THE ACTION OF SPECIAL LINEAR GROUP ON THE SET OF MUTUALLY DISTINCT TRIPLE POINTS OF CIRCLE AND INVARIANT MEASURE

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Abstract. We investigate the Möbius transformation action of $PSL(2, \mathbb{R})$ on the set of mutually distinct ordered triple points of $\mathbb{R} \cup \{\infty\}$.

1. Introduction

The upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$ has a large group of conformal automorphisms, consisting of Möbius transformations of the form

$$z \mapsto \frac{az + b}{cz + d}$$

for $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. These symmetries form the group $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I\}$.

Under the $PSL(2, \mathbb{R})$-action, $\mathbb{H}$ has an invariant metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

called the Poincaré metric. Denoting by $z = x + yi$, we have a corresponding measure

$$dA(z) = \frac{dx dy}{y^2}$$

and the distance function

$$d(z, w) = \log \frac{|z - \overline{w}| + |z - w|}{|z - \overline{w}| - |z - w|}$$

Let us denote by $T^1_+ \mathbb{H}$ the unit tangent bundle of $\mathbb{H}$, which is the bundle $\{(z, v) : z \in \mathbb{H}, v \in T_z \mathbb{H} \text{ with } |v| = 1\}$ of unit-length tangent vectors on the upper half-plane.
The action of the projective special linear group $\text{PSL}(2, \mathbb{R})$ on the unit tangent bundle $T^1 \mathbb{H}$ of the upper half plane $\mathbb{H}$ is given by (for example, see Chapter 9 of [2])

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z, v) \mapsto \left( \frac{az + b}{cz + d}, \frac{v}{cz + d} \right)^2$$

for $z \in \mathbb{H}, v \in T_z \mathbb{H}$.

In particular, the induced action of the modular group $\text{PSL}(2, \mathbb{Z})$ on $T^1 \mathbb{H}$ serves as a key example of ergodic theory on homogeneous spaces. (See also [1] for various concepts and applications for Riemann surfaces other than modular surface.)

The boundary at infinity $\partial_\infty \mathbb{H}$ may be identified with $\mathbb{R} \cup \{ \infty \}$ and hence with $S^1$. Let us say that a mutually distinct ordered triple points $p = w_1, w_2, w_3 \in (S^1)^3$ is positively ordered if one reaches $w_2$ before $w_3$ when starting counterclockwise from $w_1$.

We note that there is a bijection between the unit tangent bundle $T^1 \mathbb{H}$ of the upper half plane and the set of mutually distinct positively ordered triple points $p = w_1, w_2, w_3 \in (S^1)^3$ by the following manner. Take $w_1 = \ell(-\infty)$ and $w_3 = \ell(\infty)$ for the bi-infinite geodesic $\ell$ for which $\ell(0) = z$ and $\ell'(0) = v$. Let $w \in T_z \mathbb{H}$ be the tangent vector obtained by rotating $v$ by $\pi/2$ clockwise. Let $w_2 = \ell'(\infty)$ where $\ell'$ is the bi-infinite geodesic whose tangent vector at $z$ is $w$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{(z,v) \mapsto (w_1, w_2, w_3)}
\end{figure}

In this article, we investigate the invariant measure of the Möbius transformation action of the modular group $\text{PSL}(2, \mathbb{R})$ on the set of mutually distinct ordered triple points $p = w_1, w_2, w_3 \in (S^1)^3$.

Let $(S^1)_{\text{pos}}$ be the set of positively ordered triple points of $S^1$. Here and throughout, we identify $S^1$ with $\mathbb{R} \cup \{ \infty \}$ and pull the real distance on $\mathbb{R} \cup \{ \infty \}$ back to $S^1$.

**Theorem 1.1.** The measure $\lambda$ on $(S^1)_{\text{pos}}^3$ given by

$$d\lambda = \frac{dw_1 dw_2 dw_3}{(w_1 - w_2)(w_2 - w_3)(w_3 - w_1)}$$

is the $\text{SL}(2, \mathbb{R})$-invariant measure.

We note that $(w_1 - w_2)(w_2 - w_3)(w_3 - w_1)$ is always positive since $(w_1, w_2, w_3)$ is positively oriented.
2. Proof of Main Theorem

Given \((z, v)\) in \(T^1\mathbb{H}\), let \(z = x + yi\) \((y > 0)\) and \(\theta\) be the angle between \(v\) and the \(x\)-axis. We may identify the elements \((z, v)\) of \(T^1\mathbb{H}\) with \((x + yi, \theta)\). The Liouville measure

\[
dm = \frac{dx dy d\theta}{y^2}
\]

is the unique (up to scalar) \(PSL(2, \mathbb{R})\)-invariant measure on \(T^1\mathbb{H}\) (see Chapter 9 of [2]).

**Lemma 2.1.** The one-to-one correspondence \((x, y, \theta) \leftrightarrow (w_1, w_2, w_3)\) between \(T^1\mathbb{H}\) and \((\mathbb{R} \cup \{x\})^3_{pos}\) is explicitly given by

\[
w_1 = x + y \tan \theta - y |\sec \theta|,
\]

\[
w_2 = x - y \cot \theta + y |\csc \theta|,
\]

\[
w_3 = x + y \tan \theta + y |\sec \theta|,
\]

and

\[
x = \frac{w_1^2 w_3 + w_1 w_2^2 - 4w_1 w_2 w_3 + w_1 w_3^2 + w_2^2 w_3}{(w_1 - w_2)^2 + (w_2 - w_3)^2},
\]

\[
y = \frac{(w_1 - w_2)(w_3 - w_2)^2}{(w_1 - w_2)^2 + (w_2 - w_3)^2},
\]

\[
\theta = \arctan \left( \frac{(w_1 - w_3)(w_1 - 2w_2 + w_3)}{2(w_1 - w_2)(w_2 - w_3)^2} \right).
\]

**Proof.** First, we note that Equation (2.1) follows directly from Figure 2.

![Figure 2](image-url)

**Figure 2.** \((x + iy, \theta) \leftrightarrow (w_1, w_2, w_3)\)

Now let us recall that two Euclidean circles with Cartesian equation

\[
x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{and} \quad x^2 + y^2 + 2g'x + 2f'y + c' = 0
\]

are orthogonal if and only if \(2gg' + 2ff' = c + c'\). For the unique geodesic \(\ell\) with \(\ell(0) = z\) and \(\ell(\infty) = w_2\), let us denote by \(\ell(-\infty) = w_4\). Let \(O_1\) be the circle centered at \(\frac{w_2 + w_3}{2}\) with radius \(\frac{|w_2 - w_4|}{2}\) and let \(O_2\) be the circle centered...
at $\frac{w_1 + w_3}{2}$ with radius $\frac{|w_1 - w_3|}{2}$. Since they cut one another at right angles on $z = x + yi$, it follows that

$$\frac{(x^2 + y^2 - w_2^2)(w_1 + w_3)}{x - w_2} = 2w_1w_3 + \frac{2w_2(x^2 + y^2 - w_2^2)}{x - w_2} = 2w_2^2$$

and

$$y = \sqrt{-(x - w_1)(x - w_3)}.$$ Solving these equations yields the formula of $x$ and $y$ in Equation (2.2). From the relation

$$\tan \theta = \frac{w_1 + w_3 - 2x}{2y}$$

we obtain the formula of $\theta$ in Equation (2.2). □

Now we will give the proof of Theorem 1.1. Since

$$dm = \frac{dxdy\theta}{y^2}$$

is the $PSL(2, \mathbb{R})$-invariant measure on $T^1\mathbb{H}$ and the one-to-one correspondence $(x, y, \theta) \leftrightarrow (w_1, w_2, w_3)$ is a diffeomorphism almost everywhere, the measure on $(\mathbb{R} \cup \{\infty\})^3_{pos}$ locally given by

$$\frac{1}{y^2} \left| \frac{\partial(x, y, \theta)}{\partial(w_1, w_2, w_3)} \right| dw_1 dw_2 dw_3$$

is invariant under $PSL(2, \mathbb{R})$.

By Lemma 2.1, the Jacobian matrix

$$\begin{bmatrix}
1 & \tan \theta + |\sec \theta| & y\sec^2 \theta + y|\sec \theta|\tan \theta \\
1 & -\cot \theta + |\csc \theta| & y\csc^2 \theta - y|\csc \theta|\cot \theta \\
1 & \tan \theta - |\sec \theta| & y\sec^2 \theta - y|\sec \theta|\cot \theta
\end{bmatrix}$$

of which the determinant is $y\sec^2 \left(\frac{\theta}{2}\right) \sec^2 \theta$. Applying the formula of $y$ and $\theta$ of Equation (2.2), we get

$$\frac{1}{y^2} \left| \frac{\partial(x, y, \theta)}{\partial(w_1, w_2, w_3)} \right| dw_1 dw_2 dw_3 = \frac{dw_1 dw_2 dw_3}{(w_1 - w_2)(w_2 - w_3)(w_3 - w_1)}.$$ This completes the proof of Theorem 1.1.

3. More on parametrization and fundamental domain

**Hopf parametrization**

There is another parametrization of $(z, v) \in T^1\mathbb{H}$, called *Hopf parametrization*, by two distinct boundary points $\xi$ and $\eta$ at infinity together with a real number $t \in \mathbb{R}$. Given $(x + yi, \theta) \in T^1\mathbb{H}$, there is a unique bi-infinite parametrized geodesic $\alpha$ of $\mathbb{H}$ such that $\alpha(0) = x + yi$ and $\alpha'(0) = v$.

Let $\xi = \alpha(x)$ and $\eta = \alpha(-\infty)$. Let $o$ be the orthogonal projection point of $i \in \mathbb{H}$ onto the geodesic $\alpha$ and $t$ be the real number for which $\alpha(t) = o$. 
Under this parametrization \((x + yi, \theta) \leftrightarrow (\xi, \eta, \tau)\), the similar argument gives
\[
\frac{dxdy\theta}{y^2} = \frac{2d\xi d\eta d\tau}{(\eta - \xi)^2}
\]

**Fundamental domain**

For each \(z_0, v \in T^1 \mathbb{H}\), we can find a neighborhood of \(z_0, v\) which does not contain any other element of the \(\text{PSL}(2, \mathbb{Z})\)-orbit of \((z_0, v)\). This enables us to construct fundamental domains, which contain exactly one representative for the \(\text{PSL}(2, \mathbb{Z})\)-orbit of every \((z_0, v)\) in \(T^1 \mathbb{H}\).

There are various ways of constructing a strong fundamental domain, but a common choice is the union
\[
\{(z, v) : z \in R, v \in T^1 \mathbb{H}\} \cup \{(w_1, v) \in T^1 \mathbb{H} : 0 \leq \arg(v) < \frac{2\pi}{3}\}
\]
\[
\cup \{(w_2, v) \in T^1 \mathbb{H} : 0 \leq \arg(v) < \pi\}
\]
for two branched points \(w_1 = \frac{1}{2} + \frac{\sqrt{3}i}{2}\) and \(w_2 = i\) and the region
\[
R = \left\{ z \in \mathbb{H} : |z| > 1, -\frac{1}{2} \leq \text{Re}(z) < \frac{1}{2} \right\} \cup \left\{ z \in \mathbb{H} : |z| = 1, -\frac{1}{2} \leq \text{Re}(z) < 0 \right\}
\]
bounded by the vertical lines \(\text{Re}(z) = -\frac{1}{2}\) and \(\text{Re}(z) = \frac{1}{2}\) and the circle \(|z| = 1\).

It would be interesting to construct explicitly a fundamental domain for the action of \(\text{PSL}(2, \mathbb{R})\) on \((S^1)^3\).

We remark that the positive characteristic analogue case is attained in [3]. Namely, the author considers \(\text{PGL}(2, \mathbb{F}_q[t])\)-action on the \((q+1)\)-regular tree \(\mathcal{T}\) and its fundamental domain as a subset of \(\partial_{\mathcal{T}} \mathcal{T}_\text{dia}\), the set of mutually distinct ordered triple points on \(\partial_{\mathcal{T}} \mathcal{T}\).

**References**


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