LOCAL EXISTENCE OF CHERN-SIMONS GAUGED O(3) SIGMA EQUATIONS

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Abstract. In this paper we study the Cauchy problem for the Chern-Simons gauged $O(3)$ sigma model. We prove the local existence of solutions with low regularity initial data, observing null forms of the system and applying bilinear estimates for wave-Sobolev space $H^{s,b}$.

1. Introduction

The classical $O(3)$ sigma model originates from the description of the planar ferromagnet. The $O(3)$ sigma model in 2-dimensional Euclidean space is a popular one in theoretical physics. From the point of view of a particle physicist, the model has one important drawback: it is scale invariant and as a result its soliton solutions have arbitrary size, making them unsuitable as models for particles. The new possibilities of breaking the scale invariance of the sigma model were proposed by introducing a $U(1)$ gauge field whose dynamics is governed by Maxwell, Chern-Simons and Maxwell-Chern-Simons action. Some analysis of the self-dual equations can be found in [1, 6].

Consider the following Chern-Simons gauged $O(3)$ sigma equations,

\begin{align}
D_\mu D^\mu \phi &= -\frac{1}{\kappa^2} \phi (\langle D^\mu \phi, D_\mu \phi \rangle + \phi_3 (1 - \phi_3)^2 (1 + 2\phi_3)) + \frac{1}{\kappa^2} (0, 0, (1 - \phi_3)^2 (1 + 2\phi_3)), \\
\kappa^2 \epsilon^{\mu\nu\lambda} F_{\nu\lambda} &= -\langle n \times \phi, D^\mu \phi \rangle,
\end{align}

where $\phi$ is a three component vector with unit norm, i.e. $\langle \phi, \phi \rangle = 1$, $A_\mu : \mathbb{R}^{1,2} \rightarrow \mathbb{R}$ is the gauge field with $\mu = 0, 1, 2$, $\epsilon^{\alpha\beta\gamma}$ is the totally-antisymmetric tensor with $\epsilon^{012} = 1$, and $n = (0, 0, 1)$ is the north pole of $S^2$. The gauge covariant derivative is defined by $D_\mu \phi = \partial_\mu \phi + A_\mu (n \times \phi)$ and the Maxwell field is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The constant $\kappa > 0$ is a Chern-Simons coupling constant. Greek indices, such as $\mu, \nu$ will refer to all indices $0, 1, 2$, whereas latin indices, such as $i, j, k$, will refer only to the spatial indices $1, 2,$
unless otherwise specified. The usual inner product and cross product on $\mathbb{R}^3$ are given by

$$\langle a, b \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

$$a \times b = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

The Lagrangian for the Chern-Simons gauged $O(3)$ sigma model, proposed in [3, 9], is given by

$$\mathcal{L} = \frac{1}{2} D_\mu \phi \cdot D^\mu \phi + \frac{\kappa}{4} \epsilon^{\mu \nu \rho} A_\mu F_{\nu \rho} - \frac{1}{2\kappa^2} (1 + \phi_3)(1 - \phi_3)^3.$$  (3)

The energy density corresponding to the Lagrangian density (3) is

$$E(\phi, A) = \frac{1}{2} \left[ |D_\mu \phi|^2 + \frac{1}{\kappa^2} (1 + \phi_3)(1 - \phi_3)^3 \right].$$  (4)

The conservation of the total energy implies that

$$E(t) := \int_{\mathbb{R}} E(t, x) \, dx = \int_{\mathbb{R}} E(0, x) \, dx.$$  (5)

In the usual sigma model (called wave map), evolution problems have been studied extensively. Let us review briefly the results for global existence in time.

Wave map in 1+1 dimension extends smoothly all the time in [5, 13]. In (3+1) dimension, development of singularities from smooth initial data was shown in Section 2.
[7] by using the self-similar structure of the sigma model. Also Shatah proved that there exists global weak solution to wave map which has an \( S^n \) target manifold. The space two-dimensional case is critical. Tataru[15] and Tao[14] proved global regularity of wave maps in \((2+1)\) dimension under the assumption of small Besov norm, respectively, small energy. In [16], Tataru proved rough solutions and the continuous dependence on the initial data which is small in the critical Sobolev spaces. Then Rodnianski [12] and Tataru [11] resolves the finite time blow up solutions for the wave map problem from \( \mathbb{R}^{2+1} \to S^2 \).

The global solutions of Chern-Simons sigma equations in one space dimension was shown in [8]. The following is our main result.

\textbf{Theorem 1.1.} Let \( s > 3/2 \), consider the Cauchy problem of Chern-Simons gauged \( O(3) \) sigma equations (6)-(9), with the initial data in the following Sobolev space:

\[
A_\mu(0, \cdot) = a_\mu \in H^s(\mathbb{R}^2), \quad \phi(0, \cdot) = \phi_0 \in H^s(\mathbb{R}^2), \quad \partial_t \phi(0, \cdot) = \phi_1 \in H^{s-1}(\mathbb{R}^2)
\]

satisfying the constraint (10) and \( \langle \phi_0, \phi_1 \rangle = 0 \), then there exists a \( T > 0 \) and a solution \((A, \phi)\) of (6)-(9) in \([0, T) \times \mathbb{R}^2\) with

\[
A_\mu \in C([0, T); H^s(\mathbb{R}^2)), \quad \phi \in C([0, T]; H^s(\mathbb{R}^2)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^2)).
\]

We use \( a \lesssim b \) denote \( a \leq Cb \) for some constant \( C \). A point in the 2+1 dimensional Minkowski space is written as \((t, x) = (x^\alpha)_{0 \leq \alpha \leq 2}\). Greek indices range from 0 to 2, and Roman indices range from 1 to 2. We raise and lower indices with the Minkowski metric, diag(1, -1, -1). We write \( \partial_\alpha = \partial_{x^\alpha} \) and \( \partial_t = \partial_0 \), and we also use the Einstein notation. Therefore, \( \partial^\alpha \partial_\alpha = \partial_0^2 - \Delta = \Box \).

\section{Preliminaries}

In this section we introduce basic facts of equations, function spaces and some related theorems.

Let us consider calculus related with covariant derivative.

(1) \[ \partial_\mu \langle \phi, \psi \rangle = \langle D_\mu \phi, \psi \rangle + \langle \phi, D_\mu \psi \rangle, \]

(2) \[ D_\mu D_\nu \phi - D_\nu D_\mu \phi = F_{\mu \nu}(n \times \phi), \]

(3) \[ D_\mu (f \phi) = (\partial_\mu f) \phi + f D_\mu \phi, \]

where \( \phi, \psi \) are 3 component vector functions and \( f \) is a scalar function.

We review the constraint on the formulation of Cauchy problem (6)–(11). Using the above formula and equations (6)–(7), we can check

\[
\partial_t \left( \kappa (\partial_1 A_2 - \partial_2 A_1) + \langle n \times \phi, D_0 \phi \rangle \right)
\]

\[
= \kappa \partial_1 F_{02} - \kappa \partial_2 F_{01} + \langle D_0 (n \times \phi), D_0 \phi \rangle + \langle n \times D_0 D_0 \phi \rangle
\]

\[
= \langle D_\mu (n \times \phi), D^\mu \phi \rangle + \langle (n \times \phi), D_\mu D^\mu \phi \rangle
\]

\[
= \langle (n \times \phi), D_\mu D^\mu \phi \rangle = 0.
\]
Then (4) implies that constraint (10) is automatically satisfied at $t \geq 0$ if the initial data satisfy

$$\kappa (\partial_1 a_2 - \partial_2 a_1) + \langle n \times \phi, \phi_1 + a_0 (n \times \phi_0) \rangle = 0.$$ 

Therefore we have shown that if $(\phi, A_\mu)$ is a solution of the system (6)-(9) subject to the initial data satisfying constraint (10), then it is also a solution of equations (1)-(2) with the same initial data.

We can also check that the constraint $|\phi|^2 = 1$ is preserved as follows. If the equation (6) is satisfied in time slab $[0, T] \times \mathbb{R}^2$, then $\rho = |\phi|^2 - 1$ is the solution of the following equation

$$[\partial_\mu \partial^\mu + 2 \langle D_\mu \phi, D_\mu \phi \rangle + 2 \phi_3 (1 - \phi_3)^2 (1 + 2 \phi_3)] (|\phi|^2 - 1)
= 2 \langle D_\mu \phi, D_\mu \phi \rangle + 2 \langle \phi, D_\mu D_\mu \phi \rangle + 2 |\phi|^2 \langle D_\mu \phi, D_\mu \phi \rangle - 2 \langle D_\mu \phi, D_\mu \phi \rangle
+ 2 \phi_3 |\phi|^2 (1 - \phi_3)^2 (1 + 2 \phi_3) - 2 \phi_3 (1 - \phi_3)^2 (1 + 2 \phi_3) = 0.$$ 

This is a linear Klein-Gordon equation for the function $\rho$ with external potential $2 \langle D_\mu \phi, D_\mu \phi \rangle + 2 \phi_3 (1 - \phi_3)^2 (1 + 2 \phi_3)$. With the initial data $\rho(0) = |\phi_0|^2 - 1 = 0$ and $\partial_t \rho(0) = 2 \langle \phi_0, \phi_1 \rangle = 0$, we have $\rho = 0$ in time slab $[0, T] \times \mathbb{R}^2$.

Now we introduce function spaces as well as used. The wave-Sobolev spaces $H^{s,b}$ are $L^2$-based Sobolev spaces on the Minkowski space-time $\mathbb{R}^{1+n}$, with Fourier weights adapted to the symbol of the D’Alembertian $\Box = -\partial_t^2 + \Delta$. Specifically, for given $s, b \in \mathbb{R}$, $H^{s,b}$ is the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^{1+n})$ with respect to the norm

$$\|u\|_{H^{s,b}} = \| \langle \xi \rangle^s (|\tau| - |\xi|)^b \hat{u}(\tau, \xi)\|_{L^2_{\tau,\xi}},$$

$$\|u\|^2_{H^{s,b}} = \int \int (1 + |\xi|^2)^s (1 + |\tau| - |\xi|^2)^b \hat{u}(\tau, \xi) d\tau d\xi,$$

$$N^2_{s+1,s} = \int \int (1 + |\tau| + |\xi|)^{2s+2} (1 + |\tau| - |\xi|)^{2s} \hat{u}(\tau, \xi) d\tau d\xi,$$

$$Z^2_{s+1,s} = \int \int ((\tau + |\xi|)^2 (\xi^2 + 1)^s (\tau + |\xi|)^2 \hat{u}^2(\tau, \xi) d\tau d\xi
+ \int \int ((\tau - |\xi|)^2 (\xi^2 + 1)^s (\tau - |\xi|)^2 \hat{u}^2(\tau, \xi) d\tau d\xi,$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$ and $\hat{u}(\tau, \xi) = \int \int e^{-i(\tau x + \xi \cdot x)} u(t, x) dt dx$ is the space-time Fourier transform. 

Here the “elliptic weight” $\langle \xi \rangle^s$ is a familiar feature of the standard Sobolev space $H^s = H^s(\mathbb{R}^n)$, obtained as the completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm $\|f\|_{H^s} = \| \langle \xi \rangle^s \hat{f}(\xi)\|_{L^2_{\xi}}$, where $\hat{f}(\xi) = \int e^{-ix \cdot \xi} f(x) dx$ is the spatial Fourier transform. The “hyperbolic weight” $|\tau - |\xi||^b$, on the other hand, reflects the fact that the $H^{s,b}$-norm is adapted to $\Box$, whose symbol is $\tau^2 - |\xi|^2$. 


For $T > 0$, let $H^{s,b}(S_T)$ denote the restriction space to $S_T = (-T,T) \times \mathbb{R}^2$. We recall that fact that (see for [10])

$$H^{s,b}(S_T) \hookrightarrow C([-T,T];H^s) \text{ for } b > \frac{1}{2},$$

where $\hookrightarrow$ stands for Sobolev embedding.

We need product estimates of the form $H^{s_1,b_1} \cdot H^{s_2,b_2} \hookrightarrow H^{-s_0,-b_0}$ which means that

$$\|uv\|_{H^{-s_0,-b_0}} \leq C\|u\|_{H^{s_1,b_1}} \|v\|_{H^{s_2,b_2}} \text{ for all } u,v \in \mathcal{S}(\mathbb{R}^{1+n}),$$

where $C$ depends on the $s_\alpha, b_\alpha$ and $d$. If this holds, it is said that the exponent matrix

$$\begin{pmatrix}
s_0 & s_1 & s_2 \\
 b_0 & b_1 & b_2
\end{pmatrix}$$

is a product. In recent paper [2], the following product estimate in $\mathbb{R}^{1+2}$ is established.

**Theorem 2.1.** Assume

$$b_0 + b_1 + b_2 > \frac{1}{2},$$

$$b_0 + b_1 \geq 0,$$

$$b_1 + b_2 \geq 0,$$

$$b_0 + b_2 \geq 0,$$

$$s_0 + s_1 + s_2 > \frac{3}{2} - (b_0 + b_1 + b_2),$$

$$s_0 + s_1 + s_2 > 1 - \min(b_0 + b_1, b_0 + b_2, b_1 + b_2),$$

$$s_0 + s_1 + s_2 > \frac{1}{2} - \min(b_0, b_1, b_2),$$

$$s_0 + s_1 + s_2 > \frac{3}{4},$$

$$(s_0 + b_0) + 2s_1 + 2s_2 > 1,$$

$$2s_0 + (s_1 + b_1) + 2s_2 > 1,$$

$$2s_0 + 2s_1 + (s_2 + b_2) > 1,$$

$$s_0 + s_1 \geq \max(0, -b_2),$$

$$s_1 + s_2 \geq \max(0, -b_0),$$

$$s_0 + s_2 \geq \max(0, -b_1).$$

Then

$$\begin{pmatrix}
s_0 & s_1 & s_2 \\
 b_0 & b_1 & b_2
\end{pmatrix}$$

is a product.
Now we consider the following nonlinear Cauchy problem:

$$\Box u = F \quad (t, x) \in \mathbb{R}^{1+n},$$
$$u|_{t=0} = f, \quad \partial_t u|_{t=0} = g.$$  \hfill (5)

If $F = Q(u, v)$, where $u, v : \mathbb{R}^{1+2} \rightarrow \mathbb{R}^m$, and $Q$ is a linear combination of the three basic null forms as follows.

$$Q_0(u, v) = \partial_t u \partial_t v - \nabla u \cdot \nabla v,$$
$$Q_{ij}(u, v) = \partial_i u \partial_j v - \partial_j u \partial_i v,$$
$$Q_{ij}(u, v) = \partial_i u \partial_j v - \partial_j u \partial_i v,$$

(6)

where $\partial_j$ stands for spatial derivatives, and $\nabla$ is the spatial gradient. The following null form estimates in Sobolev space which was proven by Grigoryan and Nahmod in the $n = 2$ in [4].

**Lemma 2.2.** Let $s > \frac{3}{2}$, $b \in (\frac{1}{2}, 1)$ and $\epsilon \in [0, 1-b]$, then

$$\|Q(u, v)\|_{H^{s-1}, b-1+} \lesssim \|u\|_{H^s} \|v\|_{H^{s,b}},$$

where $Q(u, v)$ includes all cases in (6).

3. Low regularity local well-posedness

The system (6)-(9) under the Lorenz gauge condition $\partial_\mu A^\mu = 0$ can be rewritten as follows.

$$\Box \phi = -\phi Q_0(\phi, \phi) - A_\mu A^\mu \phi_3(n - \phi_3 \phi) + 2\phi_3 A^\mu \partial_\mu \phi \times \phi,$$
$$A_\mu = \epsilon_{\mu\nu\rho}Q^{\nu\rho}(n \times \phi, \phi) + 2\epsilon_{\mu\nu\rho}\phi^{\nu}(A^\nu|n \times \phi|^2),$$

where $Q_0$, and $Q^{\nu\rho}$ are the standard null forms.

$$\Box \phi = -\phi (Q_0(\phi, \phi) + A_\mu A^\mu |n \times \phi|^2) - 2A^\mu \partial_\mu (n \times \phi) - A_\mu A^\mu (\phi_1, \phi_2, 0),$$
$$\Box A_0 = Q_{12}(n \times \phi, \phi) + \partial_t (A_2 |n \times \phi|^2) - \partial_2 (A_1 |n \times \phi|^2),$$
$$\Box A_1 = Q_{02}(n \times \phi, \phi) + \partial_0 (A_2 |n \times \phi|^2) - \partial_2 (A_0 |n \times \phi|^2),$$
$$\Box A_2 = -Q_{01}(n \times \phi, \phi) - \partial_0 (A_1 |n \times \phi|^2) + \partial_1 (A_0 |n \times \phi|^2).$$

We specify data

$$A_\mu(0) \in H^s, \quad (\phi, \partial_t \phi)(0) \in H^s \times H^{s-1}.$$  \hfill (3)

The data for $\partial_t A_\mu$ are given by the constraints

$$\partial_t A_0(0) = \partial_1 A_1(0) + \partial_2 A_2(0) \in H^{s-1},$$
$$\partial_t A_j(0) = \partial_j A_0(0) - J_k(0) \in H^{s-1},$$

where $J_k = \langle n \times \phi, D_j \phi \rangle = \langle n \times \phi, \partial_j \phi \rangle + \langle n \times \phi, A_j(n \times \phi) \rangle$, hence $J_k(0) \in H^{s-1}$ with the norm bounded in terms of the norm of (3).

In the remaining part of this section, we present estimates (1)-(2) with $s > \frac{3}{2}$ and a given $b > \frac{1}{2}$. 
Proof of (1) for $\phi Q_0(\phi, \phi)$. We shall prove that
\begin{equation}
\|\phi Q_0(\phi, \phi)\|_{H^{s-1, b-1+\varepsilon}} \lesssim \|\phi\|_{s,b}^3.
\end{equation}
But (4) follows by Theorem 2.1 and Lemma 2.2,
\begin{equation}
\|\phi Q_0(\phi, \phi)\|_{H^{s-1, b-1+\varepsilon}} \lesssim \|\phi\|_{s,b}^3 \|Q_0(\phi, \phi)\|_{H^{s-1, b-1+\varepsilon}},
\end{equation}
Proof of (1) for $A_\mu A^\mu|n \times \phi|^2$ and $A_\mu A^\mu(\phi_1, \phi_2, 0)$. Trivially,
\begin{align*}
|A_\mu A^\mu|n \times \phi|^2|_{H^{s-1, b-1+\varepsilon}} & \lesssim \|A_\mu\|_{s,b}^2 \|\phi\|_{s,b}^3, \\
A_\mu A^\mu(\phi_1, \phi_2, 0)|_{H^{s-1, b-1+\varepsilon}} & \lesssim \|A_\mu\|_{s,b}^2 \|\phi\|_{s,b}.
\end{align*}
Proof of (1) for $A^\mu \partial_\mu (n \times \phi)$. By Theorem 2.1, we obtain
\begin{equation}
\|A^\mu \partial_\mu (n \times \phi)\|_{H^{s-1, b-1+\varepsilon}} \lesssim \|A_\mu\|_{s,b} \|\phi\|_{s,b}.
\end{equation}
Proof of (2) for $\epsilon_{\mu\nu\rho}Q^{\mu\rho}(n \times \phi, \phi)$. Using Lemma 2.2, we know that
\begin{align*}
|\epsilon_{\mu\nu\rho}Q^{\mu\rho}(n \times \phi, \phi)|_{H^{s-1, b-1+\varepsilon}} & \lesssim \|n \times \phi\|_{s,b} \|\phi\|_{s,b} \\
& \lesssim \|\phi\|_{s,b} \|\phi\|_{s,b}.
\end{align*}
Proof of (2) for $\epsilon_{\mu\nu\rho} \partial^\rho(A^\mu|n \times \phi|^2)$. By Leibniz’s rule, the estimates
\begin{align*}
|\epsilon_{\mu\nu\rho} \partial^\rho(A^\mu|n \times \phi|^2)|_{H^{s-1, b-1+\varepsilon}} & \lesssim \|A_\mu\|_{s,b} \|\phi\|_{s,b}^2, \\
|\epsilon_{\mu\nu\rho} \partial^\rho(A^\mu|n \times \phi|^2)|_{H^{s-1, b-1+\varepsilon}} & \lesssim \|A_\mu\|_{s,b} \|\phi\|_{s,b} \|\phi\|_{s,b} \|\phi\|_{s,b}.
\end{align*}
holds by Theorem 2.1 if $s > \frac{3}{2}$ and $b > \frac{1}{2}$.

References


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