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PRICING VULNERABLE POWER OPTION UNDER A CEV DIFFUSION

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Abstract. In the over-the-counter market, option's buyers could have a problem for default risk caused by option's writers. In addition, many participants try to maximize their benefits obviously in investing the financial derivatives. Taking all these circumstances into consideration, we deal with the vulnerable power options under a constant elasticity variance (CEV) model. We derive an analytic pricing formula for the vulnerable power option by using the asymptotic analysis, and then we verify that the analytic formula can be obtained accurately by comparing our solution with Monte-Carlo price. Finally, we examine the effect of CEV on the option price based on the derived solution.

1. Introduction

Options are financial derivatives, which mean the right to buy or sell an underlying asset at a specified price at or within an expiration date. Generally, traditional options(or vanilla options) include two widely used options, which one is European option, and another one is American option. European options allow the right to be exercised only at the maturity date and American options can be exercised at any time within the expiration date.

Before 1970s, the evaluation of the option fair price was very challenging because it really needed advanced mathematical tools. In particular, it was very hard for many researches to predict the market's behavior. Since Black and Scholes [1] has proposed firstly the geometric Brownian motions, and applied it to theory of option's pricing after 1970s, the transaction amount of many derivatives had soared and a lot of traders could do business with these contracts safely.

However, this does not take into account the counterparty risk that the option's writer does not carry out the contractual obligation with the option's

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holder in the over-the counter market. Actually, after the recent global financial crisis, there have been several studies of the financial derivatives considering the credit default risk. The option exposed to the default risk of the option writer is called a 'vulnerable option'. The model dynamics of a vulnerable option is constructed by the simultaneous consideration of the value of the underlying asset and the market value of the option writer. Johnson and Stulz [7] derived the pricing of European-style vulnerable options. Klein [9] proposed an analytic solution for the pricing of vulnerable options under Black-Scholes model, taking account of the correlation between the underlying asset of the option and the credit risk of the counterparty. Hung and Liu [6] examined the pricing of vulnerable options based on an incomplete market. Yoon and Kim [14] found European-style vulnerable options under not only constant interest rates but also stochastic interest rates and Jeon et al. [8] studied the pricing of vulnerable path-dependent options, such as vulnerable barrier, vulnerable double barrier, and vulnerable lookback options by utilizing double Mellin transforms techniques.

Recently, as the modern financial market have developed, many contractors desire to maximize their profits from the financial derivatives. It implies that the supply for other derivatives have been increasing nowadays. Then, the following questions would be raised: "*In the OCT market, how can we maximize the profit of the buyer?*" The answer is: **the power option with vulnerable model.** The power options are ones to pay compensation based on the price of the underlying asset with a positive integer index. Heynen and Kat [4] derived closed solution of a general power put option and dealt with hedging problems under the Black-Scholes model. Zhang et al. [15] investigated the evaluation of the power options for the underlying asset that, unlike the Black Scholes' environment, follow an unspecified stochastic differential equation.

In this paper, developing the vulnerable power option that we mentioned above, we present a pricing formula of vulnerable power options under a constant elasticity of volatility (CEV) diffusion. The CEV model introduced by Cox [2] is one of local volatility model. This model has some advantages to capture the skewness of the volatility and leverage effects. Especially, Park and Kim [12] presented the asymptotic expansion method to derive the option values under CEV model. By applying the asymptotic expansion method stated in Park and Kim [12] and the method of the change of variable, we obtain the analytic option pricing formula for the vulnerable power option under CEV model. Moreover, we verify the accuracy of option's formula using the Monte-Carlo methods according to the number of simulations. With this solution, we examine the influence of several parameters including CEV factors on the option.

The rest of the paper consists of as follows. In Section 2, we construct the risk-neutral underlying asset model and the payoff for the European vulnerable power option. Then, we derive the partial differential equations (PDEs) for the option. Section 3 contains the main theorems for the pricing formula of the vulnerable power option with CEV diffusion. Based on Section 3, in Section 4, we use the solution obtained by the Monte-Carlo simulation to verify the pricing accuracy and investigate the price impact for several parameters on the option. Finally, Section 5 provides the concluding remarks and future works.

2. Model formulation

In the probability space $(\Omega, \mathcal{F}, \mathbb{P}^*)$ with a filtration $\{\mathcal{F}_t : 0 \le t \le T\}$ generated by standard Brownian motions W_t^{x*} and W_t^{y*}, under the risk-neutral probability measure, we consider the following stochastic differential equations(SDEs)

$$\begin{aligned} \mathrm{d}X_t &= rX_t\mathrm{d}t + \sigma_x X_t^{\frac{\theta}{2}}\mathrm{d}W_t^{x*}, \\ \mathrm{d}Y_t &= rY_t\mathrm{d}t + \sigma_y Y_t\mathrm{d}W_t^{y*}, \end{aligned} \tag{2.1}$$

where the constant r is an interest rate, the parameter θ is an elasticity, and σ_x and σ_y are volatilities of the X_t and Y_t, respectively. Under the equivalent martingale measure $\mathbb{P}^*$, the option price is given by

$$P(t,x,y) = \mathbb{E}^*\left[e^{-r(T-t)}h(X_T,Y_T)\middle| X_t = x, Y_t = y\right]. \tag{2.2}$$

By the Feynman-Kac formula (See Øksendal [11]), the option's price $P(t,x,y)$ in (2.2) yields the following PDE:

(2.3)
$$\begin{cases} P_t + \dfrac{\sigma_x^2}{2}x^\theta P_{xx} + \dfrac{\sigma_y^2}{2}y^2P_{yy} + \rho\sigma_x\sigma_y x^{\frac{\theta}{2}}yP_{xy} + r\left(xP_x + yP_y - P\right) = 0, \\ P(T,x,y) = h(X_T,Y_T) \end{cases}$$

on the domain $\{(t,x,y) : t \in [0,T),\ x \in (0,\infty),\ y \in (0,\infty)\}$, where the payoff function $h(X_T.Y_T)$ is given by

$$h(X_T,Y_T) = (X_T^c - K)^+\left(\mathbf{1}_{\{Y_T \ge D^*\}} + \mathbf{1}_{\{Y_T < D^*\}}\frac{(1-\alpha)Y_T}{D}\right), \quad c \in \mathbb{N},$$

where K is the exercise price and D^* is the fixed default boundary. Also, D means the overall liability of the option's writer, which is greater than D^* and is an additional liability arising from the possibility that the counterparty will retain the contract even if the market value of Y_T at expiration T is less than D^*. The deadweight costs α arising from bankruptcy or reorganization of the company are expressed as a percentage of the value of the option writer's asset. At maturity T, if the option issuer's market value Y_T is equal to or greater than D^*, all payments will be paid. However, if the Y_T is less than D^*, an amount of $\frac{(1-\alpha)Y_T}{D}$ of the total payment will only be paid.

3. Price approximation

If we define the differential operator $\mathcal{L}$ by $\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma_x^2 x^\theta \frac{\partial^2}{\partial x^2} + \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2}{\partial y^2} + \rho\sigma_x\sigma_y x^{\frac{\theta}{2}} y \frac{\partial^2}{\partial x \partial y} + r\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - I\right)$, where I is an identity operator, then we can rewrite PDE (2.3) as follows:

$$\begin{cases} \mathcal{L}P = 0 & \text{in } [0,T) \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \\ P(T,x,y) = h(x,v) & \text{on } \{t = T\} \end{cases} \tag{3.1}$$

To solve the PDE (3.1), we assume that P has an asymptotic expansion with respect to $\epsilon\,(0 < \epsilon \ll 1)$ as follows:

$$P = \sum_{n=0}^{\infty} \epsilon^n P_n = P_0 + \epsilon P_1 + \epsilon^2 P_2 + \epsilon^3 P_3 + \cdots . \tag{3.2}$$

Here, if the elasticity θ is equal to 2, then (2.1) is the Black-Scholes model. Additionally, if θ is greater than 2, the inverse leverage effect occurs in the market. From the empirical studies researed by Geman and Shih [3], they said it is common that the elasticity parameter θ is less than 2 in the stock market and also, they suggested the fact that the commodity's movement (e.g. copper and gold) of the financial market is well described when the parameter θ is greater than 2. In this paper, reflecting the situation of the equity market, we assume that θ is less than 2.

Theorem 3.1. *Suppose that the vulnerable power call option price $P(t,x,y)$ has an aymptotic expansion with respect to ϵ as $P(t,x,y) = \sum_{n=0}^{\infty} \epsilon^n P_n(t,x,y)$ for $0 < \epsilon \ll 1$. Then, we obtain the following hierarchy system of PDEs:*

$$\mathcal{L}P_0(t,x,y) = 0, \tag{3.3}$$

$$\mathcal{L}P_1(t,x,y) = \frac{1}{2}\sigma_x^2 x^2 (\ln x)\frac{\partial^2 P_0}{\partial x^2} + \frac{1}{2}\rho\sigma_x\sigma_y xy(\ln x)\frac{\partial^2 P_0}{\partial x \partial y}, \tag{3.4}$$

$$\cdots$$

$$\mathcal{L}P_n(t,x,y) = g_n(t,x,y), \quad n \geq 1, \tag{3.5}$$

where

$$\begin{aligned} g_n(t,x,y) =& \frac{1}{2}\sigma_x^2 x^2 \sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}(\ln x)^{n-k}}{(n-k)!}\frac{\partial^2 P_k}{\partial x^2} \\ &+ \rho\sigma_x\sigma_y xy \sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}(\ln x)^{n-k}}{2^{n-k}(n-k)!}\frac{\partial^2 P_k}{\partial x \partial y} \end{aligned} \tag{3.6}$$

with $P_0(T,x,y) = h(x,y)$ and the terminal condition of P_n described by $P_n(T,x,y) = 0$ for $n \geq 1$.

Proof. Referring to Kim et al. [10], we put θ by $\theta = 2 - \epsilon$ and then plug it into $\mathcal{L}P = 0$ given in (3.1), and using Taylor expansions of $x^{2-\epsilon}$ and $x^{1-\frac{\epsilon}{2}}$, then we obtain the following equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_x^2 x^2 \sum_{n=0}^{\infty}\left(\epsilon^n \frac{(-1)^n(\ln x)^n}{n!}\right)\frac{\partial^2 P}{\partial x^2} + \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2 P}{\partial y^2}$$
$$+ \rho\sigma_x\sigma_y xy \sum_{n=0}^{\infty}\left(\epsilon^n \frac{(-1)^n(\ln x)^n}{2^n n!}\right)\frac{\partial^2 P}{\partial x \partial y} + r\left(x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} - P\right) = 0.$$

By applying the asymptotic expansion (3.2) to the above PDE, we have the following equation

$$\partial_t \sum_{n=0}^{\infty}\epsilon^n P_n(t,x,y) + \frac{\sigma_x^2}{2}x^2 \sum_{n=0}^{\infty}\left(\epsilon^n \frac{(-1)^n(\ln x)^n}{n!}\right)\partial_{xx}\sum_{n=0}^{\infty}\epsilon^n P_n(t,x,y)$$
$$+ \frac{\sigma_y^2}{2}y^2\partial_{yy}\sum_{n=0}^{\infty}\epsilon^n P_n(t,x,y) + \rho\sigma_x\sigma_y xy\sum_{n=0}^{\infty}\left(\epsilon^n \frac{(-1)^n(\ln x)^n}{2^n n!}\right)\partial_{xy}\sum_{n=0}^{\infty}\epsilon^n P_n(t,x,y)$$
$$+ r\left(x\partial_x\sum_{n=0}^{\infty}\epsilon^n P_n(t,x,y) + y\partial_y\sum_{n=0}^{\infty}\epsilon^n P_n(t,x,y) - \sum_{n=0}^{\infty}\epsilon^n P_n(t,x,y)\right) = 0.$$

Now, for $n \geq 0$, in the above PDE, we can classify and rearrange the PDE in terms of ϵ term as following manner:

- The zero order term : We can have the following PDE

$$\mathcal{L}P_0 = 0. \tag{3.7}$$

- The term of order ϵ : We can have the following PDE

$$\mathcal{L}P_1 = \frac{1}{2}\sigma_x^2 x^2(\ln x)\frac{\partial^2 P_0}{\partial x^2} + \frac{1}{2}\rho\sigma_x\sigma_y xy(\ln x)\frac{\partial^2 P_0}{\partial x \partial y}. \tag{3.8}$$

- The term of order ϵ^2 : We can have the following PDE

$$\begin{aligned}\mathcal{L}P_2 = &-\frac{1}{4}\sigma_x^2 x^2(\ln x)^2\frac{\partial^2 P_0}{\partial x^2} + \frac{1}{2}\sigma_x^2 x^2(\ln x)\frac{\partial^2 P_1}{\partial x^2}\\ &-\frac{1}{8}\rho\sigma_x\sigma_y xy(\ln x)^2\frac{\partial^2 P_0}{\partial x\partial y} + \frac{1}{2}\rho\sigma_x\sigma_y xy(\ln x)\frac{\partial^2 P_1}{\partial x \partial y}.\end{aligned} \tag{3.9}$$

By using the mathematical induction, we obtain the following equation in term of order ϵ^n.

- The term of order ϵ^n :

$$\begin{aligned}\mathcal{L}P_n = &\frac{1}{2}\sigma_x^2 x^2\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}(\ln x)^{n-k}}{(n-k)!}\frac{\partial^2 P_k}{\partial x^2}\\ &+\rho\sigma_x\sigma_y xy\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}(\ln x)^{n-k}}{2^{n-k}(n-k)!}\frac{\partial^2 P_k}{\partial x\partial y}\\ &\triangleq g_n(t,x,y)\end{aligned} \tag{3.10}$$

The proof of (3.10) is presented by Kim et al. [10]. So, we can obtain the desired result. □

Now, in order to find the explicit-closed solution of P_n for $n \geq 0$ given by (3.3)-(3.5),most of all, we consider the following Lemma 3.1. Lemma 3.1 presents the solution of the PDE (3.3) with the terminal condition $P_0(T, x, y) = h(x, y)$.

Lemma 3.1. *The vulnerable Black-Scholes price $P_0(t, x, y)$ is given by*

(3.11)

$$\begin{aligned} P_0(t,x,y) &= x^c e^{(c-1)\left(r+\frac{c\sigma_x^2}{2}\right)(T-t)}\mathcal{N}_2(a_1,a_2;\rho) - e^{-r(T-t)}K\mathcal{N}_2(b_1,b_2;\rho) \\ &\quad + \delta y\left(x^c e^{c(r+\rho\sigma_x\sigma_y)(T-t)}\mathcal{N}_2(c_1,c_2;-\rho) - K\mathcal{N}_2(d_1,d_2;-\rho)\right), \end{aligned}$$

where $\mathcal{N}_2$ is the bivariate normal cumulative distribution function defined by

$$\mathcal{N}_2(n_1,n_2,\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{n_1}\int_{-\infty}^{n_2}\exp\left(-\frac{p^2-2\rho pq+q^2}{2(1-\rho^2)}\right)\mathrm{d}p\mathrm{d}q$$

and

$$\begin{aligned} a_1 &= a_1(t,x) = \frac{\ln\left(\frac{x}{K^{1/c}}\right) + [r + (c-\frac{1}{2})\sigma_x^2](T-t)}{\sigma_x\sqrt{T-t}}, \\ a_2 &= a_2(t,y) = \frac{\ln\left(\frac{y}{D^*}\right) + (r - \frac{\sigma_y^2}{2} + c\rho\sigma_x\sigma_y)(T-t)}{\sigma_y\sqrt{T-t}}, \\ b_1 &= b_1(t,x) = \frac{\ln\left(\frac{x}{K^{1/c}}\right) + (r - \frac{\sigma_x^2}{2})(T-t)}{\sigma_x\sqrt{T-t}}, \\ b_2 &= b_2(t,y) = \frac{\ln\left(\frac{y}{D^*}\right) + (r - \frac{\sigma_y^2}{2})(T-t)}{\sigma_y\sqrt{T-t}}, \\ c_1 &= c_1(t,x) = \frac{\ln\left(\frac{x}{K^{1/c}}\right) + (r + (c-\frac{1}{2})\sigma_x^2 + \rho\sigma_x\sigma_y)(T-t)}{\sigma_x\sqrt{T-t}}, \\ c_2 &= c_2(t,y) = -\frac{\ln\left(\frac{y}{D^*}\right) + (r + \frac{\sigma_y^2}{2} + c\rho\sigma_x\sigma_y)(T-t)}{\sigma_y\sqrt{T-t}}, \\ d_1 &= d_1(t,x) = \frac{\ln\left(\frac{x}{K^{1/c}}\right) + (r - \frac{\sigma_x^2}{2} + \rho\sigma_x\sigma_y)(T-t)}{\sigma_x\sqrt{T-t}}, \\ d_2 &= d_2(t,y) = -\frac{\ln\left(\frac{y}{D^*}\right) + (r + \frac{\sigma_y^2}{2})(T-t)}{\sigma_y\sqrt{T-t}}, \\ \delta &\triangleq \frac{1-\alpha}{D}. \end{aligned}$$

Proof. Refer to Ha et al. [5]. □

Next, to solve the solution of P_n for $n \geq 1$ described by (3.4) and (3.5), we have the following theorem.

Theorem 3.2. *For all $n \geq 1$, we have the derivation of P_n as follows:*

$$P_n(t,x,y) = \exp\left(\phi_1 s_1(y) + \phi_2 s_2(x,y) + \phi_3 \tau\right) W(\tau, s_1^*, s_2^*), \tag{3.12}$$

where

$$\begin{aligned}
W(\tau, s_1^*, s_2^*) &= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty F(\tau - \xi_1, s_1^* - \xi_2, s_2^* - \xi_3) \tilde{g}_n(\tau, \xi_2, \xi_3) d\xi_3 d\xi_2 d\xi_1, \\
F(\tau, s_1^*, s_2^*) &= \frac{1}{4\pi\tau} \exp\left(-\frac{(s_1^*)^2 + (s_2^*)^2}{4\tau}\right), \\
\tilde{g}_n(\tau, \xi_2, \xi_3) &= -\exp\left(\phi_1 \sqrt{\psi_2}\xi_2 + \phi_2 \sqrt{\psi_4}\xi_3 + \phi_3 \tau\right) \\
&\quad \cdot g_n\left(T - \tau, \exp\left(\frac{\sigma_x}{\sigma_y^2}\sqrt{\psi_2}\xi_2 - \frac{\sqrt{\psi_4}\xi_3}{\sigma_y}\right), \exp\left(\frac{\sqrt{\psi_2}\xi_2}{\rho\sigma_y}\right)\right),
\end{aligned}$$

$$\begin{aligned}
&s_1(y) = \rho\sigma_y \ln y, \quad s_2(x,y) = \rho\sigma_x \ln y - \sigma_y \ln x, \quad \tau = T - t, \\
&\phi_1 = -\frac{\psi_1}{2\psi_2}, \quad \phi_2 = -\frac{\psi_3}{2\psi_4}, \quad \phi_3 = -\frac{\psi_1^2}{4\psi_2} - \frac{\psi_3^2}{4\psi_4} - r, \\
&s_1^*(y) = \frac{s_1(y)}{\sqrt{\psi_2}}, \quad s_2^*(x,y) = \frac{s_2(x,y)}{\sqrt{\psi_4}}, \\
&\psi_1 = r\rho\sigma_y - \frac{1}{2}\rho\sigma_y^3, \quad \psi_2 = \frac{1}{2}\rho^2\sigma_y^4, \quad \psi_3 = \frac{1}{2}\sigma_x^2\sigma_y - r\sigma_y - \frac{1}{2}\rho\sigma_x\sigma_y^2 + r\rho\sigma_x, \\
&\psi_4 = \frac{1}{2}\sigma_x^2\sigma_y^2(1-\rho), \quad \psi_5 = \sigma_y^2 AB + \rho\sigma_x\sigma_y AC.
\end{aligned}$$

Proof. From (3.5) and (3.6), and the terminal condition of P_n, $P_n(t,x,y)$ can be given by the solution of the following nonhomogeneous final value problem

$$\begin{cases} \mathcal{L}P_n = g_n(t,x,y), \\ P_n(T,x,y) = 0. \end{cases} \tag{3.13}$$

To solve the PDE (3.13), we need to the change of variables as follows:

$$\begin{aligned}
s_1 &= A \ln y, \\
s_2 &= B \ln y + C \ln x, \\
\tau &= T - t, \\
P_n(t,x,y) &= V(\tau, s_1, s_2),
\end{aligned} \tag{3.14}$$

where, A, B, and C are some possitive constants. By applying the chain rule, we can obtain the following equations.

(3.15)
$$
\begin{aligned}
&\partial_t P_n = -\partial_\tau V,\\
&\partial_x P_n = \frac{C}{x}\partial_{s_2} V,\\
&\partial^2_{xx} P_n = \frac{1}{x^2}\left(C^2\partial^2_{s_2 s_2} V - C\partial_{s_2} V\right),\\
&\partial_y P_n = \frac{1}{y}\left(A\partial_{s_1} V + B\partial_{s_2} V\right),\\
&\partial^2_{yy} P_n = \frac{1}{y^2}\left(-A\partial_{s_1} V + A^2\partial^2_{s_1 s_1} V + 2AB\partial^2_{s_1 s_2} V - B\partial_{s_2} V + B^2\partial^2_{s_2 s_2} V\right),\\
&\partial^2_{xy} P_n = \frac{1}{xy}\left(AC\partial^2_{s_1 s_2} V + BC\partial^2_{s_2 s_2} V\right).
\end{aligned}
$$

Substituting (3.15) into (3.13), $\mathcal{L}P_n = g_n(t,x,y)$ from (3.13) leads to

$$
\begin{aligned}
&-\partial_\tau V + \left(rA - \frac{1}{2}\sigma_y^2 A\right)\partial_{s_1} V + \frac{1}{2}\sigma_y^2 A^2 \partial^2_{s_1} V\\
&\quad + \left(-\frac{1}{2}\sigma_x^2 C - \frac{1}{2}\sigma_y^2 B + rB + rC\right)\partial_{s_2} V\\
&\quad + \left(\frac{1}{2}\sigma_x^2 C^2 + \frac{1}{2}\sigma_y^2 B^2 + \rho\sigma_x\sigma_y BC\right)\partial^2_{s_2} V\\
&\quad + \left(\sigma_y^2 AB + \rho\sigma_x\sigma_y AC\right)\partial^2_{s_1 s_2} V - rV\\
&= g_n(t,x,y).
\end{aligned}
\tag{3.16}
$$

For convenience, let us define some equations from (3.16) as follows.

$$
\begin{aligned}
&\psi_1 = r\rho\sigma_y - \frac{1}{2}\rho\sigma_y^3, \quad \psi_2 = \frac{1}{2}\rho^2\sigma_y^4, \quad \psi_3 = \frac{1}{2}\sigma_x^2\sigma_y - r\sigma_y - \frac{1}{2}\rho\sigma_x\sigma_y^2 + r\rho\sigma_x,\\
&\psi_4 = \frac{1}{2}\sigma_x^2\sigma_y^2(1-\rho), \quad \psi_5 = \sigma_y^2 AB + \rho\sigma_x\sigma_y AC,\\
&W = \exp\{-(\phi_1 s_1 + \phi_2 s_2 + \phi_3\tau)\}V(\tau, s_1, s_2).
\end{aligned}
$$

Then, (3.16) becomes

$$
\begin{aligned}
-\partial_\tau V + \psi_1\partial_{s_1} V + \psi_2\partial^2_{s_1 s_1} V + \psi_3\partial_{s_2} V&\\
+\psi_4\partial^2_{s_2 s_2} V + \psi_5\partial^2_{s_1 s_2} V - rV = g_n(t,x,y).&
\end{aligned}
\tag{3.17}
$$

Again, using the chain rule on (3.17), we yields

(3.18)
$$
\begin{aligned}
&\partial_\tau V = (\phi_3 W + \partial_\tau W)\, e^{\phi_1 s_1 + \phi_2 s_2 + \phi_3 \tau},\\
&\partial_{s_1} V = (\phi_1 W + \partial_{s_1} W)\, e^{\phi_1 s_1 + \phi_2 s_2 + \phi_3 \tau},\\
&\partial^2_{s_1} V = \left(\phi_1^2 W + 2\phi_1 \partial_{s_1} W + \partial^2_{s_1} W\right) e^{\phi_1 s_1 + \phi_2 s_2 + \phi_3 \tau},\\
&\partial_{s_2} V = (\phi_2 W + \partial_{s_2} W)\, e^{\phi_1 s_1 + \phi_2 s_2 + \phi_3 \tau},\\
&\partial^2_{s_2} V = \left(\phi_2^2 W + 2\phi_2 \partial_{s_2} W + \partial^2_{s_2} W\right) e^{\phi_1 s_1 + \phi_2 s_2 + \phi_3 \tau},\\
&\partial^2_{s_1 s_2} V = \left(\phi_1\phi_2 W + \phi_1 \partial_{s_2} W + \phi_2 \partial_{s_1} W + \partial^2_{s_1 s_2} W\right) e^{\phi_1 s_1 + \phi_2 s_2 + \phi_3 \tau}.
\end{aligned}
$$

Thus, substituting (3.18) into (3.17), (3.18) leads to

$$
\begin{aligned}
&-(\phi_3 W + \partial_\tau W) + \psi_1 (\phi_1 W + \partial_{s_1} W)\\
&\quad + \psi_2 \left(\phi_1^2 W + 2\phi_1 \partial_{s_1} W + \partial^2_{s_1} W\right)\\
&\quad + \psi_3 (\phi_2 W + \partial_{s_2} W) + \psi_4 \left(\phi_2^2 W + 2\phi_2 \partial_{s_2} W + \partial^2_{s_2} W\right)\\
&\quad + \psi_5 \left(\phi_1\phi_2 W + \phi_1 \partial_{s_2} W + \phi_2 \partial_{s_1} W + \partial^2_{s_1 s_2} W\right) - rW\\
&= g_n(t,x,y) e^{-(\phi_1 s_1 + \phi_2 s_2 + \phi_3 \tau)}.
\end{aligned}
\tag{3.19}
$$

Meanwhile, in order for (3.19) to be the 2-dimensional heat equation, W term, W_{s_1} term, W_{s_2} term and $W_{s_1 s_2}$ term must be zero. In other words,

$$
\begin{aligned}
&W \text{ term} : -\phi_3 + \psi_1\phi_1 + \psi_2\phi_1^2 + \psi_3\phi_2 + \psi_4\phi_2^2 - r = 0,\\
&W_{s_1} \text{ term} : \psi_1 + 2\psi_2\phi_1 = 0,\\
&W_{s_2} \text{ term} : \psi_3 + 2\psi_4\phi_2 = 0,\\
&W_{s_1 s_2} \text{ term} : \sigma_y^2 AB + \rho\sigma_x\sigma_y AC = 0.
\end{aligned}
$$

From W, W_{s_1} and W_{s_2} terms, we obtain the following constant values ϕ_1, ϕ_2 and ϕ_3:

$$
\phi_1 = -\frac{\psi_1}{2\psi_2}, \quad \phi_2 = -\frac{\psi_3}{2\psi_4}, \quad \phi_3 = -\frac{\psi_1^2}{4\psi_2} - \frac{\psi_3^2}{4\psi_4} - r,
$$

and if we set positive constant values B and C by $B = \rho\sigma_x$ and $C = -\sigma_y$,respectively, then PDE (3.19) is changed into the 2-dimensional heat equation given by

$$
\frac{\partial W}{\partial \tau} = \psi_2 \frac{\partial^2 W}{\partial s_1^2} + \psi_4 \frac{\partial^2 W}{\partial s_2^2} - g_n(t,x,y) e^{-(\phi_1 s_1 + \phi_2 s_2 + \phi_3 \tau)}
$$

with the initial condition is $W(0, s_1, s_2) = 0$.

Now, under the transformation $s_1^*(y) = s_1(y)/\sqrt{\psi_2}$ and $s_2^*(x,y) = s_2(x,y)/\sqrt{\psi_4}$, the Duhamel's principle can be used to find the fundamental solution of 2-dimensional heat equation. Hence, from $W = \exp\{-(\phi_1 s_1 + \phi_2 s_2 + \phi_3 \tau)\} V(\tau, s_1, s_2)$ and $V(\tau, s_1, s_2) = P_n$, we obtain the desired results. □

4. Numerical results

This section contains two folds: Firstly, we have to consider the accuracy of the solution P in (3.2). As shown in Kim et al. [10], the accuracy of the approximation of P is described by

$$|P - (P_0 + \epsilon P_1)| = \mathcal{O}(\epsilon^2). \tag{4.1}$$

So, for numerical experiments, we use the following approximated price

$$P_0(t, x, y) + \epsilon P_1(t, x, y) := \tilde{P}(t, x, y) \approx P(t, x, y). \tag{4.2}$$

Now, to verify the accuracy of the approximated solution (4.1), we compare the results of Monte–Carlo simulation using the stochastic dynamics (2.1) with the pricing formula presented in (4.1). we take the parameters as follows:

$$X_0 = x = K = 1,\ Y_0 = y = 100,\ r = 2\%,\ D = D^* = 85,$$
$$\rho = 0.3,\ \sigma_x = \sigma_y = 0.2,\ T - t = 1,\ \epsilon = 0.05 \text{ and } c = 3.$$

Secondly, we investigate the impact of the elasticity θ on the underlying asset price for the given parameter $c \in \mathbb{N}$.

Table 1 and Table 2 show the results of Monte-Carlo price and option's price $\tilde{P}$ in (4.2) with respect to the parameter $\theta = 1.95$ or $\theta = 2.05$. In two tables, we can notice that the difference between the Monte–Carlo solution P_{MC} and the $\tilde{P}$ goes to 0 as the number of simulation increases. So, one can observe that our solution given in $\tilde{P}$ in (4.2) is accurately derived in terms of the elasticity parameters $\theta = 1.95$ and $\theta = 2.05$.

In addition, we examine the behaviors of the option's values regarding the elasticity parameter θ. Figure 1(a) and Figure 1(b) show the price changes of $\tilde{P}$ for given elasticity parameter $\theta = 1.95$ or $\theta = 2.1$, respectively. One can notice that the approximated price $\tilde{P}$ increases as the underlying asset price increases. Furthermore, the larger c, the greater the option price.

Table 3 represents the values of the vulnerable power options in terms of the CEV parameter $\theta \in \{1.9, 1.95, 2.05, 2.1\}$ and the power index $c \in \{1, 2, 3\}$. In this table, we can find out the fact that the option's price increase as the power index c increases with respect to the fixed elasticity θ. However, the influence of the elasticity parameter θ on the option price is not sensitive to that of the power index c.

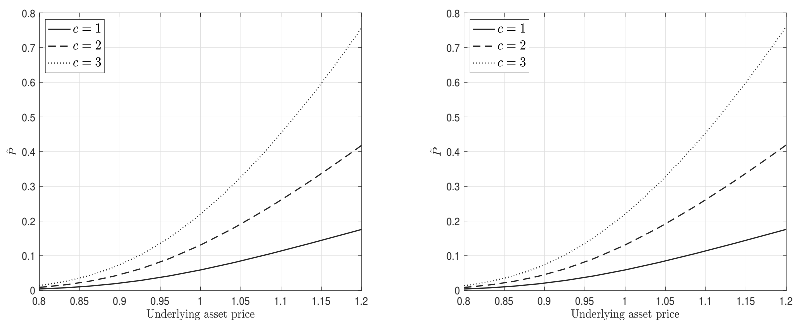
Figure 2 displays the sensitivity of θ against fixed $c \in \{1, 2, 3, 4\}$. From the Figure 2(a)–2(d), we can see that the effect of c is much more significant than that of θ on the option price.

# of simulations	P_{MC}	$\tilde{P}$	$\lvert P_{\mathrm{MC}} - \tilde{P}\rvert$
10,000	0.3403	0.3436	0.0033
20,000	0.3417	0.3436	0.0019
30,000	0.3423	0.3436	0.0013

TABLE 1. Monte-Carlo simulation results ($c = 3$ and $\theta = 1.95$)

# of simulations	P_{MC}	$\tilde{P}$	$\lvert P_{\mathrm{MC}} - \tilde{P}\rvert$
10,000	0.3380	0.3441	0.0061
20,000	0.3402	0.3441	0.0039
30,000	0.3437	0.3441	0.0004

TABLE 2. Monte-Carlo simulation results ($c = 3$ and $\theta = 2.05$)

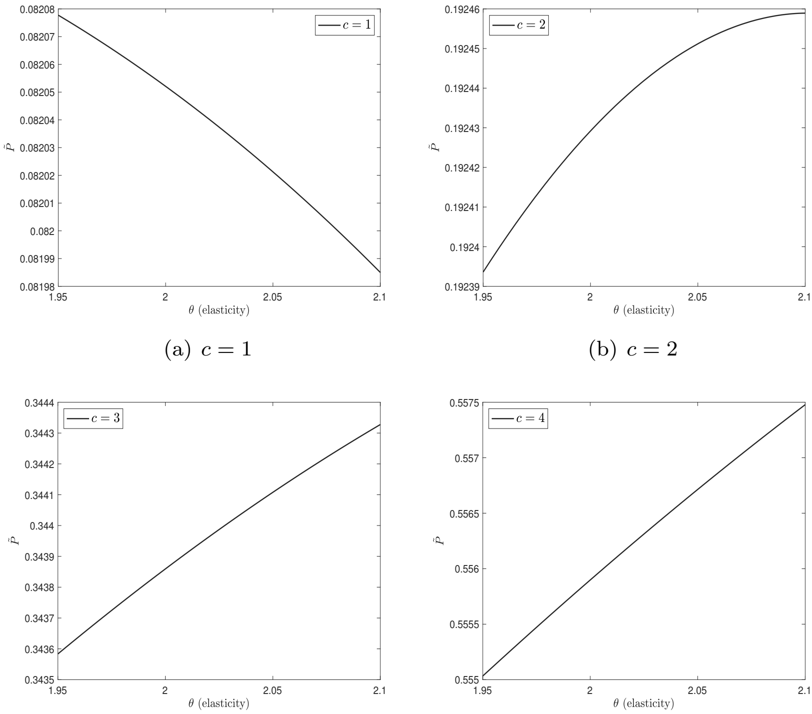


(a) $\theta = 1.95$ (b) $\theta = 2.1$

FIGURE 1. Value of $\tilde{P}$ against the underlying asset price for the given $\theta \in \{1.95, 2.1\}$.

	$\theta = 1.9$	$\theta = 1.95$	$\theta = 2.05$	$\theta = 2.1$
$c = 1$	0.0821	0.0821	0.0820	0.0820
$c = 2$	0.1923	0.1924	0.1925	0.1924
$c = 3$	0.3433	0.3436	0.3441	0.3439

TABLE 3. Option Values in terms of the elasticity parameter θ and the option's power c.



(a) $c = 1$ (b) $c = 2$

(c) $c = 3$ (d) $c = 4$

FIGURE 2. Value of $\tilde{P}$ against the underlying asset price for the given $c \in \{1, 2, 3, 4\}$.

5. Concluding remarks

In this study, we research the fair price of the vulnerable power option under a local volatility model (a CEV diffusion). Especially, assuming the elasticity parameter θ is less than 2 (i.e., $\theta := 2 - \epsilon$), we derive the analytic solution for the option price, and then analyze the price accuracy and the price change with regards to the model parameters.

Our main results consist of three folds. Firstly, we obtain the approximated price of the vulnerable power option considering a CEV diffusion. Secondly, using the Monte–Carlo methods, we verify the solution we studied is accurately derived. Finally, we conduct numerical experiments to investigate how elasticity θ affects the price of options. As a result, we can notice that the impact on option price is more sensitive to power index c than the elasticity θ.

Future works include applying the stochastic volatility model (e.g. *the Ornstein-Uhlenbeck process* or *a hybrid stochastic and local volatility*) to option's pricing model. From this advanced model, we can present the pricing formula reflecting the market's movement or the traders' behavior.

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