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ON OPTIMALITY THEOREMS FOR SEMIDEFINITE LINEAR VECTOR OPTIMIZATION PROBLEMS

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Abstract. Recently, semidefinite optimization problems have been intensively studied since many optimization problem can be changed into the problems and the the problems are very computationable. In this paper, we consider a semidefinite linear vector optimization problem (VP) and we establish the optimality theorems for (VP), which holds without any constraint qualification.

1. Introduction and Preliminaries

Semidefinite optimization problems have been intensively studied since many optimization problem can be changed into the problems which are very computationable [9]. Jeyakumar, Lee and Dinh [6] proved the sequential optimality conditions for convex optimization problem, which held without any constraint qualification and which were expressed in terms of sequences. The optimality conditions have been studied for many kinds of convex optimization problems. In particular, Lee and Lee [8] studied sequential optimality conditions for efficient solutions of an abstract convex vector optimization problems. Kim, Kim and Lee [7] investigated sequential optimality conditions for a semidefinite linear optimization problems.

In this paper, we establish sequential optimality theorems for a properly efficient solution, efficient solutions and weakly efficient solutions of (VP), which holds without any constraint qualification and which are expressed by sequences.

Let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. For a subset $D \subset X$, the closure of D, induced by the norm topology on X, is denoted by clD.

Let C be a closed convex cone in X. Then the positive dual cone of C is defined by

$$C^* := \{z \in X \ : \ \langle x, z \rangle \geqq 0 \ \ \forall x \in C\}.$$

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The indicator function $\delta_A : X \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_A := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise.} \end{cases}$$

Let $h : X \to \mathbb{R} \cup \{+\infty\}$ be a function. The conjugate function of h, $h^* : X \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$h^*(v) := \sup\{\langle v, x\rangle - h(x) \mid x \in \text{dom}h\}$$

where $\text{dom}h := \{x \in X \mid h(x) < +\infty\}$.

The function h is said to be proper if h does not take on the valued $-\infty$ and $\text{dom}h \neq \emptyset$. The epigraph of the function h is defined by

$$\text{epi}h := \{(x, r) \in X \times \mathbb{R} \ : \ h(x) \leqq r\}.$$

We say that h is proper if $h(x) > -\infty$ for all $x \in X$ and $\text{dom}h \neq \emptyset$. Moreover if $\liminf_{y \to x} h(y) \geqq h(x)$ for all $x \in X$, we say that h is lower semicontinuous. A function $h : X \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if for all $\lambda \in [0, 1]$,

$$h(\lambda x + (1-\lambda)y) \leqq \lambda h(x) + (1-\lambda)h(y) \text{ for all } x, y \in X.$$

Lemma 1.1. [1] *Let $h_1, h_2 : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. Then $epi(h_1 + h_2)^* = cl(epi\ h_1^* + epi\ h_2^*)$. If, in addition, one of h_1 and h_2 is continuous at some $x_0 \in dom\ h_1 \cap dom\ h_2$, then*

$$epi(h_1 + h_2)^* = epi\ h_1^* + epi\ h_2^*.$$

Lemma 1.2. [4] *Let I be an arbitrary index set and let $h_i : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. Suppose that there exists $x_0 \in X$ such that $\sup_{i \in I} h_i(x_0) < \infty$. Then*

$$epi(\sup_{i \in I} h_i)^* = clco \bigcup_{i \in I} epi\ h_i^*$$

where $\sup_{i \in I} h_i : X \to \mathbb{R} \cup \{+\infty\}$ is defined by $(\sup_{i \in I} h_i)(x) = \sup_{i \in I} h_i(x)$ for all $x \in X$.

TrA is the trace of $n \times n$ matrix A. For a symmetric $n \times n$ matrix A, $A \succeq 0$ means that A is positive semidefinite and $A \succ 0$ means that A is positive definite. Let $S^n_+ = \{X \in S^n \mid S \succeq 0\}$. Let S^n be the space of $n \times n$ symmetric matrices. Then $Tr(\cdot\ \cdot)$ is the inner product on S^n and S^n is the finite-dimensional Hilbert space with $Tr(\cdot\ \cdot)$ ([2]).

In this paper, we consider the semidefinite linear vector optimization problem:

$$\begin{array}{lll} \text{(VP)} & \text{Minimize} & \Big(Tr(C_1X), \cdots, Tr(C_pX)\Big) \\ & \text{subject to} & X \succeq 0,\ Tr(A_jX) = b_j,\ j = 1, \cdots, m, \end{array}$$

where $C_i \in S^n, i = 1, \cdots, p$, $A_j \in S^n$, $b_j, j = 1, \cdots, m$ are given real numbers.

Let $\triangle := \{X \in S^n \mid X \succeq 0,\ Tr(A_iX) = b_i,\ i = 1, \cdots, m\}$.

Definition 1. (1) $\bar{X} \in \triangle$ is said to be an efficient solution for (VP) if there exists no other feasible $X \in \triangle$ such that

$$\Big(Tr(C_1X), \cdots, Tr(C_pX)\Big) \leqq \Big(Tr(C_1\bar{X}), \cdots, Tr(C_p\bar{X})\Big)$$
$$\text{and} \quad \Big(Tr(C_1X), \cdots, Tr(C_pX)\Big) \neq \Big(Tr(C_1\bar{X}), \cdots, Tr(C_p\bar{X})\Big).$$

(2) [3] $\bar{X} \in \triangle$ is said to be a properly efficient solution for (VP) if it is efficient for (VP) and if there exists a scalar $M > 0$ such that for each i, we have

$$\frac{Tr(C_i\bar{X}) - Tr(C_iX)}{Tr(C_jX) - Tr(C_j\bar{X})} \leqq M$$

for some j such that $Tr(C_jX) > Tr(C_j\bar{X})$ wherever $X \in \triangle$ and $Tr(C_iX) < Tr(C_i\bar{X})$.

(3) $\bar{X} \in \triangle$ is said to be an weakly efficient solution for (VP) if there exists no other feasible $X \in \triangle$ such that

$$\Big(Tr(C_1X), \cdots, Tr(C_pX)\Big) < \Big(Tr(C_1\bar{X}), \cdots, Tr(C_p\bar{X})\Big).$$

Now we give the following necessary optimality theorems for a properly efficient solution, efficient solution, weakly efficient solution of (VP):

Theorem 1.3. *Let $\bar{X} \in \triangle$. Then the following are equivalent:*

(i) $\bar{X}$ is a properly efficient solution of (VP);

(ii) there exist $\lambda_i > 0, i = 1, \cdots, p$ ($\sum_{i=1}^p \lambda_i = 1$) such that

$$(0,0) \in \Big(\sum_{i=1}^p \lambda_i C_i, \sum_{i=1}^p \lambda_i Tr(C_i\bar{X})\Big) + \{0\} \times \mathbb{R}^+$$
$$+cl\Big(\bigcup_{\mu_j \in \mathbb{R}} \sum_{j=1}^m \mu_j(A_j, b_j) + (-S_+^n) \times \mathbb{R}^+\Big);$$

(iii) there exist $\lambda_i > 0, i = 1, \cdots, p$ ($\sum_{i=1}^p \lambda_i = 1$), $\mu_j^l \in \mathbb{R}$, $j = 1, \cdots, m$, $V^l \in S_+^n$ such that

$$\sum_{i=1}^p \lambda_i C_i + \lim_{l\to\infty}\Big[\sum_{j=1}^m \mu_j^l A_j - V^l\Big] = 0$$
$$\textit{and} \quad \lim_{l\to\infty} Tr(V^l\bar{X}) = 0.$$

Proof. ($(i) \Rightarrow (ii) \Rightarrow (iii)$) Let $\bar{X}$ be a properly efficient solution of (VP). By Theorem 2 in [3] there exists $\lambda_i > 0, i = 1, \cdots, p$ such that

$$\sum_{i=1}^p \lambda_i Tr(C_iX) \geqq \sum_{i=1}^p \lambda_i Tr(C_i\bar{X}), \ \forall X \in \triangle.$$

Let $F(X) = \sum_{i=1}^p \lambda_i Tr(C_iX)$. Then $F(X) \geqq F(\bar{X}), \ \ \forall X \in \triangle$. Since $F(X) + \delta_\triangle(X) \geqq F(\bar{X})$. Hence $Tr(0X) - \Big[F(X) + \delta_\triangle(X)\Big] \leqq -F(\bar{X}) \quad \forall X \in \triangle$.

Since $sup\Big\{Tr(0X) - \Big[F(X) + \delta_{\triangle}(X)\Big] \mid X \in \triangle\Big\} \leqq -F(\bar{X})$. Then $(F + \delta_{\triangle})^*(0) \leqq -F(\bar{X})$. Since $(0, -F(\bar{X})) \in epi(F + \delta_{\triangle})^*$, $(0, -F(\bar{X})) \in epiF^* + epi\delta_{\triangle}^*$. Here $epiF^* = \{(\sum_{i=1}^p \lambda_i C_i, 0)\} + \{0\} \times \mathbb{R}^+$ and

$$\begin{aligned}
epi\delta_{\triangle}^* &= epi\Big(\sup_{\substack{\mu_j \in \mathbb{R} \\ Z \in S_+^n}} \Big[\sum_{j=1}^m \mu_j(Tr(A_j\cdot) - b_j) + Tr(-Z\cdot)\Big]\Big)^* \\
&= clco\Big(\bigcup_{\substack{\mu_j \in \mathbb{R} \\ Z \in S_+^n}} epi\Big[\sum_{j=1}^m \mu_j(Tr(A_j\cdot) - b_j) + Tr(-Z\cdot)\Big]\Big)^* \\
&= cl\Big(\bigcup_{\mu_j \in \mathbb{R}} epi\Big[\sum_{j=1}^m \mu_j(Tr(A_j\cdot) - b_j)\Big]^* + \bigcup_{Z \in S_+^n} epi\Big[Tr(-Z\cdot)\Big]^*\Big) \\
&= cl\Big(\bigcup_{\mu_j \in \mathbb{R}} \sum_{j=1}^m \mu_j(A_j, b_j) + (-S_+^n) \times \mathbb{R}^+\Big)
\end{aligned}$$

and

$$\begin{aligned}
(0, -\sum_{i=1}^p \lambda_i Tr(C_i\bar{X})) \quad \in \quad & \Big\{\Big(\sum_{i=1}^p \lambda_i C_i, 0\Big)\Big\} + \{0\} \times \mathbb{R}^+ \\
& + cl\Big(\bigcup_{\mu_j \in \mathbb{R}} \sum_{j=1}^m \mu_j(A_j, b_j) + (-S_+^n) \times \mathbb{R}^+\Big).
\end{aligned}$$

Hence there exist $\lambda_i > 0, i = 1, \cdots, p$ ($\sum_{i=1}^p \lambda_i = 1$) such that

$$\begin{aligned}
(0,0) \in \Big(\sum_{i=1}^p \lambda_i C_i, \sum_{i=1}^p \lambda_i Tr(C_i\bar{X})\Big) + \{0\} \times \mathbb{R}^+ \\
+ cl\Big(\bigcup_{\mu_j \in \mathbb{R}} \sum_{j=1}^m \mu_j(A_j, b_j) + (-S_+^n) \times \mathbb{R}^+\Big).
\end{aligned}$$

Therefore there exist $\lambda_i > 0, i = 1, \cdots, p$ ($\sum_{i=1}^p \lambda_i = 1$), $\mu_j^l \in \mathbb{R}, V^l \in S_+^n$, $r \in \mathbb{R}^+$ and $r^l \in \mathbb{R}^+$ such that

$$\begin{aligned}
&\sum_{i=1}^p \lambda_i C_i + \lim_{l \to \infty}\Big[\sum_{j=1}^m \mu_j^l A_j - V^l\Big] = 0 \\
&\text{and} \quad \sum_{i=1}^p \lambda_i Tr(C_i\bar{X}) + r + \lim_{l \to \infty}\Big[\sum_{j=1}^m \mu_j^l b_j + r^l\Big] = 0.
\end{aligned}$$

Since

$$0 = \sum_{i=1}^p \lambda_i Tr(C_i\bar{X}) + \lim_{l \to \infty} Tr\Big(\sum_{j=1}^m \mu_j^l A_j - V^l\Big)\bar{X}$$

$$\begin{aligned} &= -\lim_{l\to\infty}\Big(\sum_{j=1}^{m}\mu_j^l b_j + r^l\Big) - r + \lim_{l\to\infty}\Big[Tr\Big(\sum_{j=1}^{m}\mu_j^l A_j\Big)\bar{X} - Tr(V^l\bar{X})\Big] \\ &= -\lim_{l\to\infty}\Big[r^l + Tr(V^l\bar{X})\Big] - r. \end{aligned}$$

Since $r^l \geqq 0$ and $Tr(V^l\bar{X}) \geqq 0$ then $r = 0$, $\lim_{l\to\infty} r^l = 0$ and $\lim_{l\to\infty} Tr(V^l\bar{X}) = 0$. Therefore there exist $\lambda_i > 0, i = 1, \cdots, p$ $(\sum_{i=1}^{p}\lambda_i = 1)$, $\mu_j^l \in \mathbb{R}$, $V^l \in S_+^n$ such that

$$\sum_{i=1}^{p}\lambda_i C_i + \lim_{l\to\infty}\Big[\sum_{j=1}^{m}\mu_j^l A_j - V^l\Big] = 0$$
$$\text{and} \quad \lim_{l\to\infty} Tr(V^l\bar{X}) = 0.$$

($(iii) \Rightarrow (i)$) Suppose that (iii) holds. Then for any $X \in \triangle$,
$\sum_{i=1}^{p}\lambda_i Tr(C_i(X-\bar{X})) + \lim_{l\to\infty}\Big[\sum_{j=1}^{m}\mu_j^l Tr(A_j(X-\bar{X})) - Tr(V^l(X-\bar{X}))\Big] = 0.$

So, $\sum_{i=1}^{p}\lambda_i Tr(C_i(X-\bar{X})) + \lim_{l\to\infty}\Big(-Tr(V^l X)\Big) = 0$ for any $X \in \triangle$. Thus $\sum_{i=1}^{p}\lambda_i Tr(C_i X) \geqq \sum_{i=1}^{p}\lambda_i Tr(C_i\bar{X})$ for any $X \in \triangle$. By Theorem 2 in [3], $\bar{X}$ is a properly efficient solution of (VP).

□

Theorem 1.4. *Let $\bar{X} \in \triangle$. Then the following are equivalent:*
(i) $\bar{X}$ is an efficient solution of (VP);
(ii) for each $i = 1, \cdots p$,

$$\begin{aligned} &(0,0) \in (C_i, Tr(C_i\bar{X})) + \{0\}\times\mathbb{R}^+ \\ &\quad + cl\Big(\bigcup_{\mu_j\in\mathbb{R}}\sum_{j=1}^{m}\mu_j(A_j, b_j) + \bigcup_{\eta_k\geqq 0}\sum_{k\neq i}\eta_k(C_k, Tr(C_k\bar{X})) + (-S_+^n)\times\mathbb{R}^+\Big); \end{aligned}$$

(iii) for each $i = 1, \cdots p$, there exist $\mu_j^l \in \mathbb{R}$, $\eta_k^l \geqq 0$, $V^l \in S_+^n$ such that

$$C_i + \lim_{l\to\infty}\Big[\sum_{j=1}^{m}\mu_j^l A_j + \sum_{k\neq i}\eta_k^l C_k - V^l\Big] = 0$$
$$\textit{and} \quad \lim_{l\to\infty} Tr(V^l\bar{X}) = 0.$$

Proof. ($(i) \Rightarrow (ii) \Rightarrow (iii)$) Let $\bar{X} \in \triangle$. Then $\bar{X}$ is an efficient solution of (VP) if and only if for each $i = 1, \cdots, p$, $\bar{X}$ is an optimal solution of the following problem $(\mathrm{VP})_i$;

$$\begin{array}{lll} (\mathrm{VP})_i & \text{Minimize} & Tr(C_i X) \\ & \text{subject to} & Tr(C_k X) \leqq Tr(C_k\bar{X}),\ k \neq i, \\ & & Tr(A_j X) = b_j,\ j = 1, \cdots, m, \\ & & X \succeq 0. \end{array}$$

Let $i \in \{1, \cdots, p\}$ be fixed. Let $F_i(X) = Tr(C_iX)$ and $G = \{X \in S^n \mid X \succeq 0,\ \ Tr(C_kX) \leqq Tr(C_k\bar{X}),\ k \neq i,\ Tr(A_jX) = b_j,\ j = 1, \cdots, m\}$. Then

$$(0, -F_i(\bar{X})) \in epiF_i^* + epi\delta_G^*. \tag{1}$$

Here $epiF_i^* = \{(C_i, 0)\} + \{0\} \times \mathbb{R}^+$ and

$$\begin{aligned}
& epi\delta_G^* \\
= \;& epi\Big(\sup_{\substack{\mu_j \in \mathbb{R} \\ \eta_k \geqq 0 \\ Z \in S_+^n}} \Big[\sum_{j=1}^m \mu_j(Tr(A_j\cdot) - b_j) + \sum_{k\neq i} \eta_k(Tr(C_k\cdot) - Tr(C_k\bar{X})) + Tr(-Z\cdot)\Big]\Big)^* \\
= \;& clco\Big(\bigcup_{\substack{\mu_j \in \mathbb{R} \\ \eta_k \geqq 0 \\ Z \in S_+^n}} epi\Big[\sum_{j=1}^m \mu_j(Tr(A_j\cdot) - b_j) + \sum_{k\neq i} \eta_k(Tr(C_k\cdot) - Tr(C_k\bar{X})) + Tr(-Z\cdot)\Big]\Big)^* \\
= \;& cl\Big(\bigcup_{\mu_j \in \mathbb{R}} epi\Big[\sum_{j=1}^m \mu_j(Tr(A_j\cdot) - b_j)\Big]^* + \bigcup_{\eta_k \geqq 0} epi\Big[\sum_{k\neq i} \eta_k(Tr(C_k\cdot) - Tr(C_k\bar{X}))\Big]^* \\
& \quad + \bigcup_{Z \in S_+^n} epi(Tr(-Z\cdot))^*\Big) \\
= \;& cl\Big(\bigcup_{\mu_j \in \mathbb{R}} \sum_{j=1}^m \mu_j(A_j, b_j) + \bigcup_{\eta_k \geqq 0} \sum_{k\neq i} \eta_k(C_k, Tr(C_k\bar{X})) + (-S_+^n) \times \mathbb{R}^+\Big).
\end{aligned}$$

So, from (1),

$$\begin{aligned}
(0, -Tr(C_i\bar{X})) \;\; \in \;\; & \{(C_i, 0)\} + \{0\} \times \mathbb{R}_+ \\
& +cl \;\; \Big(\;\; \bigcup_{\mu_j \in \mathbb{R}} \sum_{j=1}^m \mu_j(A_j, b_j) + \bigcup_{\eta_k \geqq 0} \sum_{k\neq i} \eta_k(C_k, Tr(C_k\bar{X})) + (-S_+^n) \times \mathbb{R}^+\Big).
\end{aligned}$$

Hence

$$\begin{aligned}
(0, 0) \;\; \in \;\; & (C_i, Tr(C_i\bar{X}) + \{0\} \times \mathbb{R}_+ \\
& +cl\Big(\bigcup_{\mu_j \in \mathbb{R}} \sum_{j=1}^m \mu_j(A_j, b_j) + \bigcup_{\eta_k \geqq 0} \sum_{k\neq i} \eta_k(C_k, Tr(C_k\bar{X})) + (-S_+^n) \times \mathbb{R}^+\Big).
\end{aligned}$$

Therefore there exist $r \in \mathbb{R}^+,\ \mu_j^l \in \mathbb{R},\ \eta_k^l \geqq 0,\ V^l \in S_+^n$ and $r^l \in \mathbb{R}^+$ such that

$$C_i + \lim_{l\to\infty}\Big(\sum_{j=1}^m \mu_j^l A_j + \sum_{k\neq i} \eta_k^l C_k - V^l\Big) = 0 \tag{2}$$

$$\text{and} \;\; Tr(C_i\bar{X}) + r + \lim_{l\to\infty}\Big(\sum_{j=1}^m \mu_j^l b_j + \sum_{k\neq j} \eta_k Tr(C_k\bar{X}) + r^l\Big) = 0. \tag{3}$$

From (2), $Tr(C_i\bar{X})+\lim_{l\to\infty}\Big(\sum_{j=1}^{m}\mu_j^l Tr(A_j\bar{X})+\sum_{k\neq i}\eta_k^l Tr(C_k\bar{X})-Tr(V^l\bar{X})\Big)=0$. Thus, from (3), $-r+\lim_{l\to\infty}\Big(-Tr(V^l\bar{X})-r^l\Big)=0$, that is

$$r+\lim_{l\to\infty}\Big(Tr(V^l\bar{X})+r^l\Big)=0.$$

Since $r \geqq 0$, $r^l \geqq 0$ and $Tr(V^l\bar{X}) \geqq 0$, $r=0$, $\lim_{l\to\infty}Tr(V^l\bar{X})=0$ and $\lim_{l\to\infty}r^l=0$. Therefore for each $i=1,\cdots p$, there exist $\mu_j^l\in\mathbb{R}$, $\eta_k^l\geqq 0$, $V^l\in S_+^n$ such that

$$\begin{aligned} &C_i+\lim_{l\to\infty}\Big(\sum_{j=1}^{m}\mu_j^l A_j+\sum_{k\neq i}\eta_k^l C_k-V^l\Big)=0\\ &\text{and}\quad \lim_{l\to\infty}Tr(V^l\bar{X})=0. \end{aligned}$$

($(iii)\Rightarrow(i)$) Suppose that (iii) holds. Then for any $X\in G$,

$$Tr(C_i(X-\bar{X}))+\lim_{l\to\infty}\Big(\sum_{j=1}^{m}\mu_j^l Tr(A_j(X-\bar{X}))+\sum_{k\neq i}\eta_k^l Tr(C_k(X-\bar{X}))-Tr(V^l(X-\bar{X}))\Big)=0.$$

Since $\lim_{l\to\infty}Tr(V^l\bar{X})=0$, for any $X\in G$,

$$Tr(C_i(X-\bar{X}))+\lim_{l\to\infty}\Big(\sum_{k\neq i}\eta_k^l Tr(C_k(X-\bar{X}))-Tr(V^lX)\Big)=0.$$

So, for any $X\in G$, $Tr(C_iX)\geqq Tr(C_i\bar{X})$. Thus for each $i=1,\cdots,p$, $Tr(C_iX)\geqq Tr(C_i\bar{X})$ for any $X\in G$. Hence $\bar{X}$ is an efficient solution of (VP). □

Since $\bar{X}\in\triangle$ is an efficient solution of (VP) if and only if,

$$\begin{aligned} \text{Minimize}\quad & \sum_{i=1}^{p}Tr(C_iX)\\ \text{subject to}\quad & Tr(C_iX)\leqq Tr(C_i\bar{X}),\ i=1,\cdots,p,\\ & Tr(A_jX)=b_j,\ j=1,\cdots,m,\\ & X\succeq 0, \end{aligned}$$

by proof similar to one of Theorem 1.5, we can obtain the following theorem for the efficient solution of (VP):

Theorem 1.5. *Let $\bar{X}\in\triangle$. Then the following are equivalent:*
(i) $\bar{X}$ is an efficient solution of (VP);
(ii) $(0,0)\in\Big\{\Big(\sum_{i=1}^{p}C_i, Tr(\sum_{i=1}^{p}C_i\bar{X})\Big)\Big\}+\{0\}\times\mathbb{R}^+$

$$+cl\Big(\bigcup_{\mu_j\in\mathbb{R}}\sum_{j=1}^{m}\mu_j(A_j,b_j)+\bigcup_{\eta_k\geqq 0}\sum_{k=1}^{p}\eta_k(C_k,Tr(C_k\bar{X}))+(-S_+^n)\times\mathbb{R}^+\Big);$$

(iii) there exist $\mu_j^l \in \mathbb{R},\ \eta_k^l \geqq 0$ *and* $V^l \in S_+^n$ *such that*

$$\sum_{i=1}^{p} C_i + \lim_{l\to\infty}\Big[\sum_{j=1}^{m}\mu_j^l A_j + \sum_{k=1}^{p}\eta_k^l C_k - V^l\Big] = 0$$
$$and \quad \lim_{l\to\infty} Tr(V^l\bar{X}) = 0.$$

Theorem 1.6. *Let* $\bar{X} \in \triangle$. *Then the following are equivalent:*
(i) $\bar{X}$ *is a weakly efficient solution of (VP);*
(ii) there exist $\lambda_i \geqq 0,\ i = 1, \cdots p$ $(\sum_{i=1}^{p}\lambda_i = 1)$ *such that*

$$(0,0) \in \Big(\sum_{i=1}^{m}\lambda_i C_i, \sum_{i=1}^{p}\lambda_i Tr(C_i\bar{X})\Big) + \{0\}\times\mathbb{R}^+$$
$$+cl\Big(\bigcup_{\mu_j\in\mathbb{R}}\sum_{j=1}^{m}\mu_j(A_j, b_j) + (-S_+^n)\times\mathbb{R}^+\Big);$$

(iii) there exist $\lambda_i \geqq 0, i = 1, \cdots, p$ $(\sum_{i=1}^{p}\lambda_i = 1), \mu_j^l \in \mathbb{R},\ j = 1, \cdots, m,$ $V^l \in S_+^n$ *such that*

$$\sum_{i=1}^{p}\lambda_i C_i + \lim_{l\to\infty}\Big[\sum_{j=1}^{m}\mu_j^l A_j - V^l\Big] = 0$$
$$and \quad \lim_{l\to\infty} Tr(V^l\bar{X}) = 0.$$

Proof. $((i) \Rightarrow (ii) \Rightarrow (iii))$ Let $\bar{X} \in \triangle$. Then $\bar{X}$ is a weakly efficient solution of (VP) if and only if there exist $\lambda_i \geqq 0, i = 1, \cdots, p$ $(\sum_{i=1}^{p}\lambda_i = 1)$ such that $\bar{X}$ is an optimal solution of the following problem:

$$\begin{aligned} &\text{Minimize} && \sum_{i=1}^{p}\lambda_i Tr(C_iX) \\ &\text{subject to} && Tr(A_jX) = b_j,\ j = 1, \cdots, m, \\ &&& X \succeq 0. \end{aligned}$$

Let $F(X) = \sum_{i=1}^{p}\lambda_i Tr(C_iX)$. Then $F(X) \geqq F(\bar{X}),\ \ \forall X \in \triangle$. By the method similar to the proof of Theorem 1.4, we can prove the result. □

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