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PRICING OF VULNERABLE POWER EXCHANGE OPTION UNDER THE HYBRID MODEL

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Abstract. In this paper, we deal with the pricing of vulnerable power exchange option. We consider the hybrid model as the credit risk model. The hybrid model consists of a combination of the reduced-form model and the structural model. We derive the closed-form pricing formula of vulnerable power exchange option based on the change of measure technique.

1. Introduction

Exchange option with two underlying assets was first proposed by Margrabe [8], and become one of the most popular options in the over-the-counter (OTC) market. Since the exchange option was proposed, there have been the extensions of the exchange option pricing. In particular, credit risk has been considered when the exchange option is priced because there exists the credit risk in the OTC market.

Generally, there are two kinds of approaches for modeling of credit risk: the reduced-form model approach and the structural model approach. Based on these approaches, exchange options with credit risk, which have been called vulnerable exchange option, have been studied in recent years. Under the reduced-form model of Fard [2], Huh, Jeon and Kim [3] proposed a valuation of vulnerable exchange option using the probabilistic approach. Under the structural model of Klein [7], Kim and Koo [4] derived the closed-form pricing formula of vulnerable exchange option using the partial differential equation (PDE) approach and Kim [5] used the probabilistic approach to obtain the same result. For the credit risk modeling, in this paper, we consider the hybrid model that combines the reduced-form model and the structural model. In fact, in the work of Kim [6], the hybrid model was considered to price vulnerable exchange

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option. Motivated by the work of Kim [6], we study the valuation of vulnerable power exchange option under the hybrid model.

Power exchange option is a generalization of exchange option. Blenman and Clark [1] first studied the valuation of power exchange option and provided a closed-form formula of the option. Although there have been several studies for vulnerable power exchange option, the hybrid model has not been considered for the vulnerable power exchange option pricing. In this paper, we first deal with the valuation of vulnerable power exchange option under the hybrid model.

The remainder of this paper is organized as follows. In section 2, we describe the hybrid model for vulnerable power exchange option pricing. In section 3, we derive the pricing formula of vulnerable power exchange option under the proposed model based on the change of measure technique. In section 4, we provide concluding remarks.

2. Model

We assume that a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ represents an economic environment with uncertainty, where $\{\mathcal{F}_t\}$ satisfies the usual conditions and P is the risk-neutral probability measure. Under the measure P, the processes of two risky underlying assets $S_1(t)$ and $S_2(t)$ are given by

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dW_1(t), \tag{1}$$

$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dW_2(t), \tag{2}$$

where σ_i $(i = 1, 2)$ is the volatility of the corresponding asset, and r is a risk-free interest rate. We assume that σ_i $(i = 1, 2)$ and r are positive constants. To construct a hybrid model for credit risk, we adopt the structural model of Klein [7] and the reduced-form model of Fard [2]. For the structural model of Klein, we should define the asset value process $V(t)$ of option writer. The value process $V(t)$ is assumed to be driven by

$$dV(t) = rV(t)dt + \sigma_3 V(t)dW_3(t), \tag{3}$$

where σ_3 is the volatility of asset $V(t)$ of option writer. For the reduced-form model of Fard, we define the default intensity process $\lambda(t)$ as

$$d\lambda(t) = a(b - \lambda(t))dt + \sigma_4 dW_4(t),$$

where σ_4 is the constant volatility of $\lambda(t)$. We assume that σ_i $(i = 3, 4)$, a and b are positive constants. Then the default time τ under the reduced-form model is defined by

$$P(\tau > t) = E^P\left[e^{-\int_0^t \lambda(s)ds}\right].$$

We also assume that $W_1(t), W_2(t), W_3(t)$, and $W_4(t)$ are the standard Brownian motions under the measure P with the following correlations

$$dW_i(t)dW_j(t) = \rho_{ij}dt, \quad i, j = 1, 2, 3, 4,$$

where $-1 \leq \rho_{ij} \leq 1$. Then, as in Kim [6], we construct the hybrid model which is considered both of the reduced-form model and the structural model.

Power exchange option at maturity T has the payoff of the form

$$(S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+,$$

where α_1 and α_2 are positive constants. Therefore, the initial price of vulnerable power exchange option under the hybrid model is given by

$$\begin{aligned} C &= e^{-rT}E^P\left[(S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+\right. \\ &\quad \left. \times\left(\mathbf{1}_{\{\tau>T,V(T)>D\}} + \frac{(1-\alpha)V(T)}{D}(1-\mathbf{1}_{\{\tau>T,V(T)>D\}})\right)|\mathcal{F}_0\right] \\ &= e^{-rT}E^P\left[(S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+\mathbf{1}_{\{\tau>T,V(T)>D\}}|\mathcal{F}_0\right] \\ &\quad +\frac{(1-\alpha)}{D}e^{-rT}E^P\left[V(T)(S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+|\mathcal{F}_0\right] \\ &\quad -\frac{(1-\alpha)}{D}e^{-rT}E^P\left[V(T)(S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+\mathbf{1}_{\{\tau>T,V(T)>D\}}|\mathcal{F}_0\right], \end{aligned} \tag{4}$$

where α is a deadweight cost and D is a value of the option writer's liability.

3. Power exchange option pricing

We study a valuation of exchange option with credit risk exchange option with credit risk under the hybrid model in this section. By the law of iterated conditional expectations, the price C in the equation (4) is given by

$$\begin{aligned} C &= \frac{(1-\alpha)}{D}e^{-rT}E^P\left[V(T)(S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+|\mathcal{F}_0\right] \\ &\quad +e^{-rT}E^P\left[e^{-\int_0^T\lambda(s)ds}(S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+\mathbf{1}_{\{V(T)>D\}}|\mathcal{F}_0\right] \\ &\quad -\frac{(1-\alpha)}{D}e^{-rT}E^P\left[e^{-\int_0^T\lambda(s)ds}V(T)(S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+\mathbf{1}_{\{V(T)>D\}}|\mathcal{F}_0\right]. \end{aligned}$$

In order to simplify the notations, we denote that

$$\begin{aligned} J_1 &= \frac{(1-\alpha)}{D}e^{-rT}E^P\left[V(T)(S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+|\mathcal{F}_0\right], \\ J_2 &= e^{-rT}E^P\left[e^{-\int_0^T\lambda(s)ds}(S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+\mathbf{1}_{\{V(T)>D\}}|\mathcal{F}_0\right], \\ J_3 &= \frac{(1-\alpha)}{D}e^{-rT}E^P\left[e^{-\int_0^T\lambda(s)ds}V(T)(S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+\mathbf{1}_{\{V(T)>D\}}|\mathcal{F}_0\right]. \end{aligned}$$

Then, the price C can be written as

$$C = J_1 + J_2 - J_3. \tag{5}$$

We now calculate J_1, J_2 and J_3 in the following Lemmas, respectively.

Lemma 3.1. *Let us consider J_1 in Eq. (5), then J_1 is given by*

$$\begin{aligned} J_1 &= \frac{(1-\alpha)}{D}S_1^{\alpha_1}(0)V(0)e^{\left(r+\sigma_1\sigma_3\rho_{13}-\frac{\sigma_1^2}{2}+\frac{\alpha_1\sigma_1^2}{2}\right)\alpha_1T}N(a_1) \\ &\quad -\frac{(1-\alpha)}{D}S_2^{\alpha_2}(0)V(0)e^{\left(r+\sigma_2\sigma_3\rho_{23}-\frac{\sigma_2^2}{2}+\frac{\alpha_2\sigma_2^2}{2}\right)\alpha_2T}N(a_2), \end{aligned} \tag{6}$$

where

$$a_1 = \frac{1}{\sigma\sqrt{T}} \ln \frac{S_1^{\alpha_1}(0)}{S_2^{\alpha_2}(0)} + \left(r - \frac{\sigma_1^2}{2} + \alpha_1\sigma_1^2 + \sigma_1\sigma_3\rho_{13} \right) \frac{\alpha_1\sqrt{T}}{\sigma}$$
$$- \left(r - \frac{\sigma_2^2}{2} + \alpha_1\sigma_1\sigma_2\rho_{12} + \sigma_2\sigma_3\rho_{23} \right) \frac{\alpha_2\sqrt{T}}{\sigma},$$
$$a_2 = \frac{1}{\sigma\sqrt{T}} \ln \frac{S_1^{\alpha_1}(0)}{S_2^{\alpha_2}(0)} + \left(r - \frac{\sigma_1^2}{2} + \alpha_2\sigma_1\sigma_2\rho_{12} + \sigma_1\sigma_3\rho_{13} \right) \frac{\alpha_1\sqrt{T}}{\sigma}$$
$$- \left(r - \frac{\sigma_2^2}{2} + \alpha_2\sigma_2^2 + \sigma_2\sigma_3\rho_{23} \right) \frac{\alpha_2\sqrt{T}}{\sigma},$$

with $\sigma^2 = \alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 - 2\alpha_1\alpha_2\sigma_1\sigma_2\rho_{12}$ *and* $N(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{1}{2}x^2} dx$.

Proof. We write J_1 as

$$\begin{aligned} J_1 &= \frac{(1-\alpha)}{D} e^{-rT} E^P \left[V(T) S_1^{\alpha_1}(T) \mathbf{1}_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T)\}} | \mathcal{F}_0 \right] \\ &\quad - \frac{(1-\alpha)}{D} e^{-rT} E^P \left[V(T) S_2^{\alpha_2}(T) \mathbf{1}_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T)\}} | \mathcal{F}_0 \right] \\ &:= \frac{(1-\alpha)}{D} I_1 - \frac{(1-\alpha)}{D} I_2. \end{aligned} \tag{7}$$

To calculate I_1, we define a new measure Q_1 as

$$\frac{dQ_1}{dP} = \exp\left[\alpha_1\sigma_1 W_1(T) + \sigma_3 W_3(T) - \frac{1}{2}(\alpha_1^2\sigma_1^2 + \sigma_3^2 + 2\alpha_1\rho_{13}\sigma_1\sigma_3)T \right].$$

By Girsanov's thoerem,

$$\begin{aligned} W_1^{Q_1}(T) &= W_1(T) - \alpha_1\sigma_1 T - \sigma_3\rho_{13}T, \\ W_2^{Q_1}(T) &= W_2(T) - \alpha_1\sigma_1\rho_{12}T - \sigma_3\rho_{23}T, \\ W_3^{Q_1}(T) &= W_3(T) - \sigma_3 T - \alpha_1\sigma_1\rho_{13}T \end{aligned}$$

are the standard Brownian motions under the measure Q_1. Then we have

$$\begin{aligned} I_1 &= e^{-rT} E^{Q_1} \left[\frac{dP}{dQ_1} V(T) S_1^{\alpha_1}(T) \mathbf{1}_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T)\}} | \mathcal{F}_0 \right] \\ &= S_1^{\alpha_1}(0) V(0) e^{\alpha_1 rT + \frac{\alpha_1\sigma_1^2}{2}(\alpha_1 - 1)T + \alpha_1\sigma_1\sigma_3\rho_{13}T} P^{Q_1}\left(S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T)\right) \\ &= S_1^{\alpha_1}(0) V(0) e^{\alpha_1 rT + \frac{\alpha_1\sigma_1^2}{2}(\alpha_1 - 1)T + \alpha_1\sigma_1\sigma_3\rho_{13}T} \\ &\quad \times P^{Q_1}\left(\alpha_2\sigma_2 W_2^{Q_1}(T) - \alpha_1\sigma_1 W_1^{Q_1}(T) < \ln\left(\frac{S_1^{\alpha_1}(0)}{S_2^{\alpha_2}(0)} \right) \right. \\ &\quad \left. + \left(r - \frac{\sigma_1^2}{2} + \alpha_1\sigma_1^2 + \sigma_1\sigma_3\rho_{13} \right) \alpha_1 T - \left(r - \frac{\sigma_2^2}{2} + \alpha_1\sigma_1\sigma_2\rho_{12} + \sigma_1\sigma_3\rho_{13} \right) \alpha_1 T \right) \alpha_2 T \\ &= S_1^{\alpha_1}(0) V(0) e^{\left(r + \sigma_1\sigma_3\rho_{13} - \frac{\sigma_1^2}{2} + \frac{\alpha_1\sigma_1^2}{2} \right)\alpha_1 T} N(a_1). \end{aligned} \tag{8}$$

To calculate I_2, we define a new measure Q_2 as

$$\frac{dQ_2}{dP} = \exp\left[\alpha_2\sigma_2 W_1(T) + \sigma_3 W_3(T) - \frac{1}{2}(\alpha_2^2\sigma_2^2 + \sigma_3^2 + 2\alpha_2\rho_{23}\sigma_2\sigma_3)T\right].$$

Under the measure Q_2,

$$\begin{aligned} W_1^{Q_2}(T) &= W_1(T) - \alpha_2\sigma_2\rho_{12}T - \sigma_3\rho_{13}T, \\ W_2^{Q_2}(T) &= W_2(T) - \alpha_2\sigma_2 T - \sigma_3\rho_{23}T, \\ W_3^{Q_2}(T) &= W_3(T) - \sigma_3 T - \alpha_2\sigma_2\rho_{23}T \end{aligned}$$

are the standard Brownian motions and I_2 can be calculated in a similar way above. This completes the proof. □

Lemma 3.2. *Let us consider J_2 in Eq. (5), then J_2 is given by*

$$\begin{aligned} J_2 &= e^{-rT} S_1^{\alpha_1}(0) M_1(T) e^{(r-\frac{\sigma_1^2}{2})\alpha_1 T + \frac{\alpha_1^2\sigma_1^2}{2}T - \frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds} N_2(b_1, b_2, \theta_1) \\ &\quad - e^{-rT} S_2^{\alpha_2}(0) M_1(T) e^{(r-\frac{\sigma_2^2}{2})\alpha_2 T + \frac{\alpha_2^2\sigma_2^2}{2}T - \frac{\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds} N_2(b_3, b_4, \theta_1), \end{aligned}$$

where

$$\theta_1 = \frac{(\alpha_1\sigma_1\rho_{13} - \alpha_2\sigma_2\rho_{23})}{\sigma}, \sigma^2 = \alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 - 2\alpha_1\alpha_2\sigma_1\sigma_2\rho_{12}, f(s,t,u) = 1 - e^{-u(t-s)},$$

$$M_1(T) = \exp\left[-bT - \frac{\lambda(0) - b}{a} f(0,T,a) + \frac{\sigma_4^2}{2a^2}\int_0^T f^2(s,T,a)ds\right],$$

$$b_1 = \frac{\ln\frac{S_1^{\alpha_1}(0)}{S_2^{\alpha_2}(0)} + \alpha_1\left(r - \frac{\sigma_1^2}{2}\right)T - \alpha_2\left(r - \frac{\sigma_2^2}{2}\right)T + (\alpha_1^2\sigma_1^2 - \alpha_1\alpha_2\sigma_1\sigma_2\rho_{12})T}{\sigma\sqrt{T}}$$
$$+ \frac{-\frac{\alpha_1\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds + \frac{\alpha_2\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds}{\sigma\sqrt{T}},$$

$$b_2 = \frac{\ln\frac{V(0)}{D} + \left(r - \frac{\sigma_3^2}{2}\right)T + \alpha_1\sigma_1\sigma_3\rho_{13}T - \frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds}{\sigma_3\sqrt{T}},$$

$$b_3 = \frac{\ln\frac{S_1^{\alpha_1}(0)}{S_2^{\alpha_2}(0)} + \alpha_1\left(r - \frac{\sigma_1^2}{2}\right)T - \alpha_2\left(r - \frac{\sigma_2^2}{2}\right)T + (\alpha_1\alpha_2\sigma_1\sigma_2\rho_{12} - \alpha_2^2\sigma_2^2)T}{\sigma\sqrt{T}}$$
$$+ \frac{-\frac{\alpha_1\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds + \frac{\alpha_2\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds}{\sigma\sqrt{T}},$$

$$b_4 = \frac{\ln\frac{V(0)}{D} + \left(r - \frac{\sigma_3^2}{2}\right)T + \alpha_2\sigma_2\sigma_3\rho_{23}T - \frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds}{\sigma_3\sqrt{T}},$$

and

$$N_2(n_1, n_2, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{n_1}\int_{-\infty}^{n_2} e^{-\frac{1}{2(1-\rho^2)}(x^2 - 2xy\rho + y^2)}dydx.$$

Proof. We write J_2 as

$$\begin{aligned} J_2 &= e^{-rT}E^P\left[e^{-\int_0^T \lambda(s)ds}S_1^{\alpha_1}(T)\mathbf{1}_{\{S_1^{\alpha_1}(T)>S_2^{\alpha_2}(T),V(T)>D\}}|\mathcal{F}_0\right] \\ &\quad -e^{-rT}E^P\left[e^{-\int_0^T \lambda(s)ds}S_2^{\alpha_2}(T)\mathbf{1}_{\{S_1^{\alpha_1}(T)>S_2^{\alpha_2}(T),V(T)>D\}}|\mathcal{F}_0\right] \\ &:= e^{-rT}I_3 - e^{-rT}I_4. \end{aligned} \tag{9}$$

To calculate I_4, we define a new measure Q_3 such that

$$\frac{dQ_3}{dP} = \frac{e^{-\int_0^T \lambda(s)ds}}{E[e^{-\int_0^T \lambda(s)ds}|\mathcal{F}_0]}.$$

Then, I_3 is given by

$$I_3 = M_1(T)E^{Q_3}\left[S_1^{\alpha_1}(t)\mathbf{1}_{\{S_1^{\alpha_1}(T)>S_2^{\alpha_2}(T),V(T)>D\}}|\mathcal{F}_0\right], \tag{10}$$

where $M_1(T) = E[e^{-\int_0^T \lambda(s)ds}|\mathcal{F}_0]$. To calculate Eq. (10), we define a new measure $\widetilde{Q}_3$ such that

$$\frac{d\widetilde{Q}_3}{dQ_3} = \exp\left[\alpha_1\sigma_1 W_1^{Q_3}(T) - \frac{1}{2}\alpha_1^2\sigma_1^2 T\right].$$

Then, under the measure $\widetilde{Q}_3$,

$$\begin{aligned} W_1^{\widetilde{Q}_3}(T) &= W_1(T) + \frac{\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds - \alpha_1\sigma_1 T, \\ W_2^{\widetilde{Q}_3}(T) &= W_2(T) + \frac{\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds - \alpha_1\sigma_1\rho_{12} T, \\ W_3^{\widetilde{Q}_3}(T) &= W_3(T) + \frac{\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds - \alpha_1\sigma_1\rho_{13} T, \\ W_4^{\widetilde{Q}_3}(T) &= W_4(T) + \frac{\sigma_4}{a}\int_0^T f(s,T,a)ds - \alpha_1\sigma_1\rho_{14} T \end{aligned}$$

are the standard Brownian motions. Under the measure $\widetilde{Q}_3$, we have

$$I_3 = S_1^{\alpha_1}(0)M_1(T)e^{(r-\frac{\sigma_1^2}{2})\alpha_1 T + \frac{\alpha_1^2\sigma_1^2}{2}T - \frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds}E^{\widetilde{Q}_3}\left[\mathbf{1}_{\{S_1(T)>S_2(T),V(T)>D\}}|\mathcal{F}_0\right],$$

and

$$\begin{aligned} &E^{\widetilde{Q}_3}\left[\mathbf{1}_{\{S_1^{\alpha_1}(T)>S_2^{\alpha_2}(T),V(T)>D\}}\right] \\ &= P^{\widetilde{Q}_3}\left(S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\right) \\ &= P^{\widetilde{Q}_3}\left(\alpha_2\sigma_2 W_2^{\widetilde{Q}_3}(T) - \alpha_1\sigma_1 W_1^{\widetilde{Q}_3}(T) < b_1\sigma\sqrt{T}, -\sigma_3 W_3^{\widetilde{Q}_3}(T) < b_2\sigma_3\sqrt{T}\right) \\ &= P^{\widetilde{Q}_3}\left(z_1 < b_1, z_2 < b_2\right). \end{aligned}$$

Since z_1 and z_2 are correlated standard normal variables, we obtain

$$I_3 = S_1^{\alpha_1}(0)M_1(T)e^{(r-\frac{\sigma_1^2}{2})\alpha_1 T + \frac{\alpha_1^2\sigma_1^2}{2}T - \frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds}N_2(b_1, b_2, \theta_1),$$

where θ_1 is the correlation between z_1 and z_2.

Under the measure Q_3, I_4 is represented by

$$I_4 = M_1(T)E^{Q_3}\left[S_2^{\alpha_2}(t)\mathbf{1}_{\{S_1^{\alpha_1}(T)>S_2^{\alpha_2}(T),V(T)>D\}}|\mathcal{F}_0\right]. \tag{11}$$

For the calculation of I_4, we define a new measure $\widehat{Q}_3$ such that

$$\frac{d\widehat{Q}_3}{dP} = \exp\left[\alpha_2\sigma_2 W_2^{Q_3}(T) - \frac{1}{2}\alpha_2^2\sigma_2^2 T\right].$$

Then, under the measure $\widehat{Q}_3$, we can calculate I_4 in a similar way to I_3. $\square$

Lemma 3.3. *Let us consider J_3 in the equation (5), then J_3 is given by*

$$\begin{aligned} J_3 \;=\; & \frac{(1-\alpha)}{D}S_1^{\alpha_1}(0)V(0)e^{\left(r+\sigma_1\sigma_3\rho_{13}-\frac{\sigma_1^2}{2}+\frac{\alpha_1\sigma_1^2}{2}\right)\alpha_1 T} \\ & \times e^{-\frac{\alpha_1\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds-\frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds}M_1(T)N_2(c_1,c_2,\theta_1) \\ & -\frac{(1-\alpha)}{D}S_2^{\alpha_2}(0)V(0)e^{\left(r+\sigma_2\sigma_3\rho_{23}-\frac{\sigma_2^2}{2}+\frac{\alpha_2\sigma_2^2}{2}\right)\alpha_2 T} \\ & \times e^{-\frac{\alpha_2\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds-\frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds}M_1(T)N_2(c_1,c_2,\theta_1), \end{aligned}$$

where

$$\begin{aligned} c_1 = & \frac{\ln\frac{S_1^{\alpha_1}(0)}{S_2^{\alpha_2}(0)}+(\alpha_1^2\sigma_1^2+\alpha_1\sigma_1\sigma_3\rho_{13}-\alpha_1\alpha_2\sigma_1\sigma_2\rho_{12}-\alpha_2\sigma_2\sigma_3\rho_{23})T+\alpha_1\left(r-\frac{\sigma_1^2}{2}\right)T}{\sigma\sqrt{T}} \\ & +\frac{-\alpha_2\left(r-\frac{\sigma_2^2}{2}\right)T-\frac{\alpha_1\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds+\frac{\alpha_2\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds}{\sigma\sqrt{T}}, \\ c_2 = & \frac{\ln\frac{V(0)}{D}+\left(r+\frac{\sigma_3^2}{2}\right)T+\alpha_1\sigma_1\sigma_3\rho_{13}T-\frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds}{\sigma_3\sqrt{T}}, \\ c_3 = & \frac{\ln\frac{S_1^{\alpha_1}(0)}{S_2^{\alpha_2}(0)}+(\alpha_1\alpha_2\sigma_1\sigma_2\rho_{12}+\alpha_1\sigma_1\sigma_3\rho_{13}-\alpha_2^2\sigma_2^2-\alpha_2\sigma_2\sigma_3\rho_{23})T+\alpha_1\left(r-\frac{\sigma_1^2}{2}\right)T}{\sigma\sqrt{T}} \\ & +\frac{-\alpha_2\left(r-\frac{\sigma_2^2}{2}\right)T-\frac{\alpha_1\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds+\frac{\alpha_2\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds}{\sigma\sqrt{T}}, \\ c_4 = & \frac{\ln\frac{V(0)}{D}+\left(r+\frac{\sigma_3^2}{2}\right)T+\alpha_2\sigma_2\sigma_3\rho_{23}T-\frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds}{\sigma_3\sqrt{T}}, \end{aligned}$$

and θ_1, σ, f, $M_1(T)$ and N_2 are defined in Lemma 3.2.

Proof. J_3 is represented by

$$\begin{aligned} J_3 &= \frac{(1-\alpha)}{D} e^{-rT} E^P \left[e^{-\int_0^T \lambda(s)ds} V(T) S_1^{\alpha_1}(T) \mathbf{1}_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | \mathcal{F}_0 \right] \\ &\quad - \frac{(1-\alpha)}{D} e^{-rT} E^P \left[e^{-\int_0^T \lambda(s)ds} V(T) S_2^{\alpha_2}(T) \mathbf{1}_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | \mathcal{F}_0 \right] \\ &:= \frac{(1-\alpha)}{D} I_5 - \frac{(1-\alpha)}{D} I_6. \end{aligned} \tag{12}$$

I_5 and I_6 can be calculated under the measure Q_3 defined in Lemma 3.2. Thus, under the measure Q_3, I_5 is given by

$$I_5 = M_1(T) e^{-rT} E^{Q_3} \left[V(T) S_1^{\alpha_1}(T) \mathbf{1}_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | \mathcal{F}_0 \right], \tag{13}$$

where $M_1(T)$ is defined in Lemma 3.2. With the standard Brownian motions under the measure Q_3, we define a new measure Q_4 such that

$$\frac{dQ_4}{dQ_3} = \exp\left[\alpha_1\sigma_1 W_1^{Q_3}(T) + \sigma_3 W_3^{Q_3}(T) - \frac{1}{2}(\alpha_1^2\sigma_1^2 + \sigma_3^2 + 2\alpha_1\rho_{13}\sigma_1\sigma_3)T \right].$$

By Girsanov's thoerem,

$$\begin{aligned} W_1^{Q_4}(T) &= W_1^{Q_3}(T) - \alpha_1\sigma_1 T - \sigma_3\rho_{13}T, \\ W_2^{Q_4}(T) &= W_2^{Q_3}(T) - \alpha_1\sigma_1\rho_{12}T - \sigma_3\rho_{23}T, \\ W_3^{Q_4}(T) &= W_3^{Q_3}(T) - \sigma_3 T - \alpha_1\sigma_1\rho_{13}T \end{aligned}$$

are the standard Brownian motions under the measure Q_4. Then, we obtain

$$\begin{aligned} I_5 &= S_1^{\alpha_1}(0) V(0) e^{\left(r + \sigma_1\sigma_3\rho_{13} - \frac{\sigma_1^2}{2} + \frac{\alpha_1\sigma_1^2}{2}\right)\alpha_1 T - \frac{\alpha_1\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds - \frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds} \\ &\quad \times M_1(T) E^{Q_4}\left[\mathbf{1}_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | \mathcal{F}_0 \right] \end{aligned}$$

Since $E^{Q_4}\left[\mathbf{1}_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | \mathcal{F}_0 \right] = P^{Q_4}\left(S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D \right)$, we can obtain

$$E^{Q_4}\left[\mathbf{1}_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | \mathcal{F}_0 \right] = N_2(c_1, c_2, \theta_1).$$

In a similar way, we represent I_6 under the measure Q_3 as

$$I_6 = e^{-rT} M_1(T) E^{Q_3}\left[V(T) S_2^{\alpha_2}(T) \mathbf{1}_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | \mathcal{F}_0 \right]. \tag{14}$$

To calculate I_6, we define a new measure Q_5 equivalent to Q_3 by

$$\frac{dQ_5}{dQ_3} = \exp\left[\alpha_2\sigma_2 W_2^{Q_3}(T) + \sigma_3 W_3^{Q_3}(T) - \frac{1}{2}(\alpha_2^2\sigma_2^2 + \sigma_3^2 + 2\alpha_2\rho_{23}\sigma_2\sigma_3)T \right].$$

By Girsanov's thoerem,

$$\begin{aligned} W_1^{Q_5}(T) &= W_1^{Q_3}(T) - \alpha_2\sigma_2\rho_{12}T - \sigma_3\rho_{13}T, \\ W_2^{Q_5}(T) &= W_2^{Q_3}(T) - \alpha_2\sigma_2 T - \sigma_3\rho_{23}T, \\ W_3^{Q_5}(T) &= W_3^{Q_3}(T) - \sigma_3 T - \alpha_2\sigma_2\rho_{23}T \end{aligned}$$

are the standard Brownian motions under the measure Q_5. Using these Brownian motions and a similar way to the calculation for I_5, we can calculate I_6 under the measure Q_5. This completes the proof. □

Combining the Lemmas, we finally present the closed-form formula for vulnerable power exchange option price under the hybrid model.

Theorem 3.4. *The price at time* 0 *of vulnerable power exchange option under the hybrid model is given by*

$$C = J_1 + J_2 - J_3,$$

where J_1, J_2 *and* J_3 *are defined in Lemma 3.1, Lemma 3.2 and Lemma 3.3, respectively.*

4. Concluding remarks

In this paper, we consider the hybrid model for the valuation of vulnerable power exchange option. The hybrid model is constructed as a combination of the reduced-from model and the structural model. Applying the change of measure repeatedly, we derive the closed-form pricing formula of the option. Finally, we provide the formula using the bivariate normal cumulative function and the standard normal cumulative function

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