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LOCAL EXISTENCE OF CHERN-SIMONS GAUGED $O(3)$ SIGMA EQUATIONS

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Abstract. In this paper we study the Cauchy problem for the Chern-Simons gauged $O(3)$ sigma model. We prove the local existence of solutions with low regularity initial data, observing null forms of the system and applying bilinear estimates for wave-Sobolev space $H^{s,b}$.

1. Introduction

The classical $O(3)$ sigma model originates from the description of the planar ferromagnet. The $O(3)$ sigma model in 2-dimensional Euclidean space is a popular one in theoretical physics. From the point of view of a particle physicist, the model has one important drawback: it is scale invariant and as a result its soliton solutions have arbitrary size, making them unsuitable as models for particles. The new possibilities of breaking the scale invariance of the sigma model were proposed by introducing a $U(1)$ gauge field whose dynamics is governed by Maxwell, Chern-Simons and Maxwell-Chern-Simons action. Some analysis of the self-dual equations can be found in [1, 6].

Consider the following Chern-Simons gauged $O(3)$ sigma equations,

(1)
$$D_\mu D^\mu \phi = -\frac{1}{\kappa^2}\phi\big(\langle D^\mu\phi, D_\mu\phi\rangle + \phi_3(1-\phi_3)^2(1+2\phi_3)\big) + \frac{1}{\kappa^2}(0,0,(1-\phi_3)^2(1+2\phi_3)),$$

(2)
$$\frac{\kappa}{2}\epsilon^{\mu\nu\lambda}F_{\nu\lambda} = -\langle n\times\phi, D^\mu\phi\rangle,$$

where ϕ is a three component vector with unit norm, i.e. $\langle\phi,\phi\rangle = 1$, $A_\mu : \mathbb{R}^{1,2} \to \mathbb{R}$ is the gauge field with $\mu = 0, 1, 2$, $\epsilon^{\alpha\beta\gamma}$ is the totally-antisymmetric tensor with $\epsilon^{012} = 1$, and $n = (0, 0, 1)$ is the north pole of S^2. The gauge covariant derivative is defined by $D_\mu\phi = \partial_\mu\phi + A_\mu(n\times\phi)$ and the Maxwell field is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The constant $\kappa > 0$ is a Chern-Simons coupling constant. Greek indices, such as μ, ν will refer to all indices 0, 1, 2, whereas latin indices, such as i, j, k, will refer only to the spatial indices 1, 2,

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unless otherwise specified. The usual inner product and cross product on $\mathbb{R}^3$ are given by

$$\langle a, b\rangle = a_1b_1 + a_2b_2 + a_3b_3,$$
$$a \times b = (a_2b_3 - a_3b_2,\, a_3b_1 - a_1b_3,\, a_1b_2 - a_2b_1).$$

The Lagrangian for the Chern-Simons gauged $O(3)$ sigma model, proposed in [3, 9], is given by

$$\mathcal{L} = \frac{1}{2}D_\mu\phi \cdot D^\mu\phi + \frac{\kappa}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} - \frac{1}{2\kappa^2}(1+\phi_3)(1-\phi_3)^3. \tag{3}$$

The energy density corresponding to the Lagrangian density (3) is

$$\mathcal{E}(\phi, A) = \frac{1}{2}\left[|D_\mu\phi|^2 + \frac{1}{\kappa^2}(1+\phi_3)(1-\phi_3)^3\right]. \tag{4}$$

The conservation of the total energy implies that

$$E(t) := \int_{\mathbb{R}} \mathcal{E}(t,\, x)\, dx = \int_{\mathbb{R}} \mathcal{E}(0,\, x)\, dx. \tag{5}$$

The system of equations (1)-(2) is invariant under the following gauge transformations

$$\phi = (z, \phi_3) \to (ze^{i\chi}, \phi_3), \quad A_\mu \to A_\mu - \partial_\mu\chi,$$

where χ is a real valued smooth function on $\mathbb{R}^{2+1}$ and we use the notation $z = \phi_1 + i\phi_2$. Therefore a solution of the system (1)-(2) is formed by a class of gauge equivalent pairs (ϕ, A_μ). Here we study an initial value problem of (1)-(2) under Lorenz gauge condition $\partial_\mu A^\mu = 0$ for which the system can be rewritten as follows,

(6)

$$D_\mu D^\mu\phi = -\frac{1}{\kappa^2}\phi(\langle D^\mu\phi, D_\mu\phi\rangle + \phi_3(1-\phi_3)^2(1+2\phi_3)) + \frac{1}{\kappa^2}(0, 0, (1-\phi_3)^2(1+2\phi_3)),$$

$$\kappa F_{01} = \langle n \times \phi, D_2\phi\rangle, \tag{7}$$

$$\kappa F_{02} = \langle n \times \phi, D_1\phi\rangle, \tag{8}$$

$$\partial_\mu A^\mu = 0, \tag{9}$$

supplemented by the constraint equation

$$\kappa F_{12} = -\langle n \times \phi, D_0\phi\rangle, \tag{10}$$

and the initial data

$$A_\mu(0,\cdot) = a_\mu, \quad \phi(0,\cdot) = \phi_0, \quad \partial_t\phi(0,\cdot) = \phi_1, \tag{11}$$

satisfying $\langle \phi_0,\, \phi_1\rangle = 0$. For the formulation of equations (6)–(11), we refer to Section 2.

In the usual sigma model (called wave map), evolution problems have been studied extensively. Let us review briefly the results for global existence in time. Wave map in $1+1$ dimension extends smoothly all the time in [5, 13]. In $(3+1)$ dimension, development of singularities from smooth initial data was shown in

[7] by using the self-similar structure of the sigma model. Also Shatah proved that there exists global weak solution to wave map which has an S^n target manifold. The space two-dimensional case is critical. Tataru[15] and Tao[14] proved global regularity of wave maps in (2+1) dimension under the assumption of small Besov norm, respectively, small energy. In [16], Tataru proved rough solutions and the continuous dependence on the initial data which is small in the critical Sobolev spaces. Then Rodnianski [12] and Tataru [11] resolves the finite time blow up solutions for the wave map problem from $\mathbb{R}^{2+1} \to S^2$. The global solutions of Chern-Simons sigma equations in one space dimension was shown in [8]. The following is our main result.

Theorem 1.1. *Let $s > 3/2$, consider the Cauchy problem of Chern-Simons gauged $O(3)$ sigma equations* (6)-(9)*, with the initial data in the following Sobolev space:*

$$A_\mu(0,\cdot) = a_\mu \in H^s(\mathbb{R}^2), \quad \phi(0,\cdot) = \phi_0 \in H^s(\mathbb{R}^2), \quad \partial_t\phi(0,\cdot) = \phi_1 \in H^{s-1}(\mathbb{R}^2)$$

satisfying the constraint (10) *and $\langle \phi_0,\, \phi_1 \rangle = 0$, then there exists a $T > 0$ and a solution (A, ϕ) of* (6)-(9) *in $[0, T) \times \mathbb{R}^2$ with*

$$A_\mu \in C([0, T); H^s(\mathbb{R}^2)), \quad \phi \in C([0, T]; H^s(\mathbb{R}^2)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^2)).$$

We use $a \lesssim b$ denote $a \le Cb$ for some constant C. A point in the 2+1 dimensional Minkowski space is written as $(t, x) = (x^\alpha)_{0\le\alpha\le2}$. Greek indices range from 0 to 2, and Roman indices range from 1 to 2. We raise and lower indices with the Minkowski metric, $\mathrm{diag}(1, -1, -1)$. We write $\partial_\alpha = \partial_{x^\alpha}$ and $\partial_t = \partial_0$, and we also use the Einstein notation. Therefore, $\partial^i\partial_i = \Delta$, and $\partial^\alpha\partial_\alpha = \partial_t^2 - \Delta = \Box$.

2. Preliminaries

In this section we introduce basic facts of equations, function spaces and some related theorems.

Let us consider calculus related with covariant derivative.

$$\partial_\mu\langle\phi, \psi\rangle = \langle D_\mu\phi, \psi\rangle + \langle\phi, D_\mu\psi\rangle, \tag{1}$$

$$D_\mu D_\nu\phi - D_\nu D_\mu\phi = F_{\mu\nu}(n \times \phi), \tag{2}$$

$$D_\mu(f\phi) = (\partial_\mu f)\phi + fD_\mu\phi, \tag{3}$$

where ϕ, ψ are 3 component vector functions and f is a scalar function.

We review the constraint on the formulation of Cauchy problem (6)–(11). Using the above formula and equations (6)–(7), we can check

$$\begin{aligned}
&\partial_t\big(\kappa(\partial_1A_2 - \partial_2A_1) + \langle n \times \phi, D_0\phi\rangle\big) \\
&= \kappa\partial_1F_{02} - \kappa\partial_2F_{01} + \langle D_0(n \times \phi), D_0\phi\rangle + \langle n\times, D_0D_0\phi\rangle \\
&= \langle D_\mu(n \times \phi), D^\mu\phi\rangle + \langle(n \times \phi), D_\mu D^\mu\phi\rangle \\
&= \langle(n \times \phi), D_\mu D^\mu\phi\rangle = 0.
\end{aligned} \tag{4}$$

Then (4) implies that constraint (10) is automatically satisfied at $t \geq 0$ if the initial data satisfy

$$\kappa(\partial_1 a_2 - \partial_2 a_1) + \langle n \times \phi,\, \phi_1 + a_0(n \times \phi_0)\rangle = 0.$$

Therefore we have shown that if $(\phi,\, A_\mu)$ is a solution of the system (6)-(9) subject to the initial data satisfying constraint (10), then it is also a solution of equations (1)-(2) with the same initial data.

We can also check that the constraint $|\phi|^2 = 1$ is preserved as follows. If the equation (6) is satisfied in time slab $[0,\, T] \times \mathbb{R}^2$, then $\rho = |\phi|^2 - 1$ is the solution of the following equation

$$\begin{aligned}
&[\partial_\mu \partial^\mu + 2\langle D_\mu\phi,\, D^\mu\phi\rangle + 2\phi_3(1-\phi_3)^2(1+2\phi_3)]\,(|\phi|^2 - 1)\\
&= 2\langle D_\mu\phi,\, D^\mu\phi\rangle + 2\langle\phi,\, D_\mu D^\mu\phi\rangle + 2|\phi|^2\langle D_\mu\phi,\, D^\mu\phi\rangle - 2\langle D_\mu\phi,\, D^\mu\phi\rangle\\
&\quad + 2\phi_3|\phi|^2(1-\phi_3)^2(1+2\phi_3) - 2\phi_3(1-\phi_3)^2(1+2\phi_3) = 0.
\end{aligned}$$

This is a linear Klein-Gordon equation for the function ρ with external potential $2\langle D_\mu\phi,\, D_\mu\phi\rangle + 2\phi_3(1-\phi_3)^2(1+2\phi_3)$. With the initial data $\rho(0) = |\phi_0|^2 - 1 = 0$ and $\partial_t\rho(0) = 2\langle\phi_0,\, \phi_1\rangle = 0$, we have $\rho = 0$ in time slab $[0,\, T] \times \mathbb{R}^2$.

Now we introduce function spaces as well as used. The wave-Sobolev spaces $H^{s,b} = H^{s,b}(\mathbb{R}^{1+n})$ are L^2-based Sobolev spaces on the Minkowski space-time $\mathbb{R}^{1+n}$, with Fourier weights adapted to the symbol of the D'Alembertian $\Box = -\partial_t^2 + \Delta$. Specifically, for given $s, b \in \mathbb{R}$, $H^{s,b}$ is the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^{1+n})$ with respect to the norm

$$\|u\|_{H^{s,b}} = \|\langle\xi\rangle^s\langle|\tau| - |\xi|\rangle^b \widetilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}},$$

$$\|u\|^2_{H^{s,b}} = \int\!\!\int (1+|\xi|^2)^s(1+||\tau| - |\xi||^2)^b \widetilde{u}^2(\tau,\xi)\,d\tau d\xi,$$

$$N^2_{s+1,s} = \int\!\!\int (1+||\tau| + |\xi||)^{2s+2}(1+||\tau| - |\xi||)^{2s}\widetilde{u}^2(\tau,\xi)\,d\tau d\xi,$$

$$\begin{aligned}
Z^2_{s+1,s} = &\int\!\!\int \left((\tau+|\xi|)^2\right)^2(\xi^2+1)^s(\tau+|\xi|)^2\widetilde{u}^2(\tau,\xi)\,d\tau d\xi\\
&+ \int\!\!\int \left((\tau-|\xi|)^2\right)^2(\xi^2+1)^s(\tau-|\xi|)^2\widetilde{u}^2(\tau,\xi)\,d\tau d\xi,
\end{aligned}$$

where $\langle\cdot\rangle = (1+|\cdot|^2)^{\frac{1}{2}}$ and $\widetilde{u}(\tau,\xi) = \int\int e^{-i(t\tau + x\cdot\xi)}u(t,x)\,dtdx$ is the space-time Fourier transform.

Here the "elliptic weight" $\langle\xi\rangle^s$ is a familiar feature of the standard Sobolev space $H^s = H^s(\mathbb{R}^n)$, obtained as the completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm $\|f\|_{H^s} = \|\langle\xi\rangle^s\hat{f}(\xi)\|_{L^2_\xi}$, where $\hat{f}(\xi) = \int e^{-ix\cdot\xi}f(x)dx$ is the spatial Fourier transform. The "hyperbolic weight" $\langle|\tau| - |\xi|\rangle^b$, on the other hand, reflects the fact that the $H^{s,b}$-norm is adapted to $\Box$, whose symbol is $\tau^2 - |\xi|^2$.

For $T > 0$, let $H^{s,b}(S_T)$ denote the restriction space to $S_T = (-T,T) \times \mathbb{R}^2$. We recall that fact that (see for [10])

$$H^{s,b}(S_T) \hookrightarrow C([-T,T]; H^s) \text{ for } b > \frac{1}{2},$$

where $\hookrightarrow$ stands for Sobolev embedding.

We need product estimates of the form $H^{s_1,b_1} \cdot H^{s_2,b_2} \hookrightarrow H^{-s_0,-b_0}$ which means that

$$\|uv\|_{H^{-s_0,-b_0}} \leq C\|u\|_{H^{s_1,b_1}}\|v\|_{H^{s_2,b_2}} \text{ for all } u,v \in \mathcal{S}(\mathbb{R}^{1+n}),$$

where C depends on the s_α, b_α and d. If this holds, it is said that the exponent matrix

$$\begin{pmatrix} s_0 & s_1 & s_2 \\ b_0 & b_1 & b_2 \end{pmatrix}$$

is a product. In recent paper [2], the following product estimate in $\mathbb{R}^{1+2}$ is established.

Theorem 2.1. *Assume*

$$\begin{aligned}
&b_0 + b_1 + b_2 > \frac{1}{2},\\
&b_0 + b_1 \geq 0,\\
&b_1 + b_2 \geq 0,\\
&b_0 + b_2 \geq 0,\\
&s_0 + s_1 + s_2 > \frac{3}{2} - (b_0 + b_1 + b_2),\\
&s_0 + s_1 + s_2 > 1 - \min(b_0 + b_1, b_0 + b_2, b_1 + b_2),\\
&s_0 + s_1 + s_2 > \frac{1}{2} - \min(b_0, b_1, b_2),\\
&s_0 + s_1 + s_2 > \frac{3}{4},\\
&(s_0 + b_0) + 2s_1 + 2s_2 > 1,\\
&2s_0 + (s_1 + b_1) + 2s_2 > 1,\\
&2s_0 + 2s_1 + (s_2 + b_2) > 1,\\
&s_0 + s_1 \geq \max(0, -b_2),\\
&s_1 + s_2 \geq \max(0, -b_0),\\
&s_0 + s_2 \geq \max(0, -b_1).
\end{aligned}$$

Then

$$\begin{pmatrix} s_0 & s_1 & s_2 \\ b_0 & b_1 & b_2 \end{pmatrix}$$

is a product.

Now we consider the following nonlinear Cauchy problem:

$$\begin{aligned} \Box u &= F \qquad (t,x) \in \mathbb{R}^{1+n}, \\ u|_{t=0} &= f, \quad \partial_t u|_{t=0} = g. \end{aligned} \tag{5}$$

If $F = Q(u, v)$, where u, $v : \mathbb{R}^{1+2} \to \mathbb{R}^m$, and Q is a linear combination of the three basic null forms as follows.

$$\begin{aligned} Q_0(u,\, v) &= \partial_t u \partial_t v - \nabla u \cdot \nabla v, \\ Q_i j(u,\, v) &= \partial_i u \partial_j v - \partial_j u \partial_i v, \\ Q_0 j(u,\, v) &= \partial_t u \partial_j v - \partial_j u \partial_t v, \end{aligned} \tag{6}$$

where ∂_j stands for spatial derivatives, and ∇ is the spatial gradient. The following null form estimates in Sobolev space which was proven by Grigoryan and Nahmod in the $n = 2$ in [4].

Lemma 2.2. *Let $s > \frac{3}{2}$, $b \in (\frac{1}{2}, 1)$ and $\epsilon \in [0, 1-b]$, then*

$$\|Q(u,v)\|_{H^{s-1,\, b-1+\epsilon}} \lesssim \|u\|_{H^{s,\, b}} \|v\|_{H^{s,\, b}},$$

where $Q(u,v)$ includes all cases in (6).

3. Low regularity local well-posedness

The system (6)-(9) under the Lorenz gauge condition $\partial_\mu A^\mu = 0$ can be rewritten as follows,

$$\Box \phi = -\phi Q_0(\phi, \phi) - A_\mu A^\mu \phi_3 (n - \phi_3 \phi) + 2\phi_3 A^\mu \partial_\mu \phi \times \phi, \tag{1}$$

$$\Box A_\mu = \epsilon_{\mu\nu\rho} Q^{\nu\rho}(n \times \phi, \phi) + 2\epsilon_{\mu\nu\rho} \partial^\nu (A^\rho |n \times \phi|^2), \tag{2}$$

where Q_0, and $Q^{\nu\rho}$ are the standard null forms.

$$\begin{aligned} \Box \phi &= -\phi \left(Q_0(\phi,\phi) + A_\mu A^\mu |n \times \phi|^2 \right) - 2A^\mu \partial_\mu (n \times \phi) - A_\mu A^\mu (\phi_1, \phi_2, 0), \\ \Box A_0 &= Q_{12}(n \times \phi, \phi) + \partial_1 (A_2 |n \times \phi|^2) - \partial_2 (A_1 |n \times \phi|^2), \\ \Box A_1 &= Q_{02}(n \times \phi, \phi) + \partial_0 (A_2 |n \times \phi|^2) - \partial_2 (A_0 |n \times \phi|^2), \\ \Box A_2 &= -Q_{01}(n \times \phi, \phi) - \partial_0 (A_1 |n \times \phi|^2) + \partial_1 (A_0 |n \times \phi|^2). \end{aligned}$$

We specify data

$$A_\mu(0) \in H^s, \qquad (\phi, \partial_t \phi)(0) \in H^s \times H^{s-1}. \tag{3}$$

The data for $\partial_t A_\mu$ are given by the constaints

$$\begin{aligned} \partial_t A_0(0) &= \partial_1 A_1(0) + \partial_2 A_2(0) \in H^{s-1}, \\ \partial_t A_j(0) &= \partial_j A_0(0) - J_k(0) \in H^{s-1}, \end{aligned}$$

where $J_k = \langle n \times \phi, D_j \phi \rangle = \langle n \times \phi, \partial_j \phi \rangle + \langle n \times \phi, A_j (n \times \phi) \rangle$, hence $J_k(0) \in H^{s-1}$ with the norm bounded in terms of the norm of (3).

In the remaining part of this section, we present estimates (1)-(2) with $s > \frac{3}{2}$ and a given $b > \frac{1}{2}$.

Proof of (1) **for** $\phi\, Q_0(\phi,\, \phi)$. We shall prove that

$$\|\phi\, Q_0(\phi,\, \phi)\|_{H^{s-1,b-1+\epsilon}} \lesssim \|\phi\|^3_{H^{s,\,b}}. \tag{4}$$

But (4) follows by Theorem 2.1 and Lemma 2.2,

$$\begin{aligned}\|\phi\, Q_0(\phi,\, \phi)\|_{H^{s-1,b-1+\epsilon}} &\lesssim \|\phi\|_{H^{s,\,b}} \|Q_0(\phi,\, \phi)\|_{H^{s-1,b-1+\epsilon}}, \\ &\lesssim \|\phi\|^3_{H^{s,\,b}}.\end{aligned}$$

Proof of (1) **for** $A_\mu A^\mu \phi |n \times \phi|^2$ **and** $A_\mu A^\mu (\phi_1, \phi_2, 0)$. Trivially,

$$\begin{aligned}\|A_\mu A^\mu \phi |n \times \phi|^2\|_{H^{s-1,b-1+\epsilon}} &\lesssim \|A_\mu\|^2_{H^{s,\,b}} \|\phi\|^3_{H^{s,\,b}}, \\ A_\mu A^\mu (\phi_1, \phi_2, 0)_{H^{s-1,b-1+\epsilon}} &\lesssim \|A_\mu\|^2_{H^{s,\,b}} \|\phi\|_{H^{s,\,b}}.\end{aligned}$$

Proof of (1) **for** $A^\mu \partial_\mu (n \times \phi)$. By Theorem 2.1, we obtain

$$\|A^\mu \partial_\mu (n \times \phi)\|_{H^{s-1,b-1+\epsilon}} \lesssim \|A_\mu\|_{H^{s,\,b}} \|\phi\|_{H^{s-1,\,b}}.$$

Proof of (2) **for** $\epsilon_{\mu\nu\rho} Q^{\nu\rho}(n \times \phi, \phi)$. Using Lemma 2.2, we know that

$$\begin{aligned}\|\epsilon_{\mu\nu\rho} Q^{\nu\rho}(n \times \phi, \phi)\|_{H^{s-1,b-1+\epsilon}} &\lesssim \|n \times \phi\|_{H^{s,\,b}} \|\phi\|_{H^{s,\,b}} \\ &\lesssim \|\phi\|_{H^{s,\,b}} \|\phi\|_{H^{s,\,b}}.\end{aligned}$$

Proof of (2) **for** $\epsilon_{\mu\nu\rho} \partial^\nu (A^\rho |n \times \phi|^2)$. By Leibniz's rule, the estimates

$$\begin{aligned}\|\epsilon_{\mu\nu\rho} \partial^\nu (A^\rho |n \times \phi|^2)\|_{H^{s-1,b-1+\epsilon}} &\lesssim \|A_\mu\|_{H^{s-1,\,b}} \|\phi\|^2_{H^{s,\,b}}, \\ \|\epsilon_{\mu\nu\rho} \partial^\nu (A^\rho |n \times \phi|^2)\|_{H^{s-1,b-1+\epsilon}} &\lesssim \|A_\mu\|_{H^{s,\,b}} \|\phi\|_{H^{s-1,\,b}} \|\phi\|_{H^{s,\,b}},\end{aligned}$$

holds by Theorem 2.1 if $s > \frac{3}{2}$ and $b > \frac{1}{2}$.

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