East Asian Math. J.
Vol. 37 (2021), No. 5, pp. 619–626
http://dx.doi.org/10.7858/eamj.2021.041

MATHEMATICAL MODELLING FOR THE AXIALLY MOVING PLATE WITH INTERNAL TIME DELAY

Daewook Kim

Abstract. In [1, 2], we studied the string-like system with time-varying delay. Unlike the string system, the plate system must consider both longitudinal and transverse strains. First, we consider the physical phenomenon of an axially moving plate concerning kinetic energy, potential energy, and work dones. By the energy conservation law in physics, we have a nonlinear plate-like system with internal time delay.

1. Introduction

Many string-like systems induced the mathematical modeling are studied on mathematical fields (See [1, 3, 4, 5, 7]). In the string-like system, the main equation is only expressed in a single form because the longitudinal displacement is minimal. We intend to mathematically model a system that considers both transverse and longitudinal strains. That is 'plate.' In this work, we focus on the axially moving plate.

Similar to [2], we focus on the time variable relating to the damping term. It is related to the energy generated from the boundary inward when the boundary moves. The energy generated at this time is non-conservative work done WD_{nc}. In case of the work done defined by the non conservative forces $f(x,t)$ in internal of domain and $f_c(t)$ at the boundary $x = l(t)$. In the free boundary, the axially moving plate may slip. Therefore, it is necessary to define a different time. It is necessary to express the time differently from normal time t. I will express it in τ here. An also, work done on the outward direction at the right boundary that is free is WD_{rb}. Therefore, $WD = WD_{nc} + WD_{rb}$ on the near the right boundary. Note that work done on the fixed boundary $x = 0$ is zero.

The internal time delay inside the system occurs due to interference (actually caused by the influence of both longitudinal and transverse strain of the plate) or time difference between the fixed roller and the moving roller. This phenomenon

Received September 3, 2021; Accepted September 27, 2021.
2010 *Mathematics Subject Classification.* 74J30, 35L70.
Key words and phrases. transversal displacement of plate (s), longitudinal displacement of plate (r), Kinetic energy (K), Potential energy (P), Work Done (WD).

(pISSN 1226-6973, eISSN 2287-2833)

reduces the product's value by preventing the scroll of the system from being wound constantly. So, it is necessary to control the time delay phenomenon.

Our purpose in this work derives a plate-like model with internal time delay. The plate-like model considering the longitudinal direction and the transverse direction is derived from the coupled equation considering the two displacements. It is significant for modeling industrial sites such as steel plate production. In addition, the partial differential equation derived in this way is helpful for research extending to the third-order or higher.

This work is organized into some processes. First, we check some needed physical variables. Using the variables, we consider the kinetic energy, potential energy, and work done in great detail. Next, we consider the variation for energy. Using the energy conservation and the variation lemma, Hamilton's principle, integration by parts, we get the initial-boundary problem for nonlinear third-order PDE.

2. Mathematical modelling

The plate on the mass production process is steered axially through two ends, which are spaced apart by a distance of $l(t)$. The membrane has two variables. r, s is the displacement of plate under the variables. The first variable is the spatial part x. The range of x is $0 \leq x \leq l(t)$. Indeed, we consider suitable time t of $l(t)$ is fixed. It is a part of the plate-like process. More delicately, the part is roller to roller of the axially moving plate. In the case of the first boundary $x = 0$, physically, the roller's shaft ($r(0,t)$ and $s(0,t)$) is fixed. In the other case $l(t)$, of the roller's transversal shaft ($s(1,t)$) is mechanically free. But longitudinal shaft ($r(0,t)$) is fixed. The next variable is time t Over time, the string moves in a high-speed moving axial direction. At this time, a slip phenomenon occurs in the inner area with the right border $l(t)$. For applying the variation Lemma, we set the variations. The variation of s and r is ϕ and ψ, respectively. Some variables and constants that will be used in this paper are as follows:

$$\begin{cases} \overline{v} > 0 & : \text{moving speed for axial direction;} \\ r & : \text{logitudinal displacement of the plate moving;} \\ s & : \text{transversal displacement of the plate moving;} \\ (\cdot)_t = \partial(\cdot)/\partial t & : \text{the partial derivative for time;} \\ (\cdot)_x = \partial(\cdot)/\partial x & : \text{the partial derivative for domain value;} \\ i, j & : \text{the standard basis vectors;} \\ \{\overline{v} + \overline{v} r_x + r_t\} i + \{\overline{v} s_x + s_t\} j & : \text{transversal velocity of the plate moving;} \\ C(x) & : \text{the area of cross-section;} \\ \varpi(x) & : \text{mass per unit(weight);} \\ Y & : \text{Young's elastic modulus;} \\ \sigma(x,t) & : \text{tensile stress;} \\ \zeta(x,t) & : \text{strain;} \\ \iota_0 & : \text{initial tension of plate.} \end{cases}$$

We also define certain energies and vibrations physically. K is kinetic energy. P is potential energy. δWD_{nc} is the variation of non-conservative work done. δWD_{rb} is the variation of work done at the right boundary. More specifically, all of them are given by

$$K = \frac{1}{2} \int_0^{l(t)} \varpi(x) C(x) [\{\overline{v} + \overline{v} r_x + r_t\}^2 + \{\overline{v} s_x + s_t\}^2] dx, \tag{1}$$

$$\begin{aligned} P = & \int_0^{l(t)} \left[\left(\iota_0 + \frac{YC(x)}{4} \int_0^1 \left(\frac{\partial s}{\partial x} \right)^2 dx \right) \right. \\ & \left. + \frac{1}{2} C(x) \sigma(x,t) \zeta(x,t) \right] \zeta(x,t) dx, \end{aligned} \tag{2}$$

$$\begin{aligned} \delta WD_{nc} = & \int_0^{l(t)} \left[\psi(x,t) - \frac{\varpi(x)\overline{v}}{2} s_t(x, t-\tau) \right] \delta s(x,t) dx \\ & + \psi_c(t) \delta s(l(t), t), \end{aligned} \tag{3}$$

$$\delta WD_{rb} = \varpi(l(t)) C(l(t)) \overline{v} (\overline{v} s_x(l(t), t) + s_t(l(t), t)) \delta s(l(t), t). \tag{4}$$

Now,we calculate the potential energy P in more detail. Let Z and $\zeta(x,t) = \frac{dy-dx}{dx}$(,where $|dx| = 1$) be the plate's tension and the strain under physical situations, respectively.

Then, we have

$$\begin{aligned} P &= \int_0^{l(t)} Z \zeta(x,t) dx \\ &= \int_0^{l(t)} \left[\sigma(\varepsilon(x,t)) + \frac{\sigma'(\varepsilon(x,t))}{2} \int_0^{l(t)} \left(\frac{\partial s}{\partial x} \right)^2 dx \right] \zeta(x,t) dx. \end{aligned}$$

Next we set $\sigma(\varepsilon(x,t)) = \iota_0 + \frac{YC(x)}{2}\zeta(x,t)\varepsilon(x,t)$ with ι_0. Therefore the potential energy P changes

$$P = \int_0^{l(t)} \left[\iota_0 + \frac{YC(x)}{2}\zeta(x,t)\varepsilon(x,t) + \frac{YC(x)}{4}\int_0^{l(t)} \left(\frac{\partial s}{\partial x}\right)^2 dx \right] \zeta(x,t)dx.$$

Because of $\sigma(\varepsilon(x,t))$ which is defined by $\sigma(x,t)$, we can apply $\sigma(\varepsilon(x,t)) = Y\varepsilon(x,t)$. So, we finally get

$$P = \int_0^{l(t)} \left[\left(\iota_0 + \frac{YC(x)}{4}\int_0^1 \left(\frac{\partial s}{\partial x}\right)^2 dx \right) + \frac{1}{2}C(x)\zeta(x,t)\sigma(x,t) \right] \zeta(x,t)dx.$$

By the Taylor's theorem, we get the strain

$$\begin{aligned}
\zeta(x,t) &= \frac{\left[\left\{dx + \frac{\partial r}{\partial x}dx\right\}^2 + \left\{\frac{\partial s}{\partial x}dx\right\}^2\right]^{\frac{1}{2}} - dx}{dx} \\
&= \left[1 + 2\frac{\partial r}{\partial x} + \left\{\frac{\partial r}{\partial x}\right\}^2 + \frac{1}{2}\frac{\partial r}{\partial x}\left\{\frac{\partial s}{\partial x}\right\}^2 + \frac{1}{4}\left\{\frac{\partial s}{\partial x}\right\}^4 + \cdots\right]^{\frac{1}{2}} - 1 \\
&\approx \frac{\partial r}{\partial x} + \frac{1}{2}\left(\frac{\partial s}{\partial x}\right)^2 \\
&\ll 1
\end{aligned}$$

So, we approximately have

$$\begin{aligned}
P &= \int_0^{l(t)} \left[\left(\iota_0 + \frac{YC(x)}{4}\int_0^{l(t)} \left(\frac{\partial s}{\partial x}\right)^2 dx \right) \left(\frac{\partial r}{\partial x} + \frac{1}{2}\left\{\frac{\partial s}{\partial x}\right\}^2 \right) \right. \\
&\qquad \left. + \frac{1}{2}C(x)\sigma(x,t)\left(\frac{\partial r}{\partial x} + \frac{1}{2}\left\{\frac{\partial s}{\partial x}\right\}^2 \right)^2 \right] dx.
\end{aligned}$$

From now on, we start calculating the variation between kinetic and potential energy. By using the Gâteatux derivative, we get variations of K and P like as:

$$\begin{aligned}
\delta K(r;\psi,s;\phi) &= \lim_{\varepsilon\to 0}\frac{K(r+\varepsilon\psi, s+\varepsilon\phi) - K(r,s)}{\varepsilon} \\
&= \int_0^{l(t)} \varpi(x)C(x)\left[\overline{v}(\overline{v} + \overline{v}r_x + r_t)\psi_x + (\overline{v} + \overline{v}r_x + r_t)\psi_t \right. \qquad (5) \\
&\qquad \left. + \overline{v}(\overline{v}s_x + s_t)\phi_x + (\overline{v}s_x + s_t)\phi_t\right] dx,
\end{aligned}$$

$$\begin{aligned}\delta P(r;\psi,s;\phi) &= \lim_{\varepsilon\to 0}\frac{P(r+\varepsilon\psi, s+\varepsilon\phi)-P(r,s)}{\varepsilon}\\ &= \int_0^{l(t)}\left(\iota_0+\frac{YC(x)}{4}\int_0^{l(t)}\left(\frac{\partial s}{\partial x}\right)^2 dx\right)\left[\psi_x+s_x\phi_x\right]dx\\ &\quad+\int_0^{l(t)} C(x)\sigma(x,t)\left[\left(r_x+\frac{1}{2}s_x\right)\psi_x+\left(r_x+\frac{1}{2}s_x\right)s_x\phi_x\right]dx,\end{aligned} \tag{6}$$

where ϕ is the C^1 function which depends on x and t.

Apply for the Hamilton's Principle, (3)-(4) and (5)-(6) are as follows:

$$\int_{t_0}^{t_1}(\delta K-\delta P+\delta WD_{nc}-\delta WD_{rb})dt=0,\ for\ all\ t\in[t_0,t_1] \tag{7}$$

Accordingly, (7) can be replaced with

$$\begin{aligned}&\int_{t_0}^{t_l}\int_0^{l(t)}\varpi(x)C(x)\left[\overline{v}(\overline{v}+\overline{v}r_x+r_t)\psi_x+(\overline{v}+\overline{v}r_x+r_t)\psi_t\right.\\ &\qquad\left.+\overline{v}(\overline{v}s_x+s_t)\phi_x+(\overline{v}s_x+s_t)\phi_t\right]dxdt\\ &+\int_{t_0}^{t_l}\int_0^{l(t)}\left(\iota_0+\frac{YC(x)}{4}\int_0^{l(t)}\left(\frac{\partial s}{\partial x}\right)^2dx\right)\left[\psi_x+s_x\phi_x\right]dxdt\\ &+\int_{t_0}^{t_l}\int_0^{l(t)}C(x)\sigma(x,t)\left[\left(r_x+\frac{1}{2}s_x\right)\psi_x+\left(r_x+\frac{1}{2}s_x\right)s_x\phi_x\right]dxdt\\ &+\int_{t_0}^{t_l}\int_0^{l(t)}\left[\psi(x,t)-\frac{\varpi(x)\overline{v}}{2}s_t(x,t-\tau)\right]\phi dxdt\\ &+\int_{t_0}^{t_l}\left[F_c(t)\phi(l(t),t)-\varpi(l(t))C(l(t))\overline{v}(\overline{v}s_x(l(t),t)+s_t(l(t),t))\phi(l(t),t)\right]dt=0\end{aligned} \tag{8}$$

By using the integration by parts, we deduce

$$
\begin{aligned}
&\int_{t_0}^{t_1} \Big[\varpi(l(t))C(l(t))(\overline{v}^2+\overline{v}^2 s_x(l(t),t)+\overline{v}s_t(l(t),t)) \\
&\quad -\frac{1}{2}\left(\iota_0+\frac{YC(l(t))}{4}\int_0^{l(t)}\left(\frac{\partial s}{\partial x}\right)^2 dx\right) \\
&\quad +C(l(t))\sigma(l(t),t)\left(r_x(l(t),t)+\frac{1}{2}s_x^2(l(t),t)\right)\Big]\psi(l(t),t)dt \\
&+\int_{t_0}^{t_1} \Big[\varpi(0)C(0)(\overline{v}^2+\overline{v}^2 s_x(0,t)+\overline{v}s_t(0,t)) \\
&\quad -\frac{\iota_0}{2}+C(0)\sigma(0,t)\left(r_x(0,t)+\frac{1}{2}s_x^2(0,t)\right)\Big]\psi(0,t)dt \\
&+\int_{t_0}^{t_1} \left[F_c(t)-\frac{1}{2}\left\{\left(\iota_0+\frac{YC(l(t))}{4}\int_0^{l(t)}\left(\frac{\partial s}{\partial x}\right)^2 dx\right)+\varpi(l(t))C(l(t))\right\}s_x(l(t),t)\right. \\
&\quad \left. -\varpi(l(t))C(l(t))+C(x)\sigma(x,t)\left(r_x(l(t),t)+\frac{1}{2}s_x^2(l(t),t)\right)\right]\psi(l(t),t)dt \\
&-\int_{t_0}^{t_1} \Big[\varpi(0)C(0)(\overline{v}^2 s_x(0,t)+\overline{v}s_t(0,t)) \\
&\quad -\frac{\iota_0}{2}+C(0)\sigma(0,t)\left(r_x(0,t)+\frac{1}{2}s_x^2(0,t)\right)\Big]\phi(0,t)dt \\
&\quad +\int_0^{l(t)} \varpi(x)C(x)(\overline{v}s_{xt}(x,t_1)+s_{tt}(x,t_1))\phi(x,t_1)dx \\
&\quad +\int_0^{l(t)} \varpi(x)C(x)(\overline{v}s_{xt}(x,t_0)+s_{tt}(x,t_0))\phi(x,t_0)dx \\
&\quad +\int_0^{l(t)} \varpi(x)C(x)(\overline{v}r_{xt}(x,t_1)+r_{tt}(x,t_1))\psi(x,t_1)dx \\
&\quad +\int_0^{l(t)} \varpi(x)C(x)(\overline{v}r_{xt}(x,t_0)+r_{tt}(x,t_0))\psi(x,t_0)dx \\
&-\int_{t_0}^{t_1}\int_0^{l(t)} \left[\left\{\varpi(x)C(x)\left(\overline{v}^2 r_{xx}+2\overline{v}r_{xt}-r_{tt}\right)-C(x)\sigma(x,t)\left(r_x+\frac{1}{2}s_x^2\right)_x\right\}\psi(x,t)\right. \\
&+\left\{\varpi(x)C(x)\left(\overline{v}^2-\frac{1}{\varpi(x)C(x)}\left(\iota_0+\frac{YC(l(t))}{4}\int_0^{l(t)}\left(\frac{\partial s}{\partial x}\right)^2 dx\right)s_{xx}+2\overline{v}s_{xt}+s_{tt}\right)\right. \\
&\qquad \left.\left. -\varpi(x)C(x)\left\{s_x\left(r_x+\frac{1}{2}s_x^2\right)\right\}_x-\left[\psi(x,t)-\frac{\varpi(x)\overline{v}}{2}s_t(x,t-\tau)\right]\right\}\phi(x,t)\right]dxdt=0.
\end{aligned}
$$

Applying the variational lemma, finally we get the following coupled system considering internal time delay

$$\frac{\varpi(x)}{\sigma(x,t)}\left(\overline{v}^2 r_{xx}+2\overline{v}r_{xt}-r_{tt}\right)=\left(r_x+\frac{1}{2}s_x^2\right)_x \quad \text{in } (0,l(t))\times(0,T), \tag{9}$$

$$\begin{aligned}&\overline{v}^2-\frac{1}{\varpi(x)C(x)}\left(\iota_0+\frac{YC(l(t))}{4}\int_0^{l(t)}\left(\frac{\partial s}{\partial x}\right)^2 dx\right)s_{xx}+2\overline{v}s_{xt}+s_{tt}\\ &-\frac{1}{\varpi(x)C(x)}\left[\psi(x,t)-\frac{\varpi(x)\overline{v}}{2}s_t(x,t-\tau)\right]=\left\{s_x\left(r_x+\frac{1}{2}s_x^2\right)\right\}_x\\ &\text{in } (0,l(t))\times(0,T),\end{aligned} \tag{10}$$

the boundary conditions on $(0,T)$, as

$$F_c(t)-\frac{1}{2}\left\{\left(\iota_0+\frac{YC(l(t))}{4}\int_0^{l(t)}\left(\frac{\partial s}{\partial x}\right)^2 dx\right)+\varpi(l(t))C(l(t))\right\}s_x(l(t),t)$$

$$=\varpi(l(t))C(l(t))-C(x)\sigma(x,t)\left(r_x(l(t),t)+\frac{1}{2}s_x^2(l(t),t)\right)$$

$$r(0,t)=r(l(t),t)=s(0,t)=0 \tag{11}$$

and the initial conditions as

$$\begin{aligned}s(x,0)&=s_0(x)\\ s_t(x,0)&=s_l(x)\\ r(x,0)&=r_0(x)\\ r_t(x,0)&=r_l(x)\\ &\text{in } (0,l(t)).\end{aligned} \tag{12}$$

3. Conclusion

The mathematical analysis of the coupled system (9)-(12) is significant in the field of partial differential equations. The derivation of the axially moving plate-like equation to modeling can actually be extended to the Timoshenko beam equation. (See [10]) To control the time internal delay, a mathematical analysis of the optimal control system associated with time delay is required for free boundary terms (the first equation in (11)) as future works.

4. Acknowledgement

We are very grateful to the anonymous Referees for the some valuable comments of this manuscript.

References

[1] ______, *Asymptotic behavior for the viscoelastic Kirchhoff type equation with an internal time-varying delay term*, East Asian Mathematical Journal **3**4 (2016), 399412.

[2] ______, *Mathematical modelling for the axially moving membrane with internal time delay*, East Asian Mathematical Journal **3**7 (2021), 141147.

[3] ______, *Exponential Decay for the Solution of the Viscoelastic Kirchhoff Type Equation with Memory Condition at the Boundary*, East Asian Mathematical Journal **3**4 (2018), 69–84.

[4] ______, *Asymptotic behavior of a nonlinear Kirchhoff type equation with spring boundary conditions*, Computers and Mathematics with Applications **6**2 (2011), 3004–3014.

[5] ______, *Stabilization for the Kirchhoff type equation from an axially moving heterogeneous string modeling with boundary feedback control*, Nonlinear Analysis: Theory, Methods and Applications **7**5 (2012), 3598–3617.

[6] G. Kirchhoff, *Vorlesungen über Mechanik*, Teubner, Leipzig, 1983.

[7] J. Límaco, H. R. Clark, and L. A. Medeiros, *Vibrations of elastic string with nonhomogeneous material*, Journal of Mathematical Analysis and Applications **3**44 (2008), 806–820.

[8] F. Pellicano and F. Vestroni, *Complex dynamics of high-speed axially moving systems*, Journal of Sound and Vibration, **258** (2002), 31-44.

[9] S. S. Rao, *Vibration of continuous systems*, Wiley, Florida, 2005.

[10] A. Daz-de-Anda, J. Flores, L. Gutirrez, R.A. Mndez-Snchez, G. Monsivais, and A. Morales *Experimental study of the Timoshenko beam theory predictions*, Journal of Sound and Vibration **3**4 (2012), 57325744.

Daewook Kim
Department of Mathematics and Education
Seowon University
Cheongju 28674, Korea
E-mail address: kdw@seowon.ac.kr