PRICING OF VULNERABLE POWER EXCHANGE OPTION UNDER THE HYBRID MODEL

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ABSTRACT. In this paper, we deal with the pricing of vulnerable power exchange option. We consider the hybrid model as the credit risk model. The hybrid model consists of a combination of the reduced-form model and the structural model. We derive the closed-form pricing formula of vulnerable power exchange option based on the change of measure technique.

1. Introduction

Exchange option with two underlying assets was first proposed by Margrabe [8], and become one of the most popular options in the over-the-counter (OTC) market. Since the exchange option was proposed, there have been the extensions of the exchange option pricing. In particular, credit risk has been considered when the exchange option is priced because there exists the credit risk in the OTC market.

Generally, there are two kinds of approaches for modeling of credit risk: the reduced-form model approach and the structural model approach. Based on these approaches, exchange options with credit risk, which have been called vulnerable exchange option, have been studied in recent years. Under the reduced-form model of Fard [2], Huh, Jeon and Kim [3] proposed a valuation of vulnerable exchange option using the probabilistic approach. Under the structural model of Klein [7], Kim and Koo [4] derived the closed-form pricing formula of vulnerable exchange option using the partial differential equation (PDE) approach and Kim [5] used the probabilistic approach to obtain the same result. For the credit risk modeling, in this paper, we consider the hybrid model that combines the reduced-form model and the structural model. In fact, in the work of Kim [6], the hybrid model was considered to price vulnerable exchange...
option. Motivated by the work of Kim [6], we study the valuation of vulnerable power exchange option under the hybrid model.

Power exchange option is a generalization of exchange option. Blenman and Clark [1] first studied the valuation of power exchange option and provided a closed-form formula of the option. Although there have been several studies for vulnerable power exchange option, the hybrid model has not been considered for the vulnerable power exchange option pricing. In this paper, we first deal with the valuation of vulnerable power exchange option under the hybrid model.

The remainder of this paper is organized as follows. In section 2, we describe the hybrid model for vulnerable power exchange option pricing. In section 3, we derive the pricing formula of vulnerable power exchange option under the proposed model based on the change of measure technique. In section 4, we provide concluding remarks.

2. Model

We assume that a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ represents an economic environment with uncertainty, where $\{\mathcal{F}_t\}$ satisfies the usual conditions and $P$ is the risk-neutral probability measure. Under the measure $P$, the processes of two risky underlying assets $S_1(t)$ and $S_2(t)$ are given by

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dW_1(t),$$

$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dW_2(t),$$

where $\sigma_i$ ($i = 1, 2$) is the volatility of the corresponding asset, and $r$ is a risk-free interest rate. We assume that $\sigma_i$ ($i = 1, 2$) and $r$ are positive constants. To construct a hybrid model for credit risk, we adopt the structural model of Klein [7] and the reduced-form model of Fard [2]. For the structural model of Klein, we should define the asset value process $V(t)$ of option writer. The value process $V(t)$ is assumed to be driven by

$$dV(t) = rV(t)dt + \sigma_3 V(t)dW_3(t),$$

where $\sigma_3$ is the volatility of asset $V(t)$ of option writer. For the reduced-form model of Fard, we define the default intensity process $\lambda(t)$ as

$$d\lambda(t) = a(b - \lambda(t))dt + \sigma_4 dW_4(t),$$

where $\sigma_4$ is the constant volatility of $\lambda(t)$. We assume that $\sigma_i$ ($i = 3, 4$), $a$ and $b$ are positive constants. Then the default time $\tau$ under the reduced-form model is defined by

$$P(\tau > t) = E^P \left[ e^{-\int_0^t \lambda(s)ds} \right].$$

We also assume that $W_1(t), W_2(t), W_3(t),$ and $W_4(t)$ are the standard Brownian motions under the measure $P$ with the following correlations

$$dW_i(t)dW_j(t) = \rho_{ij}dt, \quad i, j = 1, 2, 3, 4,$$

where $-1 \leq \rho_{ij} \leq 1$. Then, as in Kim [6], we construct the hybrid model which is considered both of the reduced-form model and the structural model.
Power exchange option at maturity $T$ has the payoff of the form

\[(S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+ ,\]

where $\alpha_1$ and $\alpha_2$ are positive constants. Therefore, the initial price of vulnerable power exchange option under the hybrid model is given by

\[
C = e^{-rT} E^P \left[ (S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+ \right. \\
\times \left( 1_{\{\tau>T,V(T)>D\}} + \frac{(1-\alpha)}{D} V(T) (1 - 1_{\{\tau>T,V(T)>D\}}) \right) \left| F_0 \right. \\
= e^{-rT} E^P \left[ (S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+ 1_{\{\tau>T,V(T)>D\}} \right| F_0 \\
+ \frac{(1-\alpha)}{D} e^{-rT} E^P \left[ V(T) (S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+ \right| F_0 \\
- \frac{(1-\alpha)}{D} e^{-rT} E^P \left[ V(T) (S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+ 1_{\{\tau>T,V(T)>D\}} \right| F_0 ,
\]

where $\alpha$ is a deadweight cost and $D$ is a value of the option writer’s liability.

3. Power exchange option pricing

We study a valuation of exchange option with credit risk exchange option with credit risk under the hybrid model in this section. By the law of iterated conditional expectations, the price $C$ in the equation (4) is given by

\[
C = \frac{(1-\alpha)}{D} e^{-rT} E^P \left[ V(T) (S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+ \right| F_0 \\
+ e^{-rT} E^P \left[ e^{-\int_0^T \lambda(s) ds} (S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+ \right| F_0 \\
- \frac{(1-\alpha)}{D} e^{-rT} E^P \left[ e^{-\int_0^T \lambda(s) ds} V(T) (S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+ \right| F_0 .
\]

In order to simplify the notations, we denote that

\[
J_1 = \frac{(1-\alpha)}{D} e^{-rT} E^P \left[ V(T) (S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+ \right| F_0 , \\
J_2 = e^{-rT} E^P \left[ e^{-\int_0^T \lambda(s) ds} (S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+ \right| F_0 , \\
J_3 = \frac{(1-\alpha)}{D} e^{-rT} E^P \left[ e^{-\int_0^T \lambda(s) ds} V(T) (S_1^{\alpha_1}(T) - S_2^{\alpha_2}(T))^+ \right| F_0 .
\]

Then, the price $C$ can be written as

\[
C = J_1 + J_2 - J_3. \tag{5}
\]

We now calculate $J_1, J_2$ and $J_3$ in the following Lemmas, respectively.

**Lemma 3.1.** Let us consider $J_1$ in Eq. (5), then $J_1$ is given by

\[
J_1 = \frac{(1-\alpha)}{D} S_1^{\alpha_1}(0) V(0) e^{\left( r + \sigma_1 \sigma_3 \rho_{13} - \frac{\sigma_1^2}{2} + \frac{\alpha_1 \sigma_1^2}{2} \right) a_1 T} N(a_1) \\
- \frac{(1-\alpha)}{D} S_2^{\alpha_2}(0) V(0) e^{\left( r + \sigma_2 \sigma_3 \rho_{23} - \frac{\sigma_2^2}{2} + \frac{\alpha_2 \sigma_2^2}{2} \right) a_2 T} N(a_2), \tag{6}
\]
where
\[
a_1 = \frac{1}{\sigma \sqrt{T}} \ln \frac{S_t^{a_1}(0)}{S_0^{a_2}(0)} + \left( r - \frac{\sigma^2}{2} + \alpha_1 \sigma_1^2 + \sigma_1 \sigma_3 \rho_{13} \right) \frac{\alpha_1 \sqrt{T}}{\sigma} \\
- \left( r - \frac{\sigma^2}{2} + \alpha_1 \sigma_2 \rho_{12} + \sigma_2 \sigma_3 \rho_{23} \right) \frac{\alpha_2 \sqrt{T}}{\sigma},
\]
\[
a_2 = \frac{1}{\sigma \sqrt{T}} \ln \frac{S_t^{a_1}(0)}{S_0^{a_2}(0)} + \left( r - \frac{\sigma^2}{2} + \alpha_2 \sigma_2 \rho_{12} + \sigma_1 \sigma_3 \rho_{13} \right) \frac{\alpha_1 \sqrt{T}}{\sigma} \\
- \left( r - \frac{\sigma^2}{2} + \alpha_2 \sigma_2^2 + \sigma_2 \sigma_3 \rho_{23} \right) \frac{\alpha_2 \sqrt{T}}{\sigma},
\]
with \( \sigma^2 = \alpha_1^2 \sigma_1^2 + \alpha_2^2 \sigma_2^2 - 2 \alpha_1 \alpha_2 \sigma_1 \sigma_2 \rho_{12} \) and \( N(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{1}{2}x^2} dx \).

**Proof.** We write \( J_1 \) as
\[
J_1 = \frac{(1 - \alpha)}{D} e^{-rT} E^P \left[ V(T) S_t^{a_1}(T) 1_{\{S_t^{a_1}(T) > S_0^{a_2}(T)\}} \right]_{\mathcal{F}_0} \\
- \frac{(1 - \alpha)}{D} e^{-rT} E^P \left[ V(T) S_t^{a_2}(T) 1_{\{S_t^{a_1}(T) > S_0^{a_2}(T)\}} \right]_{\mathcal{F}_0} \\
:= \frac{(1 - \alpha)}{D} I_1 - \frac{(1 - \alpha)}{D} I_2. \tag{7}
\]

To calculate \( I_1 \), we define a new measure \( Q_1 \) as
\[
\frac{dQ_1}{dP} = \exp \left[ \alpha_1 \sigma_1 W_1(T) + \sigma_3 W_3(T) - \frac{1}{2} (\alpha_1^2 \sigma_1^2 + \sigma_3^2 + 2 \alpha_1 \rho_{13} \sigma_1 \sigma_3) T \right].
\]

By Girsanov’s theorem,
\[
W_1^{Q_1}(T) = W_1(T) - \alpha_1 \sigma_1 T - \sigma_3 \rho_{13} T,
\]
\[
W_2^{Q_1}(T) = W_2(T) - \alpha_1 \sigma_2 \rho_{12} T - \sigma_3 \rho_{23} T,
\]
\[
W_3^{Q_1}(T) = W_3(T) - \sigma_3 T - \alpha_1 \sigma_1 \rho_{13} T
\]
are the standard Brownian motions under the measure \( Q_1 \). Then we have
\[
I_1 = e^{-rT} E^{Q_1} \left[ \frac{dP}{dQ_1} V(T) S_t^{a_1}(T) 1_{\{S_t^{a_1}(T) > S_0^{a_2}(T)\}} \right]_{\mathcal{F}_0} \\
= S_t^{a_1}(0) V(0) e^{\alpha_1 r T + \frac{\alpha_1^2 \sigma_1^2}{2} (\alpha_1 - 1) T + \alpha_1 \sigma_1 \sigma_3 \rho_{13} T} P^{Q_1} \left( S_t^{a_1}(T) > S_0^{a_2}(T) \right) \\
+ \left( r - \frac{\sigma^2}{2} + \alpha_1 \sigma_1^2 + \sigma_1 \sigma_3 \rho_{13} \right) \alpha_1 T - \left( r - \frac{\sigma^2}{2} + \alpha_1 \sigma_2 \rho_{12} + \sigma_1 \sigma_3 \rho_{23} \right) \alpha_2 T \\
= S_t^{a_1}(0) V(0) e^{\left( r + \sigma_1 \sigma_3 \rho_{13} - \frac{\sigma^2}{2} + \frac{\alpha_1 \sigma_1^2}{2} \right) \alpha_1 T} N(a_1). \tag{8}
\]
To calculate $I_2$, we define a new measure $Q_2$ as

$$
\frac{dQ_2}{dP} = \exp \left[ \alpha_2 \sigma_2 W_1(T) + \sigma_3 W_3(T) - \frac{1}{2} \left( \alpha_2^2 \sigma_2^2 + \sigma_3^2 + 2 \alpha_2 \rho_{23} \sigma_2 \sigma_3 \right) T \right].
$$

Under the measure $Q_2$,

\begin{align*}
W_1^{Q_2}(T) &= W_1(T) - \alpha_2 \sigma_2 \rho_{12} T - \sigma_3 \rho_{13} T, \\
W_2^{Q_2}(T) &= W_2(T) - \alpha_2 \sigma_2 T - \sigma_3 \rho_{23} T, \\
W_3^{Q_2}(T) &= W_3(T) - \sigma_3 T - \alpha_2 \sigma_2 \rho_{23} T
\end{align*}

are the standard Brownian motions and $I_2$ can be calculated in a similar way above. This completes the proof. $\square$

**Lemma 3.2.** Let us consider $J_2$ in Eq. (5), then $J_2$ is given by

\begin{align*}
J_2 &= e^{-rT} S_1^{\alpha_1}(0) M_1(T) e^{(r - \frac{\sigma_2^2}{2}) \alpha_1 T + \frac{\alpha_2^2 \sigma_2^2}{2} T - \frac{\alpha_1 \alpha_2 \rho_{14}}{a} \int_0^T f(s, T, a) ds} N_2(b_1, b_2, b_3, b_4, \theta_1) \\
&\quad - e^{-rT} S_2^{\alpha_2}(0) M_1(T) e^{(r - \frac{\sigma_3^2}{2}) \alpha_2 T + \frac{\alpha_2^2 \sigma_3^2}{2} T - \frac{\alpha_2 \rho_{34}}{a} \int_0^T f(s, T, a) ds} N_2(b_3, b_4, \theta_1),
\end{align*}

where

\begin{align*}
\theta_1 &= \frac{(\alpha_1 \sigma_1 \rho_{13} - \alpha_2 \sigma_2 \rho_{23})}{\sigma}, \\
M_1(T) &= \exp \left[ -bT - \frac{\lambda(0)}{a} f(0, T, a) + \frac{\sigma_1^2}{2a^2} \int_0^T f^2(s, T, a) ds \right], \\
b_1 &= \frac{\ln \frac{S_1^{\alpha_1}(0)}{S_2^{\alpha_2}(0)} + \alpha_1 \left( r - \frac{\sigma_2^2}{2} \right) T - \alpha_2 \left( r - \frac{\sigma_3^2}{2} \right) T + (\alpha_1^2 \sigma_1^2 - \alpha_1 \alpha_2 \sigma_1 \sigma_2 \rho_{12}) T}{\alpha_1 \sigma_1 \rho_{14}} \frac{\int_0^T f(s, T, a) ds}{\sigma \sqrt{T}} + \frac{\alpha_2 \sigma_2 \rho_{24}}{a} \int_0^T f(s, T, a) ds, \\
b_2 &= \frac{\ln \frac{V(0)}{D}}{\sigma_3 \sqrt{T}} + \left( r - \frac{\sigma_3^2}{2} \right) T + \alpha_1 \sigma_1 \sigma_3 \rho_{13} T - \frac{\sigma_3 \rho_{34}}{a} \int_0^T f(s, T, a) ds, \\
b_3 &= \frac{\ln \frac{S_1^{\alpha_1}(0)}{S_2^{\alpha_2}(0)} + \alpha_1 \left( r - \frac{\sigma_2^2}{2} \right) T - \alpha_2 \left( r - \frac{\sigma_3^2}{2} \right) T + (\alpha_1 \alpha_2 \sigma_1 \sigma_2 \rho_{12} - \alpha_2^2 \sigma_2^2) T}{\alpha_1 \sigma_1 \rho_{14}} \frac{\int_0^T f(s, T, a) ds}{\sigma \sqrt{T}} + \frac{\alpha_2 \sigma_2 \rho_{24}}{a} \int_0^T f(s, T, a) ds, \\
b_4 &= \frac{\ln \frac{V(0)}{D}}{\sigma_3 \sqrt{T}} + \left( r - \frac{\sigma_3^2}{2} \right) T + \alpha_2 \sigma_2 \sigma_3 \rho_{23} T - \frac{\sigma_3 \rho_{34}}{a} \int_0^T f(s, T, a) ds, \\
\text{and} \\
N_2(n_1, n_2, \rho) &= \frac{1}{2\pi \sqrt{1-\rho^2}} \int_{-\infty}^{n_1} \int_{-\infty}^{n_2} e^{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)} dy dx.
\end{align*}
Proof. We write $J_2$ as

$$J_2 = e^{-rT}E^P \left[ e^{-\int_0^T \lambda(s)ds} S_1^{\alpha_1}(T) 1_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | F_0 \right] - e^{-rT}E^P \left[ e^{-\int_0^T \lambda(s)ds} S_2^{\alpha_2}(T) 1_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | F_0 \right]$$

$$:= e^{-rT}I_3 - e^{-rT}I_4. \quad (9)$$

To calculate $I_4$, we define a new measure $Q_3$ such that

$$\frac{dQ_3}{dP} = \frac{e^{-\int_0^T \lambda(s)ds}}{E[e^{-\int_0^T \lambda(s)ds} | F_0]}.$$

Then, $I_3$ is given by

$$I_3 = M_1(T)E^{Q_3} \left[ S_1^{\alpha_1}(t) 1_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | F_0 \right], \quad (10)$$

where $M_1(T) = E[e^{-\int_0^T \lambda(s)ds} | F_0]$. To calculate Eq. (10), we define a new measure $\tilde{Q}_3$ such that

$$\frac{d\tilde{Q}_3}{dQ_3} = \exp \left[ \alpha_1 \sigma_1 W_1^{Q_3}(T) - \frac{1}{2} \alpha_1^2 \sigma_1^2 T \right].$$

Then, under the measure $\tilde{Q}_3$,

$$W_1^{\tilde{Q}_3}(T) = W_1(T) + \frac{\sigma_1 \rho_{14}}{a} \int_0^T f(s, T, a)ds - \alpha_1 \sigma_1 T,$$

$$W_2^{\tilde{Q}_3}(T) = W_2(T) + \frac{\sigma_1 \rho_{24}}{a} \int_0^T f(s, T, a)ds - \alpha_1 \sigma_1 \rho_{12} T,$$

$$W_3^{\tilde{Q}_3}(T) = W_3(T) + \frac{\sigma_1 \rho_{34}}{a} \int_0^T f(s, T, a)ds - \alpha_1 \sigma_1 \rho_{13} T,$$

$$W_4^{\tilde{Q}_3}(T) = W_4(T) + \frac{\sigma_1}{a} \int_0^T f(s, T, a)ds - \alpha_1 \sigma_1 \rho_{14} T$$

are the standard Brownian motions. Under the measure $\tilde{Q}_3$, we have

$$I_3 = S_1^{\alpha_1}(0) M_1(T) e^{(r-\frac{\sigma_1^2}{2}) \alpha_1 T + \frac{\sigma_1^2}{2} \rho_{14} T} f_0^T f(s, T, a)ds E^{\tilde{Q}_3} \left[ 1_{\{S_1(T) > S_2(T), V(T) > D\}} | F_0 \right],$$

and

$$E^{\tilde{Q}_3} \left[ 1_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} \right]$$

$$= P^{\tilde{Q}_3} \left( S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D \right)$$

$$= P^{\tilde{Q}_3} \left( \alpha_2 \sigma_2 W_2^{\tilde{Q}_3}(T) - \alpha_1 \sigma_1 W_1^{\tilde{Q}_3}(T) < b_1 \sigma \sqrt{T}, -\sigma_3 W_3^{\tilde{Q}_3}(T) < b_2 \sigma \sqrt{T} \right)$$

$$= P^{\tilde{Q}_3} (z_1 < b_1, z_2 < b_2).$$

Since $z_1$ and $z_2$ are correlated standard normal variables, we obtain

$$I_3 = S_1^{\alpha_1}(0) M_1(T) e^{(r-\frac{\sigma_1^2}{2}) \alpha_1 T + \frac{\sigma_1^2}{2} \rho_{14} T} f_0^T f(s, T, a)ds N_2(b_1, b_2, \theta_1),$$

where $N_2$ is the bivariate normal cumulative distribution function.
where $\theta_1$ is the correlation between $z_1$ and $z_2$.

Under the measure $Q_3$, $I_4$ is represented by

$$I_4 = M_1(T)E^{Q_3} \left[ S_2^2(t) \mathbf{1}\{S_1^2(T) > S_2^2(T), V(T) > D\} | \mathcal{F}_0 \right].$$

(11)

For the calculation of $I_4$, we define a new measure $\hat{Q}_3$ such that

$$\frac{d\hat{Q}_3}{dP} = \exp \left[ \alpha_2 \sigma_2 W^{Q_3}(T) - \frac{1}{2} \alpha_2^2 \sigma_2^2 T \right].$$

Then, under the measure $\hat{Q}_3$, we can calculate $I_4$ in a similar way to $I_3$. 

\begin{lemma}
Let us consider $J_3$ in the equation (5), then $J_3$ is given by

$$J_3 = \frac{(1 - \alpha)}{D} S_1^{\sigma_1}(0) V(0) e^{\left( r + \sigma_1 \sigma_3 \rho_{13} - \frac{\sigma_1^2}{2} + \frac{\alpha_1^2}{2} \right) T}$$

$$\times e^{-\frac{\alpha_1 \sigma_1 \sigma_3 \rho_{13}}{a} \int_0^T f(s, T, a) ds - \frac{\sigma_3^2 \rho_{34}}{a} \int_0^T f(s, T, a) ds} M_1(T) N_2(c_1, c_2, \theta_1)$$

$$- \frac{(1 - \alpha)}{D} S_2^{\sigma_2}(0) V(0) e^{\left( r + \sigma_2 \sigma_3 \rho_{23} - \frac{\sigma_2^2}{2} + \frac{\alpha_2^2}{2} \right) T}$$

$$\times e^{-\frac{\alpha_2 \sigma_2 \sigma_3 \rho_{23}}{a} \int_0^T f(s, T, a) ds - \frac{\sigma_3^2 \rho_{34}}{a} \int_0^T f(s, T, a) ds} M_1(T) N_2(c_1, c_2, \theta_1),$$

where

$$c_1 = \frac{\ln \frac{S_1^{\sigma_1}(0)}{S_2^{\sigma_2}(0)} + (\alpha_1^2 \sigma_1^2 - \alpha_1 \sigma_1 \sigma_3 \rho_{13} - \alpha_1 \alpha_2 \sigma_1 \sigma_2 \rho_{12} - \alpha_2 \sigma_2 \sigma_3 \rho_{23}) T + \alpha_1 \left( r - \frac{\sigma_1^2}{2} \right) T}{\sigma_1 \sqrt{T}}$$

$$+ \frac{-\alpha_2 \left( r - \frac{\sigma_2^2}{2} \right) T - \frac{\alpha_1 \sigma_1 \sigma_3 \rho_{13}}{a} \int_0^T f(s, T, a) ds + \frac{\alpha_2 \sigma_2 \sigma_3 \rho_{23}}{a} \int_0^T f(s, T, a) ds}{\sigma_3 \sqrt{T}},$$

$$c_2 = \frac{\ln \frac{V(0)}{D} + \left( r + \frac{\sigma_2^2}{2} \right) T + \alpha_1 \sigma_1 \sigma_3 \rho_{13} T - \frac{\sigma_3 \rho_{34}}{a} \int_0^T f(s, T, a) ds}{\sigma_3 \sqrt{T}},$$

$$c_3 = \frac{\ln \frac{S_2^{\sigma_2}(0)}{S_1^{\sigma_1}(0)} + (\alpha_1 \alpha_2 \sigma_1 \sigma_2 \rho_{12} + \alpha_1 \sigma_1 \sigma_3 \rho_{13} - \alpha_2^2 \sigma_2^2 - \alpha_2 \sigma_2 \sigma_3 \rho_{23}) T + \alpha_1 \left( r - \frac{\sigma_1^2}{2} \right) T}{\sigma_1 \sqrt{T}}$$

$$+ \frac{-\alpha_2 \left( r - \frac{\sigma_2^2}{2} \right) T - \frac{\alpha_1 \sigma_1 \sigma_3 \rho_{13}}{a} \int_0^T f(s, T, a) ds + \frac{\alpha_2 \sigma_2 \sigma_3 \rho_{23}}{a} \int_0^T f(s, T, a) ds}{\sigma_3 \sqrt{T}},$$

$$c_4 = \frac{\ln \frac{V(0)}{D} + \left( r + \frac{\sigma_2^2}{2} \right) T + \alpha_2 \sigma_2 \sigma_3 \rho_{23} T - \frac{\sigma_3 \rho_{34}}{a} \int_0^T f(s, T, a) ds}{\sigma_3 \sqrt{T}},$$

and $\theta_1$, $\sigma$, $f$, $M_1(T)$ and $N_2$ are defined in Lemma 3.2.
Proof. $J_3$ is represented by

$$J_3 = \frac{(1-\alpha)}{D} e^{-rT} E^P \left[ e^{-\int_0^T \lambda(s) ds} V(T) S_1^{\alpha_1}(T) 1_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | F_0 \right]$$

$$- \frac{(1-\alpha)}{D} e^{-rT} E^P \left[ e^{-\int_0^T \lambda(s) ds} V(T) S_2^{\alpha_2}(T) 1_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | F_0 \right]$$

$$:= \frac{(1-\alpha)}{D} I_5 - \frac{(1-\alpha)}{D} I_6. \tag{12}$$

$I_5$ and $I_6$ can be calculated under the measure $Q_3$ defined in Lemma 3.2. Thus, under the measure $Q_3$, $I_5$ is given by

$$I_5 = M_1(T) e^{-rT} E^{Q_3} \left[ V(T) S_1^{\alpha_1}(T) 1_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | F_0 \right], \tag{13}$$

where $M_1(T)$ is defined in Lemma 3.2. With the standard Brownian motions under the measure $Q_3$, we define a new measure $Q_4$ such that

$$\frac{dQ_4}{dQ_3} = \exp \left[ \alpha_1 \sigma_1 W_1^{Q_3}(T) + \sigma_3 W_3^{Q_3}(T) - \frac{1}{2} (\alpha_1^2 \sigma_1^2 + \sigma_3^2 + 2\alpha_1 \rho_{13} \sigma_1 \sigma_3) T \right].$$

By Girsanov’s theorem,

$$W_1^{Q_4}(T) = W_1^{Q_3}(T) - \alpha_1 \sigma_1 T - \sigma_3 \rho_{13} T,$$

$$W_2^{Q_4}(T) = W_2^{Q_3}(T) - \alpha_1 \sigma_1 \rho_{12} T - \sigma_3 \rho_{23} T,$$

$$W_3^{Q_4}(T) = W_3^{Q_3}(T) - \sigma_3 T - \alpha_1 \sigma_1 \rho_{13} T$$

are the standard Brownian motions under the measure $Q_4$. Then, we obtain

$$I_5 = \left. S_1^{\alpha_1}(0) V(0) e^{\left( r + \sigma_3 \rho_{13} - \frac{\sigma^2_2}{2} + \frac{\alpha_1 \sigma_1^2}{2} \right) \alpha_1 T - \frac{\sigma_3 \rho_{23}}{2} \frac{r T}{2} f(s, T, a) ds - \frac{\sigma_3 \rho_{13}}{2} \frac{r T}{2} f(s, T, a) ds} \right|_{F_0}$$

$$\times M_1(T) E^{Q_4} \left[ 1_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | F_0 \right]$$

Since $E^{Q_4} \left[ 1_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | F_0 \right] = P^{Q_4}(S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D)$, we can obtain

$$E^{Q_4} \left[ 1_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | F_0 \right] = N_2(c_1, c_2, \theta_1).$$

In a similar way, we represent $I_6$ under the measure $Q_3$ as

$$I_6 = e^{-rT} M_1(T) E^{Q_3} \left[ V(T) S_2^{\alpha_2}(T) 1_{\{S_1^{\alpha_1}(T) > S_2^{\alpha_2}(T), V(T) > D\}} | F_0 \right]. \tag{14}$$

To calculate $I_6$, we define a new measure $Q_5$ equivalent to $Q_3$ by

$$\frac{dQ_5}{dQ_3} = \exp \left[ \alpha_2 \sigma_2 W_2^{Q_3}(T) + \sigma_3 W_3^{Q_3}(T) - \frac{1}{2} (\alpha_2^2 \sigma_2^2 + \sigma_3^2 + 2\alpha_2 \rho_{23} \sigma_2 \sigma_3) T \right].$$

By Girsanov’s theorem,

$$W_1^{Q_5}(T) = W_1^{Q_3}(T) - \alpha_2 \sigma_2 \rho_{12} T - \sigma_3 \rho_{13} T,$$

$$W_2^{Q_5}(T) = W_2^{Q_3}(T) - \alpha_2 \sigma_2 T - \sigma_3 \rho_{23} T,$$

$$W_3^{Q_5}(T) = W_3^{Q_3}(T) - \sigma_3 T - \alpha_2 \sigma_2 \rho_{23} T$$
are the standard Brownian motions under the measure $Q_5$. Using these Brownian motions and a similar way to the calculation for $I_5$, we can calculate $I_6$ under the measure $Q_5$. This completes the proof. □

Combining the Lemmas, we finally present the closed-form formula for vulnerable power exchange option price under the hybrid model.

**Theorem 3.4.** The price at time 0 of vulnerable power exchange option under the hybrid model is given by

$$C = J_1 + J_2 - J_3,$$

where $J_1$, $J_2$ and $J_3$ are defined in Lemma 3.1, Lemma 3.2 and Lemma 3.3, respectively.

4. Concluding remarks

In this paper, we consider the hybrid model for the valuation of vulnerable power exchange option. The hybrid model is constructed as a combination of the reduced-from model and the structural model. Applying the change of measure repeatedly, we derive the closed-form pricing formula of the option. Finally, we provide the formula using the bivariate normal cumulative function and the standard normal cumulative function.

References
