MATHEMATICAL MODELLING FOR THE AXIALLY MOVING PLATE WITH INTERNAL TIME DELAY

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ABSTRACT. In [1, 2], we studied the string-like system with time-varying delay. Unlike the string system, the plate system must consider both longitudinal and transverse strains. First, we consider the physical phenomenon of an axially moving plate concerning kinetic energy, potential energy, and work done. By the energy conservation law in physics, we have a nonlinear plate-like system with internal time delay.

1. Introduction

Many string-like systems induced the mathematical modeling are studied on mathematical fields (See [1, 3, 4, 5, 7]). In the string-like system, the main equation is only expressed in a single form because the longitudinal displacement is minimal. We intend to mathematically model a system that considers both transverse and longitudinal strains. That is ‘plate.’ In this work, we focus on the axially moving plate.

Similar to [2], we focus on the time variable relating to the damping term. It is related to the energy generated from the boundary inward when the boundary moves. The energy generated at this time is non-conservative work done $WD_{nc}$. In case of the work done defined by the non-conservative forces $f(x, t)$ in internal of domain and $f_c(t)$ at the boundary $x = l(t)$. In the free boundary, the axially moving plate may slip. Therefore, it is necessary to define a different time. It is necessary to express the time differently from normal time $t$. I will express it in $\tau$ here. An also, work done on the outward direction at the right boundary that is free is $WD_{rb}$. Therefore, $WD = WD_{nc} + WD_{rb}$ on the near the right boundary. Note that work done on the fixed boundary $x = 0$ is zero.

The internal time delay inside the system occurs due to interference (actually caused by the influence of both longitudinal and transverse strain of the plate) or time difference between the fixed roller and the moving roller. This phenomenon

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reduces the product’s value by preventing the scroll of the system from being wound constantly. So, it is necessary to control the time delay phenomenon.

Our purpose in this work derives a plate-like model with internal time delay. The plate-like model considering the longitudinal direction and the transverse direction is derived from the coupled equation considering the two displacements. It is significant for modeling industrial sites such as steel plate production. In addition, the partial differential equation derived in this way is helpful for research extending to the third-order or higher.

This work is organized into some processes. First, we check some needed physical variables. Using the variables, we consider the kinetic energy, potential energy, and work done in great detail. Next, we consider the variation for energy. Using the energy conservation and the variation lemma, Hamilton’s principle, integration by parts, we get the initial-boundary problem for nonlinear third-order PDE.

2. Mathematical modelling

The plate on the mass production process is steered axially through two ends, which are spaced apart by a distance of \( l(t) \). The membrane has two variables, \( r, s \) is the displacement of plate under the variables. The first variable is the spatial part \( x \). The range of \( x \) is \( 0 \leq x \leq l(t) \). Indeed, we consider suitable time \( t \) of \( l(t) \) is fixed. It is a part of the plate-like process. More delicately, the part is roller to roller of the axially moving plate. In the case of the first boundary \( x = 0 \), physically, the roller’s shaft \( (r(0, t) \) and \( s(0, t) \)) is fixed. In the other case \( l(t) \), of the roller’s transversal shaft \( (s(1, t)) \) is mechanically free. But longitudinal shaft \( (r(0, t)) \) is fixed. The next variable is time \( t \) Over time, the string moves in a high-speed moving axial direction. At this time, a slip phenomenon occurs in the inner area with the right border \( l(t) \). For applying the variation Lemma, we set the variations. The variation of \( s \) and \( r \) is \( \phi \) and \( \psi \), respectively. Some variables and constants that will be used in this paper are as follows:
\[
\begin{align*}
&v > 0 : \text{moving speed for axial direction;} \\
r \quad : \text{logitudinal displacement of the plate moving;} \\
s \quad : \text{transversal displacement of the plate moving;} \\
(\cdot)_t = \partial(\cdot)/\partial t \quad : \text{the partial derivative for time;} \\
(\cdot)_x = \partial(\cdot)/\partial x \quad : \text{the partial derivative for domain value;} \\
i,j \quad : \text{the standard basis vectors;} \\
\{v + vr_x + r_t\}i + \{vs_x + s_t\}j \quad : \text{transversal velocity of the plate moving;} \\
C(x) \quad : \text{the area of cross-section;} \\
\varpi(x) \quad : \text{mass per unit(weight);} \\
Y \quad : \text{Young’s elastic modulus;} \\
\sigma(x,t) \quad : \text{tensile stress;} \\
\zeta(x,t) \quad : \text{strain;} \\
\iota_0 \quad : \text{initial tension of plate.}
\end{align*}
\]

We also define certain energies and vibrations physically. \(K\) is kinetic energy. \(P\) is potential energy. \(\delta WD_{nc}\) is the variation of non-conservative work done. \(\delta WD_{rb}\) is the variation of work done at the right boundary. More specifically, all of them are given by

\[
\begin{align*}
K &= \frac{1}{2} \int_0^{l(t)} \varpi(x)C(x)[(v + vr_x + r_t)^2 + (vs_x + s_t)^2]dx, \\
P &= \int_0^{l(t)} \left[ \left( \iota_0 + \frac{YC(x)}{4} \right) \int_0^1 \left( \frac{\partial s}{\partial x} \right)^2 dx \right] \zeta(x,t)dx + \frac{1}{2} C(x)\sigma(x,t)\zeta(x,t) \right] \zeta(x,t)dx, \\
\delta WD_{nc} &= \int_0^{l(t)} \left[ \psi(x,t) - \varpi(x)s_t(x,t - \tau) \right] \delta s(x,t)dx + \psi_c(t)\delta s(l(t),t), \\
\delta WD_{rb} &= \varpi(l(t))C(l(t))\varpi(s_x(l(t),t) + s_t(l(t),t))\delta s(l(t),t).
\end{align*}
\]

Now, we calculate the potential energy \(P\) in more detail. Let \(Z\) and \(\zeta(x,t) = \frac{dy - dx}{dx}\) (where \(|dx| = 1\)) be the plate’s tension and the strain under physical situations, respectively.

Then, we have

\[
\begin{align*}
P &= \int_0^{l(t)} Z\zeta(x,t)dx \\
&= \int_0^{l(t)} \left[ \sigma(\varepsilon(x,t)) + \frac{\sigma'(\varepsilon(x,t))}{2} \int_0^{l(t)} \left( \frac{\partial s}{\partial x} \right)^2 dx \right] \zeta(x,t)dx.
\end{align*}
\]
Next we set \( \sigma(\varepsilon(x,t)) = \iota_0 + \frac{YC(x)}{2} \zeta(x,t)\varepsilon(x,t) \) with \( \iota_0 \). Therefore the potential energy \( P \) changes

\[
P = \int_0^{l(t)} \left[ \iota_0 + \frac{YC(x)}{2} \zeta(x,t)\varepsilon(x,t) + \frac{YC(x)}{4} \int_0^{l(t)} \left( \frac{\partial s}{\partial x} \right)^2 dx \right] \zeta(x,t) dx.\]

Because of \( \sigma(\varepsilon(x,t)) \) which is defined by \( \sigma(x,t) \), we can apply \( \sigma(\varepsilon(x,t)) = Y\varepsilon(x,t) \). So, we finally get

\[
P = \int_0^{l(t)} \left[ \iota_0 + \frac{YC(x)}{4} \int_0^{1} \left( \frac{\partial s}{\partial x} \right)^2 dx + \frac{1}{2} C(x) \zeta(x,t)\sigma(x,t) \right] \zeta(x,t) dx.\]

By the Taylor’s theorem, we get the strain

\[
\zeta(x,t) = \left[ \left\{ dx + \frac{\partial r}{\partial x} dx \right\}^2 + \left\{ \frac{\partial s}{\partial x} dx \right\}^2 \right]^{\frac{1}{2}} - dx \]

\[
= \left[ 1 + 2 \frac{\partial r}{\partial x} + \left\{ \frac{\partial r}{\partial x} \right\}^2 + \frac{1}{2} \frac{\partial r}{\partial x} \left\{ \frac{\partial s}{\partial x} \right\}^2 + \frac{1}{4} \left\{ \frac{\partial s}{\partial x} \right\}^4 + \ldots \right]^{\frac{1}{2}} - 1 \]

\[
\approx \frac{\partial r}{\partial x} + \frac{1}{2} \left( \frac{\partial s}{\partial x} \right)^2 \ll 1 \]

So, we approximately have

\[
P = \int_0^{l(t)} \left[ \left( \iota_0 + \frac{YC(x)}{4} \int_0^{l(t)} \left( \frac{\partial s}{\partial x} \right)^2 dx \right) \left( \frac{\partial r}{\partial x} + \frac{1}{2} \left\{ \frac{\partial s}{\partial x} \right\}^2 \right) + \frac{1}{2} C(x) \sigma(x,t) \left( \frac{\partial r}{\partial x} + \frac{1}{2} \left\{ \frac{\partial s}{\partial x} \right\}^2 \right)^2 \right] dx.\]

From now on, we start calculating the variation between kinetic and potential energy. By using the Gâteaux derivative, we get variations of \( K \) and \( P \) like as:

\[
\delta K(r; \psi, s; \phi) = \lim_{\varepsilon \to 0} \frac{K(r + \varepsilon \psi, s + \varepsilon \phi) - K(r, s)}{\varepsilon} \]

\[
= \int_0^{l(t)} \varpi(x) C(x) \left[ \varpi (\varpi + \varpi r_x + r_t) \psi_x + (\varpi + \varpi r_x + r_t) \psi_t \right. \]

\[
+ \varpi (\varpi s_x + s_t) \phi_x + (\varpi s_x + s_t) \phi_t \left. \right] dx, \tag{5} \]
\[\delta P(r; \psi, s; \phi) = \lim_{\varepsilon \to 0} \frac{P(r + \varepsilon \psi, s + \varepsilon \phi) - P(r, s)}{\varepsilon}\]
\[= \int_0^{l(t)} \left( \iota_0 + \frac{YC(x)}{4} \int_0^{l(t)} \left( \frac{\partial s}{\partial x} \right)^2 \, dx \right) [\psi_x + s_x \phi_x] \, dx \]
\[+ \int_0^{l(t)} C(x)\sigma(x, t) \left[ \left( r_x + \frac{1}{2} s_x \right) \psi_x + \left( r_x + \frac{1}{2} s_x \right) s_x \phi_x \right] \, dx,\]

where \(\phi\) is the \(C^1\) function which depends on \(x\) and \(t\).

Apply for the Hamilton’s Principle, (3)-(4) and (5)-(6) are as follows:

\[\int_{t_0}^{t_1} (\delta K - \delta P + \delta WD_{nc} - \delta WD_{rb}) \, dt = 0, \text{ for all } t \in [t_0, t_1]\] (7)

Accordingly, (7) can be replaced with

\[\int_{t_0}^{t_1} \int_0^{l(t)} \varpi(x)C(x) \left[ \overline{v}(\overline{v} + r_x + r_t)\psi_x + (\overline{v} + \overline{v}r_x + r_t)\psi_t \right.\]
\[+ \overline{v}(\overline{v}s_x + s_t)\phi_x + (\overline{v}s_x + s_t)\phi_t \big] \, dx \, dt \]
\[+ \int_{t_0}^{t_1} \int_0^{l(t)} \left( \iota_0 + \frac{YC(x)}{4} \int_0^{l(t)} \left( \frac{\partial s}{\partial x} \right)^2 \, dx \right) [\psi_x + s_x \phi_x] \, dx \, dt \]
\[+ \int_{t_0}^{t_1} \int_0^{l(t)} C(x)\sigma(x, t) \left[ \left( r_x + \frac{1}{2} s_x \right) \psi_x + \left( r_x + \frac{1}{2} s_x \right) s_x \phi_x \right] \, dx \, dt \]
\[+ \int_{t_0}^{t_1} \int_0^{l(t)} \left[ \psi(x, t) - \frac{\varpi(x)}{2} s_t(x, t - \tau) \right] \phi \, dx \, dt \]
\[+ \int_{t_0}^{t_1} \left[ F_c(t)\phi(l(t), t) - \varpi(l(t))C(l(t))\overline{v}(\overline{v}s_x(l(t), t) + s_t(l(t), t))\phi(l(t), t) \right] \, dt = 0\]

(8)
By using the integration by parts, we deduce

\[
\int_{t_0}^{t_1} \left[ \varpi(l(t))C(l(t)) (\varpi^2 + \varpi^2 s_x l(t), t) + \varpi s_t l(t), t) \right] dt
- \frac{1}{2} \left( t_0 + \frac{YC(l(t))}{4} \int_0^{l(t)} \left( \frac{\partial s}{\partial x} \right)^2 dx \right)
+ C(l(t)) \sigma(l(t), t) \left( r_x (l(t), t) + \frac{1}{2} s_x^2 (l(t), t) \right) \psi(l(t), t) dt
+ \int_{t_0}^{t_1} \left[ \varpi(0) C(0) (\varpi^2 + \varpi^2 s_x (0, t) + \varpi s_t (0, t)) \right] \psi(0, t) dt
- \varpi(l(t)) C(l(t)) + C(x) \sigma(x, t) \left( r_x (l(t), t) + \frac{1}{2} s_x^2 (l(t), t) \right) \psi(l(t), t) dt
- \int_{t_0}^{t_1} \left[ \varpi(0) C(0) (\varpi^2 s_x (0, t) + \varpi s_t (0, t)) \right] \phi(0, t) dt
- \frac{1}{2} \left( t_0 + \frac{YC(l(t))}{4} \int_0^{l(t)} \left( \frac{\partial s}{\partial x} \right)^2 dx \right)
+ \varpi(x) C(x) (\varpi s_{xt} (x, t_1) + s_{tt} (x, t_1)) \phi(x, t_1) dx
+ \int_{t_0}^{t_1} \varpi(x) C(x) (\varpi s_{xt} (x, t_0) + s_{tt} (x, t_0)) \phi(x, t_0) dx
+ \int_{t_0}^{t_1} \varpi(x) C(x) (\varpi r_{xt} (x, t_1) + r_{tt} (x, t_1)) \psi(x, t_1) dx
+ \int_{t_0}^{t_1} \varpi(x) C(x) (\varpi r_{xt} (x, t_0) + r_{tt} (x, t_0)) \psi(x, t_0) dx
- \int_{t_0}^{t_1} \int_0^{l(t)} \left\{ \varpi(x) C(x) (\varpi^2 r_{xx} + 2 \varpi r_{tt} - r_{tt}) - C(x) \sigma(x, t) \left( r_x + \frac{1}{2} s_x^2 \right) \psi(x, t) \right\} dx dt
+ \varpi(x) C(x) \left( \varpi^2 - \frac{1}{\varpi(x) C(x)} \left( t_0 + \frac{YC(l(t))}{4} \int_0^{l(t)} \left( \frac{\partial s}{\partial x} \right)^2 dx \right) s_{xx} + 2 \varpi s_{xt} + s_{tt} \right)
- \varpi(x) C(x) \left\{ s_x \left( r_x + \frac{1}{2} s_x^2 \right) \right\} x - \left[ \psi(x, t) - \varpi(x) \varpi \sigma(x, t - \tau) \right] \phi(x, t) dx dt = 0.
\]
Applying the variational lemma, finally we get the following coupled system considering internal time delay

\[
\frac{\varpi(x)}{\sigma(x,t)} (\varpi^2 r_{xx} + 2\varpi r_{xt} - r_{tt}) = \left(r_x + \frac{1}{2} s_x^2 \right)_x
\]

in \((0, l(t)) \times (0, T)\),

\[
\varpi^2 \left(\frac{1}{\varpi(x)C(x)} \left(t_0 + \frac{YC(l(t))}{4} \int_0^{l(t(t))} \left(\frac{\partial s}{\partial x}\right)^2 dx \right) s_{xx} + 2\varpi s_{xt} + s_{tt}\right)
\]

\[
\left. - \frac{1}{\varpi(x)C(x)} \left[\psi(x,t) - \frac{\varpi(x)}{2} s_t(x, t - \tau)\right] = \left\{ s_x \left(r_x + \frac{1}{2} s_x^2 \right) \right\}_x \right) \quad (10)
\]

the boundary conditions on \((0, T)\), as

\[
F_c(t) - \frac{1}{2} \left\{ \left(t_0 + \frac{YC(l(t))}{4} \int_0^{l(t(t))} \left(\frac{\partial s}{\partial x}\right)^2 dx \right) + \varpi(l(t))C(l(t)) \right\} s_x(l(t), t)
\]

\[
= \varpi(l(t))C(l(t)) - C(x)\sigma(x, t) \left(r_x(l(t), t) + \frac{1}{2} s_x^2(l(t), t) \right)
\]

\[
r(0, t) = r(l(t), t) = s(0, t) = 0 \quad (11)
\]

and the initial conditions as

\[
s(x, 0) = s_0(x)
\]

\[
s_t(x, 0) = s_1(x)
\]

\[
r(x, 0) = r_0(x)
\]

\[
r_t(x, 0) = r_1(x)
\]

in \((0, l(t))\).

3. Conclusion

The mathematical analysis of the coupled system (9)-(12) is significant in the field of partial differential equations. The derivation of the axially moving plate-like equation to modeling can actually be extended to the Timoshenko beam equation. (See [10]) To control the time internal delay, a mathematical analysis of the optimal control system associated with time delay is required for free boundary terms (the first equation in (11)) as future works.

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References


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