BERGMAN TYPE OPERATORS ON SOME GENERALIZED CARTAN-HARTOGS DOMAINS

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Abstract. For $\mu = (\mu_1, \ldots, \mu_t)$ ($\mu_j > 0$), $\xi = (z_1, \ldots, z_t, w) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_t} \times \mathbb{C}^m$, define
\[\Omega(\mu, t) = \{ \xi \in \mathbb{B}_{n_1} \times \cdots \times \mathbb{B}_{n_t} \times \mathbb{C}^m : \|w\|^2 < C(\chi, \mu) \prod_{j=1}^t (1 - \|z_j\|^2)^{\mu_j} \},\]
where $\mathbb{B}_{n_j}$ is the unit ball in $\mathbb{C}^{n_j}$ ($1 \leq j \leq t$), $C(\chi, \mu)$ is a constant only depending on $\chi = (n_1, \ldots, n_t)$ and $\mu = (\mu_1, \ldots, \mu_t)$, which is a special type of generalized Cartan-Hartogs domain. We will give some sufficient and necessary conditions for the boundedness of some type of operators on $L^p(\Omega(\mu, t), \omega)$ (the weighted $L^p$ space of $\Omega(\mu, t)$ with weight $\omega$, $1 < p < \infty$). This result generalizes the works from certain classes of generalized complex ellipsoids to the generalized Cartan-Hartogs domain $\Omega(\mu, t)$.

1. Introduction

Let $\Omega$ be a domain in $\mathbb{C}^n$ and $\omega(z)$ be a non-negative measurable function on $\Omega$. For $p \in [1, \infty)$, let $L^p(\Omega, \omega)$ denote the set of all complex measurable functions $f$ for which
\[\left( \int_{\Omega} |f(z)|^p \omega(z) dV(z) \right)^{\frac{1}{p}} < \infty,\]
where $dV(z)$ is the ordinary Lebesgue measure on $\Omega$. We call $\omega(z)$ a weight on $\Omega$ and $L^p(\Omega, \omega)$ the weighted $L^p$ space of $\Omega$. The norm of $L^p(\Omega, \omega)$ is defined as follows:
\[\|f\|_{L^p, \omega} = \int_{\Omega} |f(z)|^p \omega(z) dV(z).\]
If $p = 2$, $L^2(\Omega, \omega)$ is a Hilbert space with the inner product:

\[(f, g)_\omega = \int_{\Omega} f(z) \overline{g(z)} \omega(z) dV(z).\]

The weighted Bergman space of $\Omega$ with weight $\omega$ is defined by

\[A^p(\Omega, \omega) := \mathcal{O}(\Omega) \cap L^p(\Omega, \omega),\]

where $\mathcal{O}(\Omega)$ is the space of all holomorphic functions on $\Omega$. $A^2(\Omega, \omega)$ is a subspace of holomorphic functions in $L^2(\Omega, \omega)$. By the result in [16] we know that if $\omega$ is continuous and never vanishes inside $\Omega$, then $A^2(\Omega, \omega)$ is a closed subspace of $L^2(\Omega, \omega)$ and there exists the orthogonal projection, called the weighted Bergman projection:

\[P_\omega : L^2(\Omega, \omega) \rightarrow A^2(\Omega, \omega).\]

This projection is an integral operator with the weighted Bergman kernel, denoted by $K_\omega(z, w)$:

\[P_\omega f(z) := \int_{\Omega} K_\omega(z, w) f(w) \omega(w) dV(w).\]

When $\omega(z) \equiv 1$, the weighted Bergman kernel $K_\omega$ and the weighted Bergman projection $P_\omega$ will degenerate to the ordinary Bergman kernel $K$ and the ordinary Bergman projection $P$, respectively. If $\{\phi_j : j = 0, 1, 2, \ldots\}$ is an orthonormal basis of $A^2(\Omega, \omega)$, then the weighted kernel function satisfies

\[K_\omega(z, w) = \sum_{j=0}^{\infty} \phi_j(z) \overline{\phi_j(w)}.\]

It is of great interest to find an explicit form of the Bergman kernel for a domain $\Omega$. Many results have been obtained when $\Omega$ has nice properties. Gindikin [7] obtained explicit forms of the Bergman kernels for homogeneous domains. For non-homogeneous domains, there are many results on the explicit forms of the Bergman kernels obtained in [1, 3, 8, 12–15, 18–22].

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ and $p : \Omega \rightarrow (0, \infty)$ a continuous function. Define a Hartogs type domain $\tilde{\Omega}$ by

\[\tilde{\Omega} := \{(z, w) \in \Omega \times \mathbb{C}^m : \|w\|^2 < p(z)\},\]

where $\|\cdot\|$ is the standard Hermitian norm on $\mathbb{C}^m$. By the Forelli-Rudin construction, Ligocka [10] studied the properties of holomorphic functions on Hartogs type domains and obtained the regularity of weighted Bergman projections on weakly regular domains. Using similar method, Charpentier-Dupain-Mounkaila [4] investigated the regularity properties of weighted Bergman projections for smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^n$.

Let $\Omega$ be an irreducible bounded symmetric domain in $\mathbb{C}^n$ of genus $\gamma$, the generic norm of $\Omega$ is defined by

\[N(z, w) := (V(\Omega) K(z, w))^{-\frac{1}{\gamma}},\]
where $V(\Omega)$ is the volume of $\Omega$ with respect to the Euclidean measure of $\mathbb{C}^n$ and $K(z, w)$ is the Bergman kernel. Ahn-Park [1] introduced the generalized Cartan-Hartogs domain as follows:

$$\tilde{\Omega}_m := \left\{ (z, w) \in \Omega_1 \times \cdots \times \Omega_t \times \mathbb{C}^m : \|w\|^2 < \prod_{j=1}^{t} N_{\Omega_j}(z_j, z_j)^{\mu_j}, \mu_j > 0 \right\},$$

where $z_j \in \Omega_j$, $\Omega_j$ is one of bounded symmetric domains of six types, and $N_{\Omega_j}(z_j, z_j)$ is the corresponding generic norm of $\Omega_j$ $(1 \leq j \leq t)$. This type of domains generalises the Cartan-Hartogs domain introduced by Yin and Roos [19–21]. Using the virtual Bergman kernel, Ahn-Park [1] obtained the Bergman projection is bounded only depending on $\chi$ where $B_z$ where $N$.

Zhao [23] studied the boundedness of this type of operators from $L^p$ to $L^q$ spaces on the unit ball with possibly different measures, for $1 \leq p \leq q < \infty$. Liu-Stoll [11] and Shi [17] generalized the results to the generalized complex ellipsoids as follows:

$$\Omega_n := \{ \xi = (z, w) \in \mathbb{C}^{n+m} : z \in \mathbb{C}^n, w \in \mathbb{C}^m, \|z\|^2 + \|w\|^2 < 1 \}.$$
where $a > 0$. Motivated by their works, in this paper we study the mapping properties for the Bergman type operators on the generalized Cartan-Hartogs domain $\Omega(\mu, t)$, which can naturally be viewed as complements to Liu-Stoll [11] and Shi [17].

Define
\[
\rho(\xi) = 1 - \frac{||w||^2}{C(\chi, \mu) \prod_{j=1}^t (1 - ||z_j||^2)\mu_j}.
\]

For $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_t)$, $\sigma_0 > -1$, $\sigma_j > -\mu_j m - 1$, $1 \leq j \leq t$, define
\[
\omega_\sigma(\xi) = \rho^{\sigma_0}(\xi) \prod_{j=1}^t (1 - ||z_j||^2)^{\sigma_j}.
\]

We will first give a formula for the weighted Bergman kernel of $A^2(\Omega(\mu, t), \omega_\sigma)$ as follows.

**Theorem 1.1.** Let $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_t)$, $\sigma_0 > -1$, $\sigma_j > -\mu_j m - 1$, $1 \leq j \leq t$. Then the weighted Bergman kernel of the Hilbert space $A^2(\Omega(\mu, t), \omega_\sigma)$ is given by
\[
K_{\omega_\sigma}(\xi, \xi') = \sum_{k=0}^{n_1 + \cdots + n_t} C_k \frac{\prod_{j=1}^t (1 - \langle z_j, z'_j \rangle)\mu_j^{k+\sigma_0+1} - \sigma_j - n_j - 1}{[C(\chi, \mu) \prod_{j=1}^t (1 - \langle z_j, z'_j \rangle)\mu_j - \langle w, w' \rangle]^{k+m+\sigma_0+1}},
\]
where $\xi = (z_1, \ldots, z_t, w)$, $\xi' = (z'_1, \ldots, z'_t, w') \in \Omega(\mu, t)$ and $C_k$ are constants depending on $m, n_j, \sigma_0, \sigma_j, \mu_j (1 \leq j \leq t)$.

Let $s = (s_0, s_1, \ldots, s_t)$, $s_0 = \sigma_0 + ia_0$, $\sigma_0 > -1$, $a_0 \in \mathbb{R}$ and $s_j = \sigma_j + ia_j$, $\sigma_j > -\mu_j m - 1$, $a_j \in \mathbb{R}$, $1 \leq j \leq t$. Let $K_{\omega_s}$ be the kernel obtained from $K_{\omega_\sigma}$ by replacing $\sigma$ with $s$ in (1).

Let $v = (v_1, \ldots, v_t)$. For $u, v_1, \ldots, v_t \geq 0$, we define
\[
K_{s,u,v}(\xi, \xi') = \sum_{k=0}^{n_1 + \cdots + n_t} C_k \frac{\prod_{j=1}^t (1 - \langle z_j, z'_j \rangle)\mu_j^{k+s_0+1} - s_j - n_j - 1 - v_j}{[C(\chi, \mu) \prod_{j=1}^t (1 - \langle z_j, z'_j \rangle)\mu_j - \langle w, w' \rangle]^{k+m+s_0+1+u}}.
\]

We can see that when $u = v_1 = \cdots = v_t = 0$, $K_{s,u,v}(\xi, \xi') = K_{\omega_s}(\xi, \xi')$.

Using this kernel, we will study the action of the following operator $T_{s,u,v,\epsilon}$ on the space $L^p(\Omega(\mu, t), \omega_\lambda)$:
\[
(T_{s,u,v,\epsilon}f)(\xi) = \rho^{u+v}(\xi) \prod_{j=1}^t (1 - ||z_j||^2)^{\sigma_j} \int_{\Omega(\mu, t)} \omega_\lambda(\xi') K_{s,u,v}(\xi, \xi') f(\xi')dV(\xi'),
\]
where $\epsilon \geq 0$ and we will obtain the following results.

**Theorem 1.2.** Let $s = (s_0, s_1, \ldots, s_t)$, $s_0 = \sigma_0 + ia_0$, $\sigma_0 > -1$, $a_0 \in \mathbb{R}$ and $s_j = \sigma_j + ia_j$, $\sigma_j > -\mu_j m - 1$, $a_j \in \mathbb{R}$, $1 \leq j \leq t$.

(i) For $1 < p < \infty$, $T_{s,u,v,\epsilon}$ is a bounded operator on $L^p(\Omega(\mu, t), \omega_\lambda)$ if and only if

$$
\lambda_j > 1, \mu_j > 1, \sigma_j > -1, \hfill (3)
$$

(ii) Suppose $p = 1$, if $T_{s,u,v,\epsilon}$ is a bounded operator on $L^1(\Omega(\mu, t), \omega_\lambda)$, then

$$
\lambda_j > 1, \mu_j > 1, \sigma_j > -1, \hfill (4)
$$

Conversely, if

$$
\lambda_j < 1, \mu_j < 1, \sigma_j < -1, \hfill (5)
$$

then $T_{s,u,v,\epsilon}$ is a bounded operator on $L^1(\Omega(\mu, t), \omega_\lambda)$.

**Remark 1.3.** (i) Suppose $t = \mu_1 = 1$, $m = \epsilon = 0$. Then $C_k = 0$ for $k = 0, 1, \ldots, n_1 - 1$ and $C_{n_1} = \frac{\Gamma(n_1 + 1)}{\pi^{n_1}}$. Therefore,

$$
(T_{s,u,v,\epsilon}f)(\xi) = \frac{\Gamma(n_1 + 1)}{\pi^{n_1}} V(\mathbb{B}_{n_1}) \frac{\|z_i^j\|^{n_1 + n_2 + n_3 + 1}}{(1 - \|z_i^j\|^2)^{u + v_1}} \times \int_{\mathbb{B}_{n_1}} \frac{(1 - \|z_i^j\|^2)^{l_1} f(z_i^j)}{(1 - (z_i^j, z_i^j))^{s_1 + s_2 + s_3 + 1 + u + v_1}} dV(z_i^j).
$$

If $u = v_1 = 0$, then $T_{s,u,v,\epsilon}$ is the operator studied by Forelli-Rudin [6] and Zhu [24]. If $n_1 = 1, s_1 = \sigma_1$, then $T_{s,u,v,\epsilon}$ is the operator studied by Zhu [25].

(ii) Suppose $t = 1, \epsilon = \frac{\mu_1}{\mu_2}, s_1 = \mu_1 s_0$. Then $T_{s,u,v,\epsilon}$ is the operator studied by Shi [17].

If $A$ and $B$ are functions of several variables, we use $A \lesssim B$ to denote that $A \leq K \cdot B$ for a positive constant $K$. Also we use $A \simeq B$ to denote that $A \lesssim B \lesssim A$.

2. Preliminaries

**Lemma 2.1** (See [2]). For $\alpha > -1$, the following multiple integration exists and

$$
\int_0^1 dx_m \cdots \int_0^{1 - \sum_{i=1}^m x_i} \left(1 - \sum_{i=1}^m x_i\right)^\alpha \prod_{i=1}^m x_i^{q_i} dx_1 = \frac{\prod_{i=1}^m \Gamma(q_i + 1)^\alpha}{\Gamma(\alpha + \sum_{i=1}^m q_i + m + 1)}.
$$
Lemma 2.2 (See [5]). For $||x|| < 1$ and $s > 0$, we have

$$\sum_{\alpha \in \mathbb{R}^+} \frac{x^{2\alpha}}{\Gamma(s) \prod_{i=1}^{n} \Gamma(\alpha_{i} + 1)} = \frac{1}{(1 - ||x||^2)^s}.$$ 

Lemma 2.3 (See [5]). Suppose that $q_d$ is a polynomial of degree $d$ in one variable. Then there are constants $c_k$ such that

$$\sum_{\gamma \in \mathbb{R}^+} \frac{x^{\gamma}}{\Gamma(s) \prod_{i=1}^{m} \Gamma(\gamma_{i} + 1)} q_d(\gamma) x^\gamma = \sum_{k=0}^{d} c_k \left( \frac{1}{1 - ||x||^2} \right)^{s+k}.$$ 

Remark 2.4. Although the parameter $\alpha$ in Lemma 2.1 is a real number bigger than $-1$, we can see that Lemma 2.1 is also valid for $\alpha \in \mathbb{C}$ with $\Re \alpha > -1$. Similarly, Lemma 2.2 and Lemma 2.3 are also valid for $s \in \mathbb{C}, \Re s > 0$.

Lemma 2.5. Let $s = (s_0, s_1, \ldots, s_t)$, $s_0 = \sigma_0 + ia_0$, $\sigma_0 > -1$, $a_0 \in \mathbb{R}$ and $s_j = \sigma_j + i\alpha_j$, $\sigma_j > -\mu_j m - 1$, $\alpha_j \in \mathbb{R}$, $1 \leq j \leq t$. $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{j_n}) \in \mathbb{N}^n$, $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}^m$. Then

$$\int_{\Omega(\mu, t)} \rho^{\alpha_0}(\xi) \prod_{j=1}^{t} (1 - ||z_j||^2)^{\alpha_j} z_1^{\alpha_{j1}} \cdots z_t^{\alpha_{jt}} w^\beta dV(\xi) = \frac{\pi^{n_1 + \cdots + n_t + m} C(\chi, \mu)^{m+|\beta|} \Gamma(s_0 + 1)}{\Gamma(m + s_0 + |\beta| + 1)} \times \prod_{j=1}^{t} \frac{(\alpha_j) \Gamma(\mu_j(m + |\beta|) + s_j + 1)}{\Gamma(\mu_j(m + |\beta|) + s_j + |\alpha_j| + n_j + 1)},$$

where $(\alpha_j)! = \alpha_{j1}! \cdots \alpha_{jn_j}!$, $1 \leq j \leq t$.

Proof. By using the polar coordinates $z_{jl} = r_{jl} e^{i\theta_{jl}}$ ($1 \leq j \leq t, 1 \leq l \leq n_j$), we have

$$\int_{\Omega(\mu, t)} \rho^{\alpha_0}(\xi) \prod_{j=1}^{t} (1 - ||z_j||^2)^{\alpha_j} z_1^{\alpha_{j1}} \cdots z_t^{\alpha_{jt}} w^\beta dV(\xi)$$

$$= (2\pi)^{n_1 + \cdots + n_t} \int_{[r_1 < 1]} \int_{[r_2 > 0]} \cdots \int_{[r_t < 1]} \int_{[r_1 > 0]} \cdots \int_{[r_t > 0]} \prod_{j=1}^{t} (1 - ||r_j||^2)^{\alpha_j} \times \frac{1}{C(\chi, \mu)^{m}} \prod_{j=1}^{t} (1 - ||r_j||^2)^{\mu_j} |w|^{2} dV(w) dV(r_1, \ldots, r_t),$$

where $W = \{w \in \mathbb{C}^m : ||w||^2 < C(\chi, \mu)^{m} \prod_{j=1}^{t} (1 - ||r_j||^2)^{\mu_j} \}$. Denote the integral in the last line of the above equation by $I$. Let $S_m$ denote the surface of the
unit ball in $\mathbb{C}^m$ and $d\sigma$ denote the normalized measure of $S_m$. Using the polar coordinates we have

$$I = \frac{2\pi^m}{\Gamma(m)} \int_{\mathbb{R}^m} \left( \frac{1 - r^2}{C(\chi, \mu)} \prod_{j=1}^t (1 - \|r_j\|^2)^{\mu_j} \right)^{s_0} r^{2|\beta|} |\zeta|^2 d\sigma(\zeta)$$

$$= \frac{2\pi^m}{\Gamma(m)} \frac{\Gamma(m)\beta!}{\Gamma(m + |\beta|)} \int_{\mathbb{R}^m} \left( \frac{1 - r^2}{C(\chi, \mu)} \prod_{j=1}^t (1 - \|r_j\|^2)^{\mu_j} \right)^{s_0} r^{2|\beta|+s_{\beta}+1} d\sigma(\zeta)$$

By setting $\tilde{r} = C(\chi, \mu)^{-1} \prod_{j=1}^t (1 - \|r_j\|^2)^{-\mu_j} r^2$, we have

$$I = \frac{\pi^m\beta!}{\Gamma(m + |\beta|)} \left( \frac{C(\chi, \mu)^{m+|\beta|}}{\prod_{j=1}^t (1 - \|r_j\|^2)^{\mu_j}} \right)^{s_0} \int_0^1 \tilde{r}^{m+|\beta|-1}(1 - \tilde{r})^{s_0} d\tilde{r}$$

Putting (7) into (6), and setting $\tilde{r}_j = (\tilde{r}_j, \ldots, \tilde{r}_{j-1}) = (r_j^2, \ldots, r_w^2)$, $1 \leq j \leq t$, we obtain

$$\int_{\Omega_{(\mu, t)}} \rho^{s_0}(\xi) \prod_{j=1}^t (1 - \|z_j\|^2)^{s_j} |z_1^{\alpha_1} \cdots z_n^{\alpha_n} w^\beta|^2 dV(\xi)$$

$$= \frac{\pi^{n1+\cdots+n1+m} C(\chi, \mu)^{m+|\beta|}}{\Gamma(m + s_0 + |\beta| + 1)}$$

$$\times \prod_{j=1}^t \left( \sum_{\tilde{r}_j \geq 0} \right) \left( 1 - \sum_{\tilde{r}_j} \right)^{\mu_j(m+|\beta|)+s_j} \tilde{r}_j^{s_j} d\tilde{r}_j$$

$$= \frac{\pi^{n1+\cdots+n1+m} C(\chi, \mu)^{m+|\beta|}}{\Gamma(m + s_0 + |\beta| + 1)}$$

$$\times \prod_{j=1}^t \left( \alpha_j \right)^{m+|\beta|+s_j} d\tilde{r}_j$$

where the last equality holds by Lemma 2.1.

Let $\xi = (z_1, \ldots, z_t, w)$, $\xi' = (z_1', \ldots, z_t', w') \in \Omega(\mu, t)$, and denote

$$\psi_{t,c}(\xi, \xi') = \frac{(w, w')^t}{\prod_{j=1}^t (1 - (z_j, z_j'))^{e_j}},$$

$$\psi_{t,c}(\xi, \xi') = \frac{(w, w')^t}{\prod_{j=1}^t (1 - (z_j, z_j'))^{e_j}},$$

$$\psi_{t,c}(\xi, \xi') = \frac{(w, w')^t}{\prod_{j=1}^t (1 - (z_j, z_j'))^{e_j}}.$$
where \( l \) is a non-negative integer and \( c = (c_1, \ldots, c_t) \in \mathbb{C}^t \) with \( \Re c_j > 0 \) (\( 1 \leq j \leq t \)) and the complex power is taken to be the principal branch.

**Lemma 2.6.** Let \( b, c \in \mathbb{C}^t, s = (s_0, s_1, \ldots, s_t) \), \( s_0 = \sigma_0 + ia_0 \), \( \sigma_0 > -1 \), \( a_0 \in \mathbb{R} \) and \( s_j = \sigma_j + ia_j \), \( \sigma_j > -\mu_j, m - 1 \), \( a_j \in \mathbb{R}, \Re b_j > 0, \Re c_j > 0, 1 \leq j \leq t. \)

(i) If \( l \neq k \), then

\[
(8) \quad \int_{\Omega(\mu, t)} \rho^s(\xi) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{s_j} \psi_{\alpha, c}(\xi, \xi') \overline{\psi_{\beta, b}(\xi, \xi')} dV(\xi') = 0.
\]

(ii)

\[
\int_{\Omega(\mu, t)} \rho^s(\xi) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{s_j} |\psi_{\alpha, c}(\xi, \xi')|^2 dV(\xi')
\]

\[
(9) \quad = B(s_0, l, c) \|w\|^{2l} \prod_{j=1}^{t} \sum_{l_j=0}^{\infty} \frac{|\Gamma(l_j + c_j)|^2 \|z_j\|^2_{\alpha, \beta}}{\Gamma(l_j + 1) \Gamma(\mu_j + m + l)_j + s_j + l_j + n_j + 1},
\]

where

\[
B(s_0, l, c) = \frac{\pi^{n_1 + \cdots + n_t + m} C(\chi, \mu)^{m+l} \Gamma(s_0 + 1)}{\Gamma(m + s_0 + l + 1)} \prod_{j=1}^{t} \frac{\Gamma(\mu_j + m + l + s_j + 1)}{|\Gamma(c_j)|^2}.
\]

**Proof.** Using polar coordinates, it is easy to obtain (8). Now we prove (9).

\[
\int_{\Omega(\mu, t)} \rho^s(\xi) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{s_j} |\psi_{\alpha, c}(\xi, \xi')|^2 dV(\xi')
\]

\[
= \int_{\Omega(\mu, t)} \rho^s(\xi) \sum_{|\beta| = l_0} \frac{\Gamma^2(l + 1)}{\prod_{i=1}^{m} \Gamma^2(\beta_i + 1)} |w^{\alpha, \beta}|^2 \prod_{j=1}^{t} (1 - \|z_j\|^2)^{s_j}
\]

\[
\times \sum_{l_j=0}^{\infty} \sum_{|\alpha_j| = l_j} \left| \frac{\Gamma(l_j + c_j)}{\prod_{i=1}^{m} \Gamma(\alpha_j i + 1) \Gamma(c_j)} \right|^2 |z_j^{\alpha_j} \overline{z_j}^{\alpha_j}|^2 dV(\xi').
\]

By Lemma 2.2 and Lemma 2.5, we have

\[
\int_{\Omega(\mu, t)} \rho^s(\xi) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{s_j} |\psi_{\alpha, c}(\xi, \xi')|^2 dV(\xi')
\]

\[
= \sum_{|\beta| = l_0} \prod_{j=1}^{t} \sum_{l_j=0}^{\infty} \sum_{|\alpha_j| = l_j} \frac{\Gamma^2(l + 1)}{\prod_{i=1}^{m} \Gamma^2(\beta_i + 1)} |w^{\alpha, \beta}|^2 |z_j^{\alpha_j} \overline{z_j}^{\alpha_j}|^2 \left| \frac{\Gamma(l_j + c_j)}{\prod_{i=1}^{m} \Gamma(\alpha_j i + 1) \Gamma(c_j)} \right|^2
\]

\[
\times \frac{\pi^{n_1 + \cdots + n_t + m} C(\chi, \mu)^{m+l} \beta \Gamma(s_0 + 1)}{\Gamma(m + s_0 + |\beta| + 1)} \frac{(\alpha) \Gamma(\mu_j + |\beta| + s_j + 1)}{\Gamma(\mu_j + |\beta| + s_j + |\alpha_j| + n_j + 1)}
\]

\[
= \frac{\pi^{n_1 + \cdots + n_t + m} C(\chi, \mu)^{m+l} \Gamma(s_0 + 1)}{\Gamma(m + s_0 + l + 1)} \|w\|^{2l} \prod_{j=1}^{t} \sum_{l_j=0}^{\infty} \frac{\Gamma(\mu_j + m + l + s_j + 1)}{|\Gamma(c_j)|^2}.
\]
\[
\times \frac{|\Gamma(l_j + c_j)|^{2} \|z_j\|^{2l_j}}{\Gamma(l_j + 1)\Gamma(\mu_j(m + l) + s_j + l_j + n_j + 1)}.
\]

Let
\[
G_{k,s,u,v}(\xi, \xi') = C(\chi, \mu)^{\frac{k}{2}} \prod_{j=1}^{t} \left(1 - (\langle z_j, z'_j \rangle )\right)^{b_j} \left(1 - \frac{\langle w, w' \rangle}{C(\chi, \mu)\prod_{j=1}^{t}(1 - (\langle z_j, z'_j \rangle )^{\mu_j}}\right)^{-\frac{d}{2}},
\]
where \(b_j = \mu_j(m + u) + s_j + n_j + v_j + 1, 1 \leq j \leq t, d = s_0 + m + k + u + 1.\) Then
\[
K_{s,u,v}(\xi, \xi') = \sum_{k=0}^{\infty} C_k G_{k,s,u,v}(\xi, \xi').
\]

**Lemma 2.7.** If \(-1 < x_0 < u + \sigma_0, -\mu_j m - 1 < x_j < \mu_j u + \sigma_j + v_j, 1 \leq j \leq t,\) then
\[
\int_{\Omega(\mu, t)} \rho^{x_0}(\xi') \prod_{j=1}^{t} \left(1 - \|z_j\|^2 \right)^{x_j} |G_{k,s,u,v}(\xi, \xi')|^2 dV(\xi') \lesssim \rho^{x_0 - \sigma_0 - u}(\xi') \prod_{j=1}^{t} \left(1 - \|z_j\|^2 \right)^{x_j - \mu_j u - \sigma_j - v_j}.
\]

**Proof.** Since \(\xi = (z, w), \xi' = (z', w') \in \Omega_{\mu},\) we can see that
\[
|\langle w, w' \rangle| < \left|C(\chi, \mu) \prod_{j=1}^{t} (1 - \langle z_j, z'_j \rangle)^{\mu_j}\right|.
\]
Therefore \(G_{k,s,u,v}\) can be written as
\[
G_{k,s,u,v}(\xi, \xi') = \sum_{l=0}^{\infty} \frac{\Gamma(l_j + 1)}{\Gamma(l_j + 1)} C(\chi, \mu)^{\frac{k}{2} - l} \psi_{l,c}(\xi, \xi'),
\]
where \(c_j = \mu_j l + \frac{b_j}{2}, 1 \leq j \leq t.\) By Lemma 2.6, we have
\[
\int_{\Omega(\mu, t)} \rho^{x_0}(\xi') \prod_{j=1}^{t} \left(1 - \|z_j\|^2 \right)^{x_j} |G_{k,s,u,v}(\xi, \xi')|^2 dV(\xi') = \sum_{l=0}^{\infty} \frac{\Gamma(l_j + 1)^{2} \|C(\chi, \mu)^{d - 2l} \int_{\Omega(\mu, t)} \rho^{x_0}(\xi') \prod_{j=1}^{t} \left(1 - \|z_j\|^2 \right)^{x_j} \times |\psi_{l,c}(\xi, \xi')|^2 dV(\xi')}{|\Gamma(l_j + 1)|^2 |\Gamma(l_j + 1)|^2}.
\]
Let \(J_1(\xi)\) denote the integral on the right hand side of the above equation. By Lemma 2.6, we have
\[
J_1(\xi) = B(x_0, l, c) \||w||^{2l_j} \prod_{j=1}^{t} \sum_{l_j=0}^{\infty} \frac{|\Gamma(l_j + c_j)|^{2} \|z_j\|^{2l_j}}{\Gamma(l_j + 1)\Gamma(\mu_j(m + l) + x_j + l_j + n_j + 1)}.
\]
Using Stirling’s formula $\Gamma(z) = \sqrt{2\pi}e^{-z}z^{z-\frac{1}{2}}(1 + O(\frac{1}{z}))$ ($|\text{arg}(z)| < \pi, |z| \to \infty$) in (11), we have

$$J_l(\xi) \approx B(x_0, l, c)||w||^{2l} \prod_{j=1}^{t} \sum_{l_j=0}^{\infty} \{\xi_j^{l_j} + \frac{l_j + \mu_j(l + u) + \sigma_j + v_j - x_j}{l_j!}\}$$

(12)  

$$\approx B(x_0, l, c)||w||^{2l} \prod_{j=1}^{t} \sum_{l_j=0}^{\infty} \Gamma(\eta_j)(1 - ||z_j||^2)^{-\eta_j},$$

where $\eta_j = \mu_j(l + u) + \sigma_j + v_j - x_j$, $1 \leq j \leq t$. Putting (12) into (10), we have

$$\int_{\Omega(\mu, t)} \rho^{\tau_0}(\xi') \prod_{j=1}^{t} (1 - ||z_j'||^2)^{x_j} |G_{k, s, u, v}(\xi, \xi')|^2 dV(\xi')$$

$$\approx \sum_{l=0}^{\infty} \frac{\Gamma((l + \frac{1}{2})^2)}{\Gamma(l^2)!} B(x_0, l, c) C(\chi, \mu)^{-l} ||w||^{2l} \prod_{j=1}^{t} \sum_{l_j=0}^{\infty} \Gamma(\eta_j)(1 - ||z_j||^2)^{-\eta_j}$$

$$\approx \prod_{j=1}^{t} (1 - ||z_j||^2)^{x_j-\mu_j u - \sigma_j - v_j} \sum_{l=0}^{\infty} \frac{\Gamma((l + \frac{1}{2})^2)}{\Gamma(l + \frac{b_j}{2})^2} \frac{\Gamma(\mu_j l + x_j + 1)}{\Gamma(\mu_j l + 1) } (\frac{||w||^2}{C(\chi, \mu) \prod_{i=1}^{t} (1 - ||z_i||^2)^{\mu_i}})^l$$

(13)  

$$\times \sum_{l=0}^{\infty} \frac{\Gamma(l + \sigma_0 + u - x_0)}{l!} \frac{1}{n_{l_1 + \cdots + n_{l_t} - k}} (\frac{||w||^2}{C(\chi, \mu) \prod_{i=1}^{t} (1 - ||z_i||^2)^{\mu_i}})^l.$$  

Note that $k \leq n_1 + \cdots + n_t$, by (13) we obtain

$$\int_{\Omega(\mu, t)} \rho^{\tau_0}(\xi') \prod_{j=1}^{t} (1 - ||z_j'||^2)^{x_j} |G_{k, s, u, v}(\xi, \xi')|^2 dV(\xi')$$

$$\leq \prod_{j=1}^{t} (1 - ||z_j||^2)^{x_j-\mu_j u - \sigma_j - v_j} \left(1 - \frac{||w||^2}{C(\chi, \mu) \prod_{i=1}^{t} (1 - ||z_i||^2)^{\mu_i}}\right)^{x_0 - \sigma_0 - u}. \quad \square$$

Using Lemma 2.7, we have following results, which are needed in the proof of Theorem 1.2.

**Lemma 2.8.** If $-1 < x_0 < u + \sigma_0$, $-\mu_j m - 1 < x_j < \mu_j u + \sigma_j + v_j$, $1 \leq j \leq t$, then

$$\int_{\Omega(\mu, t)} \rho^{\tau_0}(\xi') \prod_{j=1}^{t} (1 - ||z_j'||^2)^{x_j} |K_{s, u, v}(\xi, \xi')| dV(\xi')$$
Proof. Since $K_{\eta,\mu_1,\nu}(\xi,\xi') = \sum_{k=0}^{n_1 \cdots + n_n} C_k G_{k,\eta,\mu_1,\nu}(\xi,\xi')$, by Lemma 2.7 we have

$$\int_{\Omega(\mu,\eta)} \rho^{\sigma_0}(\xi) \prod_{j=1}^{\ell} (1 - \|z_j\|^2)^{h_j} |K_{\sigma,\mu_1,\nu}(\xi,\xi')|dV(\xi') \leq \sum_{k=0}^{n_1 \cdots + n_n} |C_k| \int_{\Omega(\mu,\eta)} \rho^{\sigma_0}(\xi) \prod_{j=1}^{\ell} (1 - \|z_j\|^2)^{h_j} |G_{k,\eta,\mu_1,\nu}(\xi,\xi')|^2dV(\xi') \leq \rho^{\sigma_0}\prod_{j=1}^{\ell} (1 - \|z_j\|^2)^{h_j}. \tag{**}
$$

Lemma 2.9. If $-1 < x_0 < u + \sigma_0$, $-\mu_j m - 1 < x_j < \mu_j u + \sigma_j + v_j$, $1 \leq j \leq t$, then

$$\int_{\Omega(\eta,\nu)} \rho^{\sigma_0}(\xi') \prod_{j=1}^{\ell} (1 - \|z_j\|^2)^{h_j} |K_{\sigma,\mu_1,\nu}(\xi,\xi')|dV(\xi) \leq \rho^{\sigma_0}\prod_{j=1}^{\ell} (1 - \|z_j\|^2)^{h_j}. \tag{***}
$$

Proof. Noting that $|K_{\sigma,\mu_1,\nu}(\xi,\xi')| = \nabla K_{\sigma,\mu_1,\nu}(\xi,\xi') = |K_{\sigma,\mu_1,\nu}(\xi,\xi')|$, by Lemma 2.8 with $\xi$ and $\xi'$ exchanged we obtain the desired inequality. \hfill \Box

Lemma 2.10. If $s = (s_0, s_1, \ldots, s_t)$, $s_0 = \sigma_0 + ia_0$, $\sigma_0 > -1$, $a_0 \in \mathbb{R}$ and $s_j = \sigma_j + ia_j$, $\sigma_j > -\mu_j m - 1$, $a_j \in \mathbb{R}$, $1 \leq j \leq t$, then there exists a constant $K$ independent of $\xi$ such that

$$\int_{\Omega(\mu,\eta)} \rho^{\sigma_0}(\xi') \prod_{j=1}^{\ell} (1 - \|z_j\|^2)^{h_j} K_{\sigma,\mu_1,\nu}(\xi,\xi')dV(\xi') = K. \tag{****}
$$

Proof. Since $K_{\sigma,\mu_1,\nu}(\xi,\xi') = \sum_{k=0}^{n_1 \cdots + n_n} C_k \sum_{l=0}^{\infty} \frac{\Gamma(d+l)}{\Gamma(d)!} \psi_{\mu_1,\nu}(\xi,\xi')$, where $c_j = \mu_j l + b_j$, $1 \leq j \leq t$, by Lemma 2.6 we have

$$\int_{\Omega(\mu,\eta)} \rho^{\sigma_0}(\xi') \prod_{j=1}^{\ell} (1 - \|z_j\|^2)^{h_j} K_{\sigma,\mu_1,\nu}(\xi,\xi')dV(\xi') = \int_{\Omega(\mu,\eta)} \rho^{\sigma_0}(\xi') \prod_{j=1}^{\ell} (1 - \|z_j\|^2)^{h_j} \sum_{k=0}^{n_1 \cdots + n_n} C_k \sum_{l=0}^{\infty} \frac{\Gamma(d+l)}{\Gamma(d)!} \psi_{\mu_1,\nu}(\xi,\xi')dV(\xi') = \sum_{k=0}^{n_1 \cdots + n_n} C_k \int_{\Omega(\mu,\eta)} \rho^{\sigma_0}(\xi') \prod_{j=1}^{\ell} (1 - \|z_j\|^2)^{h_j} \sum_{l_j=0}^{\infty} \sum_{|\alpha_j|=l_j}^{\infty} \frac{\Gamma(l_j + b_j)}{\Gamma(\alpha_j + 1)}.$$
× \sum_{\alpha_1, \ldots, \alpha_t} \int_{\Omega(\mu, t)} \rho^{\alpha_1}(\xi) \prod_{j=1}^{t}(1 - \|z_j\|^2)^{\mu_j} dV(\xi) = \mathcal{K}.

\]

Finally, we will give the Schur’s lemma, which is a common tool to prove boundedness of an integral operator.

**Lemma 2.11** (See [26]). Let \((\Omega, \kappa)\) be a measure space and \(K\) be a nonnegative measurable function on \(\Omega \times \Omega\). Let \(T\) be the integral operator with kernel \(K\), and let \(1 < p < \infty\) with \(\frac{1}{p} + \frac{1}{q} = 1\). If there exist positive constants \(C_1\) and \(C_2\) and a positive measurable function \(h\) on \(\Omega\) such that

\[
\int_{\Omega} K(x, y) h^q(y) d\kappa(y) \leq C_1 h^q(x)
\]

for almost every \(x\) in \(\Omega\) and

\[
\int_{\Omega} K(x, y) h^p(x) d\kappa(x) \leq C_2 h^p(y)
\]

for almost every \(y\) in \(\Omega\), then \(T\) is a bounded linear operator on \(L^p(\Omega, \kappa)\).

3. **Proof of the main result**

3.1. **Proof of Theorem 1.1**

Since \(\Omega(\mu, t)\) is a Reinhardt domain, \(\left\{ \frac{z_1^{\alpha_1} \cdots z_t^{\alpha_t}}{\|z_1^{\alpha_1} \cdots z_t^{\alpha_t}\|_2}, w^\beta \right\}\) constitutes an orthonormal basis of \(A^2(\Omega(\mu, t), w^\sigma)\). Then

\[
K_{\omega_{\sigma}}(\xi, \xi) = \sum_{\alpha_j \in \mathbb{N}^t, 1 \leq j \leq t, \beta \in \mathbb{N}^m} \frac{|z_1^{\alpha_1} \cdots z_t^{\alpha_t} w^\beta|^2}{\|z_1^{\alpha_1} \cdots z_t^{\alpha_t} w^\beta\|^2_2, w^\sigma}.
\]

By Lemma 2.2 and Lemma 2.5, we have

\[
K_{\omega_{\sigma}}(\xi, \xi) = \frac{1}{\pi^{n_1 + \cdots + n_t + m} \Gamma(\sigma_0 + 1)} \prod_{j=1}^{t} \sum_{\alpha_j \in \mathbb{N}^t, 1 \leq j \leq t} \frac{\Gamma(m + \sigma_0 + |\beta| + 1)}{C(\chi, \mu)^{m + |\beta|} |\beta|!} \times \frac{\Gamma(\mu_j(m + |\beta|) + \sigma_j + |\alpha_j| + n_j + 1)}{\Gamma(\mu_j(m + |\beta|) + \sigma_j + 1)|\alpha_j|!} |z_j^{\alpha_j}|^2 |w^\beta|^2
\]

\[
= \frac{1}{\pi^{n_1 + \cdots + n_t + m} \Gamma(\sigma_0 + 1)} \prod_{j=1}^{t} \sum_{\beta \in \mathbb{N}^m} \frac{\Gamma(m + \sigma_0 + |\beta| + 1)}{C(\chi, \mu)^{m + |\beta|} |\beta|!} \times \frac{\Gamma(\mu_j(m + |\beta|) + \sigma_j + |\alpha_j| + n_j + 1)}{\alpha_j!} |z_j^{\alpha_j}|^2
\]

\[
\times \sum_{\alpha_j \in \mathbb{N}^t} |z_1^{\alpha_1} \cdots z_t^{\alpha_t}|^2 |w^\beta|^2
\]
(16) \[ \frac{1}{\pi^{n_1 + \cdots + n_t + m} C(\chi, \mu)^m} \prod_{j=1}^{t} \sum_{\beta \in \mathbb{N}^m} \frac{\Gamma(m + \sigma_0 + |\beta| + 1) \Gamma(\zeta_j) \mu_0}{\Gamma(\mu_j (m + |\beta|) + \sigma_j + 1) \beta!} |w_\beta|^2 \times (1 - \|z_j\|^2)^{-\zeta_j} C(\chi, \mu)^{-|\beta|}, \]

where \(\zeta_j = \mu_j (m + |\beta|) + \sigma_j + n_j + 1, j = 1, \ldots, t.\)

We note that \(\frac{1}{\pi^{n_1 + \cdots + n_t + m} C(\chi, \mu)^m} \prod_{j=1}^{t} \Pi_{\beta}^0 (\mu_j (m + |\beta|) + \sigma_j + n_j)\) is a polynomial of degree \(\sum_{j=1}^{t} n_j\) in \(|\beta|\), and we denote it by \(P_{n_1 + \cdots + n_t}(|\beta|)\). By (16) and Lemma 2.3, we have

\[ K_{\omega_\nu}(\xi, \xi) = \sum_{\beta \in \mathbb{N}^m} \frac{\Gamma(m + \sigma_0 + |\beta| + 1)}{\beta! C(\chi, \mu)^{|\beta|}} P_{n_1 + \cdots + n_t}(|\beta|) |w_\beta|^2 \prod_{j=1}^{t} (1 - \|z_j\|^2)^{-\zeta_j} \times P_{n_1 + \cdots + n_t}(|\beta|) |w_\beta|^2 \left( C(\chi, \mu) \prod_{i=1}^{t} (1 - \|z_i\|^2)^{\mu_i} \right)^{-|\beta|} \]

\[ = \sum_{n_1 + \cdots + n_t} C_k \left( \frac{1}{C(\chi, \mu) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{\mu_j}} \right)^{-k - \sigma_{n_0} - 1} \times \left( \prod_{j=1}^{t} (1 - \|z_j\|^2)^{\mu_j (k + \sigma_0 + 1) - \sigma_j - n_j - 1} \right) \]

By exchanging the second variable with \(\xi'\) in \(K_{\omega_\nu}(\xi, \xi)\) in the above calculations, we obtain (1).

3.2. Proof of Theorem 1.2

If \(1 < p < \infty\), we first prove the necessity of condition (3) for the boundedness of \(T_{s,u,v,c}^\ast\) on \(L^p(\Omega(\mu, t), \omega_A)\).

Choose \(\delta_0 > \max \left( -1 - \sigma_0, -\frac{1 - \lambda_a}{p} \right), \delta_j > \max \left( -1 - \sigma_j - \mu_j m, -\frac{1 - \lambda_j - \mu_j m}{p} \right), 1 \leq j \leq t.\) Let \(g(\xi) = \rho^{\mu_0}(\xi) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{\delta_j}\). Then \(g \in L^p(\Omega(\mu, t), \omega_A)\). By the definition of \(T_{s,u,v,c}^\ast\) we have

\[ (T_{s,u,v,c}^\ast g)(\xi) = \rho^{\mu + \sigma}(\xi) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{\mu_j + v_j} \]
\[
\times \int_{\Omega(\mu,t)} \rho^{s_0+\delta_j}(\xi') \prod_{i=1}^{t}(1-\|z_i\|^2)^{s_i+\delta_j} K_{s,u,v}(\xi,\xi')dV(\xi').
\]

By Lemma 2.10, there exists a constant \( K \) independent of \( \xi \) such that
\[
(T_{s,u,v,\epsilon} g)(\xi) = K \rho^{u+}(\xi) \prod_{j=1}^{t}(1-\|z_j\|^2)^{\mu_j+\lambda_j}.
\]

Since \( T_{s,u,v,\epsilon} \) is bounded on \( L^p(\Omega(\mu,t),\omega_\lambda) \), we have \( p(u+\epsilon) + \lambda_0 > -1, p(\mu_j u + v_j) + \lambda_j > -\mu_j m - 1, 1 \leq j \leq t \). To prove the right side of (3) we will look at \( T_{s,u,v,\epsilon}^* \), the adjoint of \( T_{s,u,v,\epsilon} \). \( T_{s,u,v,\epsilon}^* \) is given by
\[
(T_{s,u,v,\epsilon}^* g)(\xi) = \rho^{\lambda_0-\lambda}(\xi) \prod_{j=1}^{t}(1-\|z_j\|^2)^{\lambda_j} \int_{\Omega(\mu,t)} \rho^{u+\lambda_0}(\xi') \]
\[
\times \prod_{i=1}^{t}(1-\|z_i\|^2)^{\mu_i u + v_i + \lambda_i} K_{s,u,v}(\xi,\xi') g(\xi')dV(\xi').
\]

We know that if \( T_{s,u,v,\epsilon} \) is bounded on \( L^p(\Omega(\mu,t),\omega_\lambda) \), then \( T_{s,u,v,\epsilon}^* \) is bounded on \( L^q(\Omega(\mu,t),\omega_\lambda) \), where \( \frac{1}{p} + \frac{1}{q} = 1 \) (See [26]). Choose \( \delta_0 > \max \left( -1 - u - \lambda_0, -\lambda_0 - \epsilon, -\lambda_0, -\mu_0, -\mu_j, -\mu_j m, -\frac{1-\lambda_0-\mu_j m}{\eta} \right) \), \( \delta_j > \max \left( -1 - \lambda_j, -v_j - \mu_j (m + u), -\frac{1-\lambda_0-\mu_j m}{\eta} \right) \), \( 1 \leq j \leq t \).

Let \( g(\xi) = \rho^{\delta}(\xi) \prod_{j=1}^{t}(1-\|z_j\|^2)^{\delta_j} \). Then \( g \in L^q(\Omega(\mu,t),\omega_\lambda) \). By definition of \( T_{s,u,v,\epsilon}^* \) and Lemma 2.10, there exists a constant \( K \) independent of \( \xi \) such that
\[
(T_{s,u,v,\epsilon}^* g)(\xi) = K \rho^{\lambda_0-\lambda}(\xi) \prod_{j=1}^{t}(1-\|z_j\|^2)^{\lambda_j}.
\]

Since \( T_{s,u,v,\epsilon}^* \) is bounded on \( L^q(\Omega(\mu,t),\omega_\lambda) \), we have \( q(\sigma_j - \lambda_j) + \lambda_j > -\mu_j m - 1, 1 \leq j \leq t \).

Next we prove the sufficiency of condition (3) for the boundedness of \( T_{s,u,v,\epsilon} \) on \( L^p(\Omega(\mu,t),\omega_\lambda) \). We can see that conditions
\[
\lambda_0 + 1 < p(\sigma_0 + 1), \quad \lambda_j + \mu_j + 1 < p(\sigma_j + \mu_j m + 1), \quad 1 \leq j \leq t
\]
and \( \frac{1}{p} + \frac{1}{q} = 1 \) imply that
\[
-1 - \sigma_0 < \frac{\sigma_0 - \lambda_0}{p}, \quad -1 - \sigma_j - \mu_j m < \frac{\sigma_j - \lambda_j}{p}, \quad 1 \leq j \leq t.
\]

Similarly, conditions
\[
-p(u+\epsilon) < \lambda_0 + 1, \quad -p(\mu_j u + v_j) < \lambda_j + \mu_j m + 1, \quad 1 \leq j \leq t
\]
and \( \frac{1}{p} + \frac{1}{q} = 1 \) imply that
\[
-1 - u - \lambda_0 - \epsilon < \frac{u+\epsilon}{q}, \quad -1 - v_j - \lambda_j - \mu_j (m + u) < \frac{\mu_j u + v_j}{q}, \quad 1 \leq j \leq t.
\]
Therefore, 
\[ A_0 := \left( \frac{-1 - \sigma_0}{q}, \frac{u + \epsilon}{q} \right) \cap \left( \frac{-1 - u - \lambda_0 - \epsilon}{p}, \frac{\sigma_0 - \lambda_0}{p} \right) \neq \emptyset, \]
and for \( j = 1, 2, \ldots, t \) 
\[ A_j := \left( \frac{-1 - \sigma_j - \mu_j m}{q}, \frac{\mu_j u + v_j}{q} \right) \cap \left( \frac{-1 - v_j - \lambda_j - \mu_j (m + u)}{p}, \frac{\sigma_j - \lambda_j}{p} \right) \neq \emptyset. \]

Then we can take \( \tau_0 \in A_0, \tau_j \in A_j, 1 \leq j \leq t \). Let \( g(\xi) = \rho^{\sigma_0}(\xi) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{\tau_j} \) and write 
\[
(T_{s,u,v,\epsilon} f)(\xi) = \int_{\Omega(\mu,t)} \rho^{u+c}(\xi) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{\mu_j u + v_j} \rho^{\sigma_0 - \lambda_0}(\xi') \\
\times \prod_{i=1}^{t} (1 - \|z_i\|^2)^{\lambda_i} K_{s,u,v}(\xi,\xi') f(\xi') d\omega_{\lambda}(\xi').
\]
Let \( T \) be the integral operator with kernel 
\[
K(\xi,\xi') = \rho^{u+c}(\xi) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{\mu_j u + v_j} \rho^{\sigma_0 - \lambda_0}(\xi') \prod_{i=1}^{t} (1 - \|z_i\|^2)^{\lambda_i} \times |K_{s,u,v}(\xi,\xi')|.
\]
It is easy to see that if \( T \) is bounded on \( L^p(\Omega(\mu,t),\omega_{\lambda}) \), then \( T_{s,u,v,\epsilon} \) is bounded on \( L^p(\Omega(\mu,t),\omega_{\lambda}) \). We denote 
\[
A_{01} = \left( \frac{-1 - \sigma_0}{q}, \frac{u}{q} \right) \cap \left( \frac{-1 - u - \lambda_0 - \epsilon}{p}, \frac{\sigma_0 - \lambda_0 - \epsilon}{p} \right), \\
A_{02} = \left( \frac{-1 - \sigma_0}{q}, \frac{u}{q} \right) \cap \left( \frac{-1 - u - \lambda_0}{p}, \frac{\sigma_0 - \lambda_0}{p} \right), \\
A_{00} = \left( \frac{-1 - \sigma_0 + \epsilon}{q}, \frac{u + \epsilon}{q} \right) \cap \left( \frac{-1 - u - \lambda_0 - \epsilon}{p}, \frac{\sigma_0 - \lambda_0 - \epsilon}{p} \right), \\
A_{02} = \left( \frac{-1 - \sigma_0 + \epsilon}{q}, \frac{u + \epsilon}{q} \right) \cap \left( \frac{-1 - u - \lambda_0}{p}, \frac{\sigma_0 - \lambda_0}{p} \right).
\]
We can see that \( A_0 = A_{01} \cup A_{02} \cup A_{00} \cup A_{02} \).
If \( \tau_0 \in A_{01}, \tau_j \in A_j, j = 1, 2, \ldots, t \), by Lemma 2.8, we obtain 
\[
\int_{\Omega(\mu,t)} K(\xi,\xi') g^{\theta}(\xi') d\omega_{\lambda}(\xi') \\
= \rho^{u+c}(\xi) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{\mu_j u + v_j} \int_{\Omega(\mu,t)} \rho^{\sigma_0 + \theta \tau_0}(\xi') \prod_{i=1}^{t} (1 - \|z_i\|^2)^{\lambda_i + \theta \tau_i} \\
\times |K_{s,u,v}(\xi,\xi')| dV(\xi').
\]
Similarly, by Lemma 2.9, we obtain
\[ g^q(\xi) \]

Thus (14) and (15) hold for
\[ \tau_j \in A_j, j = 1, 2, \ldots, t, \]
for the same reason as the case \( \tau_0 \in A_{01} \), \( \tau_j \in A_j, j = 1, 2, \ldots, t, \) we have
\[ \int_{\Omega(\mu, t)} K(\xi, \xi')g^p(\xi)d\omega(\xi) \lesssim g^p(\xi'). \]

By Lemma 2.9, we obtain
\[ \int_{\Omega(\mu, t)} K(\xi, \xi')g^p(\xi)d\omega(\xi) \]
\[ \lesssim p^{u-\lambda_0}(\xi') \prod_{j=1}^{t}(1 - \|z_j\|^2)^{\sigma_j - \lambda_j} \int_{\Omega(\mu, t)} \]
\[ \times p^{u+\lambda_0+p\tau}(\xi) \prod_{i=1}^{t}(1 - \|z_i\|^2)^{\mu_i+\nu_i+\lambda_i+p\tau_i}|K_{s,u,v}(\xi, \xi')|dV(\xi) \]
\[ \lesssim p^{p\eta}(\xi') \prod_{j=1}^{t}(1 - \|z_j\|^2)^{p\tau_j} \]
\[ = g^p(\xi'). \]

Similarly, we can check that (14) and (15) hold for \( \tau_0 \in A_{01}'' \cup A_{02}'', \tau_j \in A_j, j = 1, 2, \ldots, t. \) Thus (14) and (15) hold for \( \tau_0 \in A_0, \tau_j \in A_j, j = 1, 2, \ldots, t. \) By Lemma 2.11, we know that \( T \) is bounded on \( L^p(\Omega(\mu, t), \omega\lambda) \). Therefore, if (3) holds, then \( T_{s,u,v,c} \) is bounded on \( L^p(\Omega(\mu, t), \omega\lambda) \).

When \( p = 1 \), if \( T_{s,u,v,c} \) is a bounded operator on \( L^1(\Omega(\mu, t), \omega\lambda) \), we will prove that (4) hold. By the same reason as in the case \( 1 < p < \infty \), we can prove the left side of (4). Since \( T_{s,u,v,c} \) is bounded on \( L^1(\Omega(\mu, t), \omega\lambda) \), then \( T_{s,u,v,c} \) is bounded on \( L^\infty(\Omega(\mu, t)) \). Choose \( \delta_0 > \max(0, -1 - \epsilon - \lambda_0), \delta_j > \max(0, -\nu_j - \lambda_j - \mu_j(m + u)), 1 \leq j \leq t. \) Let \( g(\xi) = \rho^{h_0}(\xi) \prod_{j=1}^{t}(1 - \|z_j\|^2)^{\delta_j} \).

Then \( g \in L^\infty(\Omega(\mu, t)) \). By (17),
\[ (T_{s,u,v,c})g(\xi) = \rho^{\sigma - \lambda_0}(\xi) \prod_{j=1}^{t}(1 - \|z_j\|^2)^{\sigma_j - \lambda_j} \int_{\Omega(\mu, t)} \rho^{u+\lambda_0+\delta_j}(\xi') \]
\[ \lesssim g^q(\xi). \]

Similarly, by Lemma 2.9, we obtain
\[ \int_{\Omega(\mu, t)} K(\xi, \xi')g^p(\xi)d\omega(\xi) \lesssim g^p(\xi'). \]

If \( \tau_0 \in A_{02}', \tau_j \in A_j, j = 1, 2, \ldots, t, \) for the same reason as the case \( \tau_0 \in A_{01}, \tau_j \in A_j, j = 1, 2, \ldots, t, \) we have
\[ \int_{\Omega(\mu, t)} K(\xi, \xi')g^p(\xi)d\omega(\xi) \lesssim g^p(\xi). \]
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\[ \times \prod_{i=1}^{t} (1 - \| z_i' \|^2)^{\mu_i u + v_i + \lambda_i} K_{s,u,v}(\xi, \xi') dV(\xi'). \]

By Lemma 2.10, there exists a constant \( K \) independent of \( \xi \) such that

\[ (T^*_{s,u,v,e} g)(\xi) = K \rho^{\omega - \lambda_0}(\xi) \prod_{i=1}^{t} (1 - \| z_i' \|^2)^{\sigma_i} K_{s,u,v}(\xi, \xi') dV(\xi'). \]

Since \( T^*_{s,u,v,e} \) is bounded on \( L^\infty(\Omega(\mu,t)) \), we have \( \sigma_0 \geq \lambda_0, \sigma_j \geq \lambda_j, 1 \leq j \leq t. \)

Next we will prove that if (5) holds, then \( T_{s,u,v,e} \) is a bounded operator on \( L^1(\Omega(\mu,t), \omega_\lambda) \). Given \( f \in L^1(\Omega(\mu,t), \omega_\lambda) \), we have

\[ \int_{\Omega(\mu,t)} |(T_{s,u,v,e} f)(\xi)| d\omega_\lambda(\xi) \leq \int_{\Omega(\mu,t)} (1 - \| z_i' \|^2)^{\sigma_i} K_{s,u,v}(\xi, \xi') |f(\xi')| dV(\xi') dV(\xi) \]

(18)

\[ = \int_{\Omega(\mu,t)} (1 - \| z_i' \|^2)^{\sigma_i} K_{s,u,v}(\xi, \xi') |f(\xi')| dV(\xi') dV(\xi) \]

Since (5) holds, by Lemma 2.9 and (18) we obtain

\[ \int_{\Omega(\mu,t)} |(T_{s,u,v,e} f)(\xi)| d\omega_\lambda(\xi) \leq \int_{\Omega(\mu,t)} \rho^{\omega + \lambda_0}(\xi') \prod_{i=1}^{t} (1 - \| z_i' \|^2)^{\sigma_i} |f(\xi')| d\omega_\lambda(\xi'). \]

Thus \( T_{s,u,v,e} \) is a bounded operator on \( L^1(\Omega(\mu,t), \omega_\lambda) \).

References


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