to $u \in L(\mathbb{R})$ denoted by $D-\lim\limits_{n\to\infty} u_n = u$ if $\lim_{n\to\infty} D(u_n, u) = 0$, i.e., for all given $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon) \in \mathbb{R}$ such that $D(u_n, u) < \varepsilon$, for $n > N$

Let X be a vector space over $\mathbb{R}$, $\|.\| : X \to L^*(\mathbb{R})$ and the mappings $L, R : [0,1] \times [0,1] \to [0,1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$.

The quadruple $(X, \|.\|, L, R)$ is called fuzzy normed linear space (FNS) and $\|.\|$ is a fuzzy norm if the following axioms are satisfied

(i) $\|x\| = \widetilde{0}$ iff $x = \theta$,

(ii) $\|rx\| = |r| \odot \|x\|$ for $x \in X$, $r \in \mathbb{R}$,

(iii) For all $x, y \in X$

(a) $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$,

whenever $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s+t \leq \|x+y\|_1^-$,

(b) $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$,

whenever $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s+t \geq \|x+y\|_1^-$.

When $L = \min$ and $R = \max$ are taken in above (iii), triangle inequalities become

$$\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^- \quad \text{and} \quad \|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+,$$

for all $\alpha \in (0,1]$ and $x, y \in X$. Since they fulfil the other conditions of norm, $\|x\|_\alpha^-$ and $\|x\|_\alpha^+$ can be seen as ordinary norms on X.

Example 1.1. *Let $(X, \|.\|_C)$ be an ordinary normed linear space. Then, a fuzzy norm $\|.\|$ on X can be obtained*

$$\|x\|(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq a\|x\|_C \text{ or } t \geq b\|x\|_C, \\ \frac{t}{(1-a)\|x\|_C} - \frac{a}{1-a}, & \text{if } a\|x\|_C \leq t \leq \|x\|_C, \\ \frac{-t}{(b-1)\|x\|_C} + \frac{b}{b-1}, & \text{if } \|x\|_C \leq t \leq b\|x\|_C, \end{cases}$$

where $\|x\|_C$ is the ordinary norm of x $(\neq \theta)$, $0 < a < 1$ and $1 < b < \infty$. For $x = \theta$, define $\|x\| = \widetilde{0}$. Hence $(X, \|.\|)$ is a fuzzy normed linear space.

Throughout paper, let $(X, \|.\|)$ be an fuzzy normed linear space (FNS).

Let us consider the topological structure of an FNS $(X, \|.\|)$. For any $\varepsilon > 0, \alpha \in [0,1]$ and $x \in X$, the $(\varepsilon, \alpha)-$ neighborhood of x is the set

$$\mathcal{N}_x(\varepsilon, \alpha) := \{y \in X : \|x-y\|_\alpha^+ < \varepsilon\}.$$

A sequence $(x_n)_{n=1}^\infty$ in X is convergent to $x \in X$ with respect to the fuzzy norm on X and we denote by $x_n \overset{FN}{\to} x$, provided that

$$(D) - \lim_{n\to\infty} \|x_n - x\| = \widetilde{0},$$

i.e., for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that

$$D(\|x_n - x\|, \widetilde{0}) < \varepsilon,$$

for all $n \geq N(\varepsilon)$. This means that for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N(\varepsilon)$,

$$\sup_{\alpha \in [0,1]} \|x_n - x\|_\alpha^+ = \|x_n - x\|_0^+ < \varepsilon.$$

In terms of neighborhoods, for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $x_n \in \mathcal{N}_x(\varepsilon, 0)$ for $n \geq N(\varepsilon)$.

Let σ be a mapping of the positive integers into itself. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean if and only if

(i) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
(ii) $\phi(e) = 1$, where $e = (1, 1, 1...)$,
(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and satisfied the condition $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Invariant mean, ϕ, is a extension of the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. The sequence is called invariant convergent when its invariant means are equal. In case $\sigma(n) = n + 1$, the σ mean is often called a Banach limit and invariant convergent is almost convergent.

A bounded sequence (x_n) is σ-convergent to the number L if and only if $\lim_{m \to \infty} t_{mn} = L$ uniformly in m, where

$$t_{mn} = \frac{x_n + x_{\sigma(n)} + x_{\sigma^2(n)} + \cdots + x_{\sigma^m(n)}}{m+1}.$$

2. Main Results

Definition 2.1. *A sequence $x = (x_n)$ in X is said to be invariant convergent to $L \in X$ with respect to fuzzy norm and denoted by $x_n \stackrel{\sigma - FN}{\longrightarrow} L$ if*

$$(D) - \lim_{m \to \infty} \|t_{mn} - L\| = \widetilde{0}$$

uniformly in n, where

$$t_{mn} = \frac{x_n + x_{\sigma(n)} + x_{\sigma^2(n)} + \cdots + x_{\sigma^m(n)}}{m+1}.$$

Namely, for given $\varepsilon > 0$ there exists a $m_0 = m_0(\varepsilon) \in \mathbb{N}$ such that for all $m > m_0$ and every $n \in \mathbb{N}$,

$$D(\|t_{mn} - L\|, \widetilde{0}) = \sup_{\alpha \in [0,1]} \|t_{mn} - L\|_\alpha^+ = \|t_{mn} - L\|_0^+ < \varepsilon.$$

In terms of neighborhood, for given $\varepsilon > 0$ there exists a $m_0 = m_0(\varepsilon) \in \mathbb{N}$ such that for all $m > m_0$ and every $n \in \mathbb{N}$,

$$t_{mn} \in \mathcal{N}_L(\varepsilon, 0).$$

Theorem 2.2. *Let $x = (x_n)$ be a sequence in X. If x is invariant convergent, then it's limit is unique with respect to fuzzy norm.*

Proof. Let $x_n \overset{\sigma-FN}{\longrightarrow} L_1$ and $x_n \overset{\sigma-FN}{\longrightarrow} L_2$, where $L_1 \neq L_2$. Then, for every $\varepsilon > 0$ there exists a $m_1 = m_1(\varepsilon) \in \mathbb{N}$ such that for all $m > m_1$,

$$D(\|t_{mn} - L_1\|, \widetilde{0}) = \sup_{\alpha\in[0,1]} \|t_{mn} - L_1\|_\alpha^+ = \|t_{mn} - L_1\|_0^+ < \frac{\varepsilon}{2}$$

and also, for every $\varepsilon > 0$ there exists a $m_2 = m_2(\varepsilon) \in \mathbb{N}$ such that for all $m > m_2$ and every $n \in \mathbb{N}$,

$$D(\|t_{mn} - L_2\|, \widetilde{0}) = \sup_{\alpha\in[0,1]} \|t_{mn} - L_2\|_\alpha^+ = \|t_{mn} - L_2\|_0^+ < \frac{\varepsilon}{2}.$$

Therefore, let $m_0 = \max\{m_1, m_2\}$ such that for all $m > m_0$ and every $n \in \mathbb{N}$,

$$\begin{aligned} \|L_1 - L_2\|_0^+ &= \|L_1 - t_{mn} + t_{mn} - L_2\|_0^+ \\ &\leq \|L_1 - t_{mn}\|_0^+ + \|t_{mn} - L_2\|_0^+ \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and so, we have $L_1 = L_2$. □

Theorem 2.3. *Let $x = (x_n)$ and $y = (y_n)$ be two sequences in X. If*

$$x_n \overset{\sigma-FN}{\longrightarrow} L_1 \text{ and } y_n \overset{\sigma-FN}{\longrightarrow} L_2,$$

then

$$x_n + y_n \overset{\sigma-FN}{\longrightarrow} L_1 + L_2.$$

Proof. Assume that

$$x_n \overset{\sigma-FN}{\longrightarrow} L_1 \text{ and } y_n \overset{\sigma-FN}{\longrightarrow} L_2.$$

Then, for every $\varepsilon > 0$ there exists a $m_1 = m_1(\varepsilon) \in \mathbb{N}$ such that for all $m > m_1$,

$$D(\|t_{mn} - L_1\|, \widetilde{0}) = \sup_{\alpha\in[0,1]} \|t_{mn} - L_1\|_\alpha^+ = \|t_{mn} - L_1\|_0^+ < \frac{\varepsilon}{2}$$

and also, for every $\varepsilon > 0$ there exists a $m_2 = m_2(\varepsilon) \in \mathbb{N}$ such that for all $m > m_2$,

$$D(\|k_{mn} - L_2\|, \widetilde{0}) = \sup_{\alpha\in[0,1]} \|k_{mn} - L_2\|_\alpha^+ = \|k_{mn} - L_2\|_0^+ < \frac{\varepsilon}{2},$$

for every $n \in \mathbb{N}$. Therefore, let $m_0 = \max\{m_1, m_2\}$ such that for all $m > m_0$ and every $n \in \mathbb{N}$,

$$\begin{aligned}\| (t_{mn} + k_{mn}) - (L_1 + L_2) \|_0^+ &= \|t_{mn} - L_1 + k_{mn} - L_2\|_0^+ \\ &\leq \|t_{mn} - L_1\|_0^+ + \|k_{mn} - L_2\|_0^+ \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

and so, we have

$$x_n + y_n \overset{\sigma-FN}{\longrightarrow} L_1 + L_2.$$

□

Theorem 2.4. *Let $x = (x_n)$ be a sequence in X and c be a scalar. If $x_n \overset{\sigma-FN}{\longrightarrow} L$, then $c\,x_n \overset{\sigma-FN}{\longrightarrow} c\,L$.*

Proof. Assume that $x_n \overset{\sigma-FN}{\longrightarrow} L$ and c be a scalar. The proof is clear for $c = 0$. Let $c \neq 0$. Then, for every $\varepsilon > 0$ there exists a $m_0 = m_0(\varepsilon) \in \mathbb{N}$ such that for all $m > m_0$ and every $n \in \mathbb{N}$,

$$\|t_{mn} - L\|_0^+ < \frac{\varepsilon}{|c|}.$$

Therefore, we have

$$\begin{aligned}\|c\,t_{mn} - c\,L\|_0^+ &= |c|\,\|t_{mn} - L\|_0^+ \\ &< |c|\,\frac{\varepsilon}{|c|} = \varepsilon\end{aligned}$$

and so,

$$c\,x_n \overset{\sigma-FN}{\longrightarrow} c\,L.$$

□

Definition 2.5. *A sequence $x = (x_n)$ in X is said to be invariant Cauchy sequence with respect to fuzzy norm if for every $\varepsilon \geq 0$ there exists a $m_0 = m_0(\varepsilon) \in \mathbb{N}$ such that for all $m, p > m_0$,*

$$(1) \qquad D(\|t_{mk} - t_{pl}\|, \widetilde{0}) = \sup_{\alpha\in[0,1]} \|t_{mk} - t_{pl}\|_\alpha^+ = \|t_{mk} - t_{pl}\|_0^+ < \varepsilon,$$

for every $k, l \in \mathbb{N}$.

Theorem 2.6. *Let $x = (x_n)$ be a sequence in X. x is invariant convergent if and only if it is invariant Cauchy sequence with respect to fuzzy norm.*

Proof. Assume x be invariant convergent to L in fuzzy normed space X. That is, for every $\varepsilon > 0$ there exists a $m_0 = m_0(\varepsilon) \in \mathbb{N}$ such that for all $m > m_0$

$$D(\|t_{mk} - L\|, \widetilde{0}) = \sup_{\alpha\in[0,1]} \|t_{mk} - L\|_\alpha^+ = \|t_{mk} - L\|_0^+ < \frac{\varepsilon}{2},$$

for every $k \in \mathbb{N}$. Clearly for every $k, l \in \mathbb{N}$, the inequality

$$\begin{aligned} \|t_{mk} - t_{pl}\|_0^+ &\leq \|t_{mk} - L\|_0^+ + \|t_{pl} - L\|_0^+ \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

is satisfied whenever $m, p > m_0$.

Conversely, let $x = (x_n)$ be invariant Cauchy sequence with respect to fuzzy norm, i.e., for every $\varepsilon > 0$ there exists a $m_0 = m_0(\varepsilon) \in \mathbb{N}$ such that for all $m, p > m_0$

$$D(\|t_{mk} - t_{pl}\|, \widetilde{0}) = \sup_{\alpha \in [0,1]} \|t_{mk} - t_{pl}\|_\alpha^+ = \|t_{mk} - t_{pl}\|_0^+ < \frac{\varepsilon}{2},$$

for every $k, l \in \mathbb{N}$. Fix n, say $n = n_0$. So (1) implies taking $k = l = n_0$ that $(t_{mn_0})_{m=0}^{\infty}$ is a Cauchy sequence and so has a limit, say L. For every $\varepsilon > 0$, there exists a $m_1 = m_1(\varepsilon) \in \mathbb{N}$ such that for all $m > m_1$,

$$\|t_{mn_0} - L\|_0^+ < \frac{\varepsilon}{2}.$$

Thus, for given $\varepsilon > 0$ and arbitrary $n \in \mathbb{N}$, when $m > \max\{m_0, m_1\}$ is taken,

$$\begin{aligned} \|t_{mn} - L\|_0^+ &\leq \|t_{mn} - t_{mn_0}\| + \|t_{mn_0} - L\|_0^+ \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So it can be seen that the sequence x is invariant convergent to L. □

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