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A REMARK ON L^2 EXTENSION THEOREMS

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Abstract. Using recent L^2 extension theorems, we give an analytic proof under some conditions of Zariski's theorem on zero-dimensional base loci. This motivates further discussions on the crucial curvature conditions in L^2 extension theorems.

1. Introduction

In several complex variables, L^2 extension theorems of Ohsawa-Takegoshi type for $Y \subset X$ extend holomorphic sections of line bundles from a submanifold Y to the ambient complex manifold X together with crucial L^2 norm estimates for the extended holomorphic sections. Such theorems have seen numerous applications in complex analysis and algebraic geometry (cf. [1,3,8,9,16,19,20] and many others).

Recently considerable attention has been paid to the 'optimal constants' versions of L^2 extension theorems especially in the domain case: 'optimal' here referring to the constant which appears in the L^2 estimates (such as in (3), (14)). On the other hand, in the compact manifold case (cf. [3,5,8,12,17] and others), current understanding about the necessary curvature conditions and the L^2 norm conditions in L^2 extension theorems seems to be rather far from optimal, leaving much to be investigated further. Indeed, on a compact complex manifold, there exist no nonconstant plurisubharmonic functions, which makes it often highly nontrivial to satisfy the curvature conditions. Also it is well known in algebraic geometry that working with semipositive curvature is much more subtle and difficult than with strictly positive curvature (cf. [4], [22] [1]).

This concern is well illustrated when we apply recent L^2 extension theorems to obtain an analytic proof, under some hermitian semipositivity conditions, of the following theorem due to Zariski [24] on linear systems in algebraic

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[1] From [22, p. 319], 'Applications to produce holomorphic sections from only nonnegative curvature condition remain mostly an area yet to be explored.'

geometry. See also [7], [15, Remark 2.1.32]. (We will use additive notation for line bundles.)

Theorem 1.1. *Let X be a compact projective complex manifold and let L be a holomorphic line bundle with the property that the base locus* $\mathrm{Bs}(L)$ *is a finite set. Suppose that L and $bL - K_X$ is hermitian semipositive where $b > 0$ is a rational number and K_X is the canonical bundle. Suppose that, for every integer $m \geq 1$, the linear system of mL is generic in the sense of Definition 2.3. Then L is semiample, i.e., the tensor power mL is base point free for some $m \geq 1$.*

Here $bL - K_X$ is a $\mathbf{Q}$-line bundle: hermitian semipositivity of a $\mathbf{Q}$-line bundle is defined as the existence of a smooth hermitian metric with semipositive curvature. The proof uses the recent L^2 extension theorem of Demailly [5, Thm. 2.8] (which generalizes the L^2 extension theorem of Ohsawa [17]) combined with the theory of log canonical centers [10] from algebraic geometry. The generalization in [5] is crucial here since the sections in the linear system do not necessarily cut out the base locus transversally: hence the 'Ψ function' we can construct from the conditions of Theorem 1.1 does not necessarily satisfy the requirements (see Condition 1 below) in [17].

We note that recently there was another completely different analytic proof due to [23] of this result without the additional conditions in Theorem 1.1. The proof in [23] depends on several highly nontrivial technical ingredients including the main result of [2]. We believe that the virtue of our proof of Theorem 1.1 lies in that the idea of proof is rather clear and immediate from applying recent L^2 extension Theorem 2.1 due to [5]. We hope that this will make Theorem 1.1 a nice testing ground for various existing and future formulations of L^2 extension theorems in the compact case. Also to our knowledge, Theorem 1.1 is among the first geometric applications of Theorem 2.1 [5], which requires the crucial examination (as in Lemma 2.4) of the residual volume form $dV_M[\Psi]$. [2]

It would be certainly a natural problem to remove the hermitian semipositivity of L in (our proof of) Theorem 1.1. The obstacle for this problem is in fact common when one tries to apply the L^2 extension theorems in the compact case only using semipositivity. As we will explain, the obstacle to be investigated comes from the well-known curvature conditions in L^2 extension which appear in both Theorem 2.1 [5] and Theorem 1.2 [17] (and many others). For the purpose of this investigation, the simpler setting of Theorem 1.2 will be sufficient in this introduction.

Let M be a compact projective complex manifold. Let $S \subset M$ be a closed connected complex submanifold of dimension $k \leq n := \dim M$. Let E be a holomorphic line bundle on M together with a smooth hermitian metric h.

Ohsawa [17] formulated his L^2 extension theorem in terms of a real-valued function Ψ which defines the submanifold S, replacing the role of holomorphic

[2] On the other hand, the generic condition on mL in Definition 2.3 is on the uniqueness of the involved log canonical places, which is of mild nature and in fact expected to be removed in the future.

equations defining S by such a more flexible function Ψ. Let $\Psi : M \to [-\infty, 0)$ be a quasi-psh function such that $\Psi \in C^\infty(M \setminus S)$ and $\Psi^{-1}(-\infty) = S$. As in [17, p. 4], assume the following:

Condition 1. The quasi-psh function Ψ is locally written (near any point on S) as the sum of $(n-k)\log(|z_{k+1}|^2 + \cdots + |z_n|^2)$ and a smooth bounded function, where $(z_1, \ldots, z_k, z_{k+1}, \ldots, z_n)$ are local coordinates adapted to the submanifold S. The function Ψ induces the residual volume form $dV_M[\Psi]$ as defined in [17, p. 4]. The L^2 extension theorem due to Ohsawa [17] is as follows (a slightly restricted version for the purpose of this paper: see [17] for the full statement).

Theorem 1.2 (Ohsawa, [17, Theorem 4 and erratum], cf. [18, Theorem 3.5])**.** *Let $M, S, (E, h)$ and Ψ be as above. Let K_M be the canonical line bundle of M. Let dV_M be a smooth volume form on M. Suppose that, for some $\delta > 0$, we have curvature conditions*

$$\Theta_{he^{-\Psi}} \geq 0 \quad \text{and} \quad \Theta_{he^{-(1+\delta)\Psi}} \geq 0. \tag{1}$$

Let f be a holomorphic section of the line bundle $(K_M \otimes E)|_S$ on S. If f has the L^2 norm given by

$$\int_S |f|^2_{h \cdot dV_M^{-1}} \, dV_M[\Psi] < \infty, \tag{2}$$

then there exists $F \in H^0(M, K_M \otimes E)$ such that $F|_S = f$ and

$$\int_M |F|^2_{h \cdot dV_M^{-1}} \, dV_M \leq (C + \delta^{-\frac{3}{2}}) \int_S |f|^2_{h \cdot dV_M^{-1}} \, dV_M[\Psi]. \tag{3}$$

Note that the line bundles K_M and E are equipped with the hermitian metrics $(dV_M)^{-1}$ and h, respectively, in (3). It is well known that both L^2 norms in (3) do not depend on the choice of dV_M. The statement with $\delta = 0$ has counterexamples (see e.g. [12, Example 4.3]).

In order to apply Theorem 1.2 (and Theorem 2.1) to given manifolds and line bundles, it is absolutely important to find the right pair of h and Ψ. In practice, perhaps the only way to find h and Ψ may be to construct $he^{-\Psi}$ first, as in (4) in the below, using some holomorphic sections $s_1, \ldots, s_m$ (if exist) of E which generate the ideal sheaf of the submanifold S. Then for a choice of a smooth hermitian metric h (without requiring curvature conditions), one can determine Ψ by the same relation (4) (which is equality of singular hermitian metrics of E):

$$he^{-\Psi} = \frac{1}{\sum_{j=1}^m |s_j|^2}. \tag{4}$$

But there still remains the problem of satisfying the second curvature condition of (1) which can be written as $\Theta_{he^{-\Psi}} + \delta\sqrt{-1}\partial\overline{\partial}\Psi \geq 0$. In the compact case, this is usually not easy to satisfy one reason being that the only global psh functions are constant. We will use hermitian semipositivity of line bundles to satisfy this in the proof of Theorem 1.1 (see also Remark 2.5). Then what can one do in general to satisfy these curvature conditions, in the absence of such hermitian semipositivity? Since we observe that geometrically intrinsic statements often lead to optimal curvature conditions in algebraic geometry (cf. [4]), we raise the following:

Question 1.3. Does there exist a variant/generalization of Theorem 1.2 (and Theorem 2.1) whose statement depends only on $he^{-\Psi}$?

In other words, here we mean a statement which is invariant when we change the pair (h, Ψ) to $(\tilde{h}, \tilde{\Psi})$ satisfying $he^{-\Psi} = \tilde{h}e^{-\tilde{\Psi}}$. We give the following partial evidence to Question 1.3.

Theorem 1.4. *In Theorem 1.2 of Ohsawa* [17]*, the L^2 norm* (2) *of the section f on S,*

$$\int_S |f|^2_{h \cdot dV_M^{-1}} \, dV_M[\Psi] \tag{5}$$

depends only on $he^{-\Psi}$, not separately on h and on $e^{-\Psi}$. In other words, the value of (5) *is invariant when we change (h, Ψ) to $(\tilde{h}, \tilde{\Psi})$ satisfying $he^{-\Psi} = \tilde{h}e^{-\tilde{\Psi}}$.*

In fact, we do not need projectivity or compactness of M in Theorem 1.4. The proof uses results from [8].

Remark 1.5. In Question 1.3, one needs also to ask whether the L^2 norm of the extended section $\int_M |F|^2_{h \cdot dV_M^{-1}} \, dV_M$ in (3) (which does not involve Ψ) can also be replaced by a variant that depends (only) on $he^{-\Psi}$. For this, we note that there is a counterpart L^2 norm on the LHS of (8) in Theorem 2.1 [5] which involves the extra pole of $\gamma(\delta\Psi)e^{-\Psi}$. Such a counterpart will be more suitable for this aspect of the question.

The rest of this paper is organized as follows. In Section 2, we give the proof of Theorem 1.1 using the L^2 extension Theorem 2.1 together with algebro geometric ideas involving log canonical centers. In Section 3, we give the proof of Theorem 1.4.

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2. Proof of Theorem 1.1

We will use the following generalization of Theorem 1.2 due to Demailly [5]. See [5, Theorem 2.8] for the full statement (which is in fact stated for a weakly pseudoconvex Kähler manifold) and more details.

Theorem 2.1 (Demailly, [5, Thm. 2.8]). *Let $M,(E,h)$, K_M, dV_M be as in Theorem 1.2. Let $\Psi : M \to [-\infty,\infty)$ be a quasi-psh function with log canonical neat analytic singularities, so that the subscheme S associated to the multiplier ideal $\mathcal{J}(\Psi)$ is a reduced subvariety. Suppose that, for some $\delta > 0$, we have curvature conditions*

$$\Theta_{he^{-(1+\alpha)\Psi}} \geq 0 \tag{6}$$

for all $\alpha \in [0,\delta]$. Let f be a holomorphic section of the line bundle $(K_M \otimes E)|_S$ on S^0, the regular locus of S. If f has the L^2 norm given by

$$\int_{S^0} |f|^2_{h\cdot dV_M^{-1}}\, dV_M[\Psi] < \infty, \tag{7}$$

then there exists $F \in H^0(M, K_M \otimes E)$ such that $F|_S = f$ and

$$\int_M \gamma(\delta\Psi)e^{-\Psi}\, |F|^2_{h\cdot dV_M^{-1}}\, dV_M \leq \frac{34}{\delta}\int_{S^0} |f|^2_{h\cdot dV_M^{-1}}\, dV_M[\Psi], \tag{8}$$

where γ is as given in [5, Thm. 2.8].

We note that the curvature condition (6) is of the similar nature as (1). The important difference compared to Theorem 1.2 is that now Ψ does not have to satisfy Condition 1 (which should be viewed as the simplest case of log canonical singularity).

Now toward the proof of Theorem 1.1, let L on X be as in Theorem 1.1. We may assume $\dim X \geq 2$. Denote the $\mathbf{Q}$-line bundle $bL - K_X$ by F. Let $s_1,\dots,s_m \in H^0(X,L)$ be a basis of the linear system. Define a singular hermitian metric g of L by

$$g = \frac{1}{\sum_{i=1}^m |s_i|^2}. \tag{9}$$

It is easy to see that g is uniquely defined up to equivalence of singularities (which is as defined in [4]) independent of the choice of a basis. By the condition that the base locus of the linear system $H^0(X,L)$ is a finite set of points (say $Z := \{p_1,\dots,p_k\} \subset X$), the metric g is locally bounded outside these points.

At this point, we recall the theory of singularity of pairs from birational geometry as in [10] (cf. [13], [15, §9]) in this special case (in particular, X is a smooth variety). Since the metric g is defined by sections, it has a log-resolution $\mu : X' \to X$, i.e., a projective birational morphism from a smooth projective X' such that μ is isomorphism outside an snc (simple normal crossing) divisor and μ^*g has poles along an snc divisor which we will denote by $[\mu^*g]$. We can write down the equality of finite sums of snc divisors

$$K_{X'/X} - [\mu^*g] = \sum c_i E_i, \tag{10}$$

where E_i's are exceptional divisors for μ. The coefficient c_i is called the discrepancy of E_i with respect to the pair (X,g). We will say that the pair (X,g) is log canonical if all $c_i \geq -1$.

Definition 2.2. We will say that the pair (X, g) is log canonical if all $c_i \geq -1$. Moreover, for i with $c_i = -1$, we will say that the image $\mu(E_i)$ (which is a point among $p_1, \ldots, p_k$ in this case) is a log canonical center, or for short, an lc center of the pair (X, g).

Following [10], we will also call the exceptional divisor E_i an lc place of the lc center $\mu(E_i)$.

The pair (X, g) given from (9) may not be log canonical, but it is clear from the log-resolution $\mu : X' \to X$ that there exists a rational number $a > 0$ such that, for the metric g^a for the $\mathbf{Q}$-line bundle aL, the pair (X, g^a) is log canonical. Let Y denote the (reduced) subvariety associated to the multiplier ideal of g^a. Then Y is a subset $Y := \{p_1, \ldots, p_l\}$ of Z where $1 \leq l \leq k$. In the language of log canonical centers (cf. [10]), each point in Y is a minimal lc center (i.e., minimal with respect to inclusion of lc centers) since it is 0-dimensional. If a minimal lc center has a unique lc place, it is called an *exceptional* minimal lc center [14]. Contrary to the adjective, an exceptional minimal lc center is usually considered as the 'generic' case of a minimal lc center (cf. [14]). In view of this, we define the following mild condition which was used in the statement of Theorem 1.1.

Definition 2.3. We will say that the linear system of L is generic if, for the above g^a, every point in Y has a unique lc place (i.e., exceptional divisor with discrepancy -1).

It is clear that this property depends only on L, not on the choice of a basis in (9).

Proof of Theorem 1.1. Recall that $a, b > 0$ are rational numbers which appeared in the above: in particular, $bL - K_X$ is hermitian semipositive.

Let $c > 0$ be the smallest positive rational number such that $d := a+b+c \geq 1$ is an integer. We have the equality of $\mathbf{Q}$-line bundles $dL = (a + b + c)L = K_X + F + aL + cL$. Since F and L are hermitian semipositive, we can fix smooth hermitian metrics with semipositive curvature, (F, ϕ) and (L, η). Define the quasi-psh function Ψ by

$$g^a = \eta^a e^{-\Psi} \tag{11}$$

which is an equality of two singular hermitian metrics of the $\mathbf{Q}$-line bundle aL.

Let E be the line bundle $F + aL + cL$ equipped with the smooth hermitian metric $h := \eta^{a+c}\phi$. Taking $\delta = \frac{c}{a} > 0$, this satisfies the curvature condition (6) since

$$\Theta_{he^{-(1+\alpha)\Psi}} = (1 + \alpha)\Theta_{\eta^a e^{-\Psi}} + (c - a\alpha)\Theta_\eta + \Theta_\phi \geq 0$$

for every $0 \leq \alpha \leq \delta$.

Now we can apply Theorem 2.1 (taking $S = Y$) to extend arbitrary holomorphic functions f on $Y = \{p_1, \ldots, p_l\}$ to X: thanks to Lemma 2.4, there are no restrictions on the values $f(p_i), (1 \leq i \leq l)$ for the extension. When we take $f(p_j) = 1$ for $1 \le j \le l$, this means that $p_1, \dots, p_l$ are not base points of the linear system dL.

Starting from the linear system of L with k base points, we obtained dL whose linear system has $k - l$, or less, base points where $0 \le k - l \le k - 1$ (recall that the base locus of dL is contained in that of L). By repeating the same arguments to dL, it is clear that L is semiample. □

This proof used the following lemma on $dV_M[\Psi]$, which can be viewed as analogous to Kawamata subadjunction [11] in this special case. From Theorem 2.1, in general, $dV_M[\Psi]$ is a singular volume form defined on each connected component of the regular locus of the (possibly reducible) reduced subvariety S.

Note that S itself is not necessarily connected. Assume that S has a connected component W which is a minimal lc center of the pair (X, Ψ) (in the place of the pair (X, g^a) in our case) having a unique lc place $T \subset X'$ in a log resolution $\mu : X' \to X$ of the pair. Then, thanks to [5, Proof of Prop. 4.5(a)], $dV_M[\Psi]$ is given by the fiberwise integration of a singular volume form ω (which can be specified) on T along the morphism $T \to W$ which is induced from the log resolution $\mu : X' \to X$. We will apply this when the dimension of W is zero.

Lemma 2.4. *For Ψ defined by* (11) *under the conditions of Theorem 1.1, the residual volume form $dV_M[\Psi]$ is locally integrable at each point of $Y = \{p_1, \dots, p_l\}$.*

Proof. We will follow the argument (and some notation) of [5, Proof of Prop. 4.5(a)]. The singular volume form $dV_M[\Psi]$ is simply given by pointwise masses m_j supported at each point p_j so that $\int_Y f dV_M[\Psi] = m_1 f(p_1) + \cdots + m_l f(p_l)$. Among these points, consider p_1 without loss of generality, taking $W = \{p_1\}$ in the above paragraph. We rewrite (10) for g^a in the place of g as follows taking T to be E_1:

$$K_{X'/X} - [\mu^* g^a] = \sum c_i E_i = -E_1 + \sum_{i \neq 1} c_i E_i. \tag{12}$$

The fiberwise integration we need to consider along $T \to W$ simplifies to the usual integration on T since W is a point. From [5, Proof of Prop. 4.5(a)] (using its notations), the singular volume form ω on T is given by restriction to T of a singular volume form on X' which has only divisorial poles. The divisorial poles can be locally written as

$$\frac{\left|w^{ca-b}\right|^2}{\left|w_k\right|^2}$$

in local coordinates $w = (w_1, \dots, w_n)$. Here $|w^{ca}|^2 := \prod |w_i|^{2ca_i}$ is determined by $\mu^* \Psi = c \log |w^a|^2 + u(w)$ where u is a smooth function. Also $w^b := \prod {w_i}^{b_i}$ is a local defining function of the relative canonical divisor $K_{X'/X}$ and $w_k = 0$

is a local defining function of the divisor $E_1 \subset X'$. Hence $\frac{w^{ca-b}}{w_k}$ is a (locally written) function with poles along an snc divisor Γ which can be determined from (12): namely we have $\Gamma = \sum_{i\neq 1}(-c_i)E_i$.

Therefore the singular volume form ω on $T = E_1$ is the one with poles along the restriction $\Gamma|_{E_1} = \sum_{i\neq 1}(-c_i)(E_i)|_{E_1}$. While $-c_i \leq 1$ for every i, if $(E_i)|_{E_1}$ is nonzero (i.e., $E_i \cap E_1 \neq \emptyset$), then $-c_i < 1$ since W is an exceptional minimal lc center. Thus ω is locally integrable and it follows that its fiberwise integration, i.e., the residual volume form $dV_M[\Psi]$ is also locally integrable at p_1. In other words, $dV_M[\Psi]$ are represented by nonnegative real numbers $m_1, \ldots, m_l$ (given in the beginning), not by infinity. □

Remark 2.5. If we allow η to be a singular hermitian metric with semipositive curvature (instead of smooth), then from (11), Ψ should be the difference of two (quasi-)psh functions. Such a case is not allowed in [5, (2.8), (2.9)(b)].

Remark 2.6. It is easy to see that similar (but much simpler, only using Theorem 1.2) arguments used in the proof of Theorem 1.1 can be applied to improve the extension result in the proof of Theorem 1.11 of [6]: the hypothesis that the 0-dimensional Y is a complete intersection can be removed.

Remark 2.7. In [21, 1.8.4], using the Skoda division theorem (another major consequence of L^2 estimates for $\overline{\partial}$ which indeed influenced the discovery and development of L^2 extension theorems according to Ohsawa), Yum-Tong Siu showed the following result which is related to the setting of Theorem 1.1. Let L be a semiample holomorphic line bundle on a compact projective complex manifold X. Suppose that $a \geq 1$ and $b \geq 0$ are integers such that aL and $bL - K_X$ are base points free on X. Then the section ring $\oplus_{m=0}^{\infty} H^0(X, mL)$ is generated by those sections in $H^0(X, mL)$ for $0 \leq m \leq (n+2)a + b - 1$ where $n = \dim X$.

3. Proof of Theorem 1.4 and further remarks

3.1. Proof of Theorem 1.4

Proof. Let $he^{-\Psi} = \tilde{h}e^{-\tilde{\Psi}}$ be two possible choices of the pair (h, Ψ). Let $\beta = \frac{\tilde{h}}{h}$ be the smooth function defined as the quotient of the smooth hermitian metrics. We will show the equality of the integrands

$$|f|^2_{h \cdot dV_M^{-1}}\, dV_M[\Psi] = |f|^2_{\tilde{h} \cdot dV_M^{-1}}\, dV_M[\tilde{\Psi}].$$

Note that the pointwise lengths $|f|$ are with respect to $h \cdot (dV_M)^{-1}$ and $\tilde{h} \cdot (dV_M)^{-1}$, respectively, two smooth hermitian metrics of the line bundle $K_M \otimes E$ and its restriction to S.

Thus it suffices to show that $dV_M[\tilde{\Psi}]$ is equal to $\frac{1}{\beta}$ times $dV_M[\Psi]$. In order to show this, let U be a neighborhood of an arbitrary point $p \in S$ with adapted local coordinates $(z_1, \ldots, z_n)$ such that $U \cap S$ is defined by $z_{k+1} = \cdots = z_n = 0$.

On U, we can write $dV_M = g(z)\,|dz_1 \wedge \cdots \wedge dz_n|^2$ for some smooth function g. By the conditions on Ψ and $\tilde{\Psi}$, we have (on U)

$$\Psi = (n-k)\log(|z_{k+1}|^2 + \cdots + |z_n|^2) + \varphi,$$
$$\tilde{\Psi} = (n-k)\log(|z_{k+1}|^2 + \cdots + |z_n|^2) + \tilde{\varphi}$$

for some smooth bounded functions $\varphi, \tilde{\varphi}$ on U. From [8, Lemma 4.14] (which generalizes the codimension 1 case in [17]), we have

$$dV_M[\Psi] = e^{-\varphi(z)} g(z) d\lambda_{z'}, \quad dV_M[\tilde{\Psi}] = e^{-\tilde{\varphi}(z)} g(z) d\lambda_{z'},$$

where $d\lambda_{z'} = |dz_1 \wedge \cdots \wedge dz_k|^2$ is the Lebesgue measure on $S \cap U$. Now from the condition $he^{-\Psi} = \tilde{h}e^{-\tilde{\Psi}}$, we have $e^{-\tilde{\varphi}(z)} = \frac{1}{\beta} e^{-\varphi(z)}$, which completes the proof. □

3.2. Further remarks on the L^2 extension of Guan and Zhou

Theorem 1.4 also applies to the following 'optimal constants' version of L^2 extension Theorem 3.1 from [8]. In order to have comparison with Theorem 1.2, we will take the auxiliary function c_A in [8, Theorem 2.1] to be the constant function 1.

Theorem 3.1 (Guan and Zhou, [8, Theorem 2.1])**.** *Let M, (E, h) and S be as in Theorem 1.2. Let $\Psi : M \to [-\infty, A)$ be as in* [8, Theorem 2.1] *where $A \in \mathbf{R}$. Replace* (1) *by the following curvature condition in* [8, Theorem 2.1] (*which is more complicated than* (1)): *for some $\delta > 0$,*

$$\sqrt{-1}\Theta_{he^{-\Psi}} \geq 0, \quad \sqrt{-1}\Theta_{he^{-\Psi}} + \frac{1}{a(-\Psi)}\sqrt{-1}\partial\bar{\partial}\Psi \geq 0, \tag{13}$$

where $a(t)$ is a continuous function for $t \in (-A, +\infty]$ which satisfies

$$0 < a(t) \leq s(t) := \frac{(\delta+1)(t+A)e^A + \delta(e^{-t} - e^A) + \frac{1}{\delta}e^A}{e^A + \delta(e^A - e^{-t})}$$

(*note that $s(t) > 0$ as seen from its definition as in* [8, Theorem 2.1]). *Then we have the same kind of extension statement as in Theorem 1.2 with the estimate*

$$\int_M |F|^2_{h\cdot dV_M^{-1}}\, dV_M \leq (1 + \frac{1}{\delta})e^A \frac{\pi^{n-k}}{(n-k)!}\int_S |f|^2_{h\cdot dV_M^{-1}}\, dV_M[\Psi]. \tag{14}$$

Note that Ψ in this theorem is not restricted to be negative valued (as in Theorem 1.2). From this, we observe that the fact from Theorem 1.4 that $\int_M |F|^2_{h\cdot dV_M^{-1}}\, dV_M$ does not involve Ψ, does not lead to inconsistency: suppose that while we keep $he^{-\Psi}$ the same, we multiply h by a constant $\alpha > 0$ and $e^{-\Psi}$ by $\frac{1}{\alpha}$.

From the estimates (3) and (14), we do not have inconsistency since in Theorem 3.1, we have the factor e^A in (14) while in Theorem 1.2, the range of the function Ψ is limited to be $[-\infty, 0)$ (i.e., $A = 0$ in the notation of [8]). In

Theorem 3.1, Ψ becomes $\Psi + \log \alpha$ and A becomes $A + \log \alpha$, hence both sides of (14) are multiplied by α.

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